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GENERALIZED SPECTRUM AND NUMERICAL RANG OF MATRIX THE LORENTZIAN OSCILLATOR GROUP OF DIMENSION FOUR

RAFIK DERKAOUI*, **ABDERRAHMANE SMAIL**

University of Oran 1, Laboratory GEANLAB, P.O. Box 1524, Oran, 31000, Algeria

* *Corresponding author:* rafikderkaoui27@yahoo.com

Abstract. In this paper, we find the spectrum, pseudo-spectrum and numerical rang of matrix of the metric g_a.

1. Introduction

Connected Lie groups that admit a bi-invariant Lorentzian metric were determined by the first of the authors in [14]. Among them, those that are solvable, non-commutative, and simply connected are called oscillator groups. This group has many properties useful both in geometry and physics.

We study here the geometry of these groups and their networks, i.e their discrete sub-groups co-compact. If G is an oscillator group, its networks determine compact homogeneous Lorentz manifolds, on which G acts by isometries.

Let $H_{2k+1} = \mathbb{R} \times \mathbb{C}^k$ be the Heisenberg group and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ k be strictly positive real numbers. Let the additive group $\mathbb{R}$ act on H_{2k+1} by the action:

$$\rho(t)(u, (z_j)) = (u, (e^{i\lambda_j t} z_j)).$$

The group $G_k(\lambda)$, a semi-direct product of $\mathbb{R}$ by H_{2k+1} following ρ, is provided with a bi-invariant Lorentz metric. Here is how it is built:

$$\mathfrak{g} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2k}$$

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is the tangent space at the origin. Let us extend the usual scalar product of $\mathbb{R}^{2k}$ into a symmetric bilinear form over $\mathfrak{g}$ so that the plane $\mathbb{R} \times \mathbb{R}$ is hyperbolic and orthogonal to $\mathbb{R}^{2k}$. This form defines an invariant Lorentz metric on the left on $G_k(\lambda)$, it is also invariant on the right because the adjoint operators on $\mathfrak{g}$ are antisymmetric [15].

Groups $G_k(\lambda)$ are characterized [14] by:

Theorem 1.1. *The groups $G_k(\lambda)$ are the only Lie group simply connected , resolvable and noncommutative which admit a bi-invariant Lorentz metric.*

Remark 1.1. *it is easy to see that the groups $G_1(\lambda)$ are isomorphous; the group $G_1 = G_1(1)$ is usually known as the oscillator group [20].*

Since [1], [2] et [3] the oscillator group has been generalized to a dimension equal to an even number $2n$ with $n \geq 2$, plus this provides a known example of homogeneous space-time [6].

For $n = 2$, the oscillator group of dimension 4 admits a Lorentzian metric invariant on the left and on the right (bi-invariant). This bi-invariant metric has been generalized a family g_a, $-1 < a < 1$, invariant Lorentzian metrics on the left. For $a = 0$, the metric g_0 become or the only example of Lorentzian bi-invariant metric [7]

The researchers Giovani and Zaeim extracted three vectors feilds from the oscillator group, which are: Killing vector feild, Affine vector feild, parallel vector feild (see [4]).

and also Giovani and Zaeim classified the totally geodesic and parallel hypersurfaces of four-dimensional groups (see [3]).

Varah published an article entitled "On the separation of two matrices" in which he defined with standard 2 the pseudospectrum using the smallest singular value $\sigma_{\min}(zI - A)$ under the notion $\Lambda_\epsilon(A)$ see [23]. In the 1960s the pseudospectrum was studied in several by L. N. Trefethen [19], [21].

In recent years the study of the pseudospectrum has been very active, many contributions related to the pseudospectrum have been made by various researchers, for example, J. S. Baggett, A. Bottcher, M. Embree, L. N. Trefethen, L. Reichel, S.C.Reddy, T.A. Driscoll.

The pseudospectrum of a normal matrix A consists of circles of radius ϵ around each eigenvalue. For non-nirmal matrices, the pseudospectrum takes different forms in the complex plane. in [19] The pseudospectrum of thirteen highly non-normal matrices is presented.

2. Preliminaries

At the moment we consider on G_λ a family parametre of left-invariant Lorentzian metrics g_a. With respect to coordinates (x_1, x_2, x_3, x_4), this metric g_a is explicitly given by

$$g_a = adx_1^2 + 2ax_3dx_1dx_2 + (1 + ax_3^2)dx_2^2 + dx_3^2 + 2dx_1dx_4 + 2x_3dx_2dx_4 + adx_4^2,$$

with $-1 < a < 1$.

Note that for $a = 0$ and $\lambda = 1$ we have the bi-invariant metric on the oscillator group G_1 [7]. In all other cases, g_a is only invariant on the left.

The matrix of the metric g_a is given by

$$A_a = \begin{pmatrix} a & ax_3 & 0 & 1 \\ ax_3 & 1 + ax_3^2 & 0 & x_3 \\ 0 & 0 & 1 & 0 \\ 1 & x_3 & 0 & a \end{pmatrix}.$$

Numerical rang

Definition 2.1. *Let A be an $n \times n$ complex matrix. Then the numerical rang of A, $W(A)$, is defined to be*

$$W(A) = \left\{ \frac{x^* A x}{x^* x}, \quad x \in \mathbb{C}^n, \ x \neq 0 \right\}.$$

where x^* denotes the conjugate transpose of the vector x.

Proposition 2.1. *Based on the definition of the numerical range, one can now fairly easily deduce the following basic properties; for details see primarily [[9], Chapter 1] but also [8].*

$1-$ *For any $A \in M_n(\mathbb{C})$ and for any $a, b \in \mathbb{C}$, $W(aA + bI_n) = aW(A) + b$.*

$2-$ *For any $A, B \in M_n(\mathbb{C})$, $W(A + B) \subseteq W(A) + W(B)$.*

$3-$ *For any $A \in M_n(\mathbb{C})$, $W(A)$ contains the convex hull of the eigenvalues of A. If A is normal, i.e., $A^*A = AA^*$, then $W(A)$ equals the convex hull of $\sigma(A)$.*

$4-$ *For any $A \in M_n(\mathbb{C})$, $W(A) \subset \mathbb{R}$ if and only if A is* ***Hermitian****, i.e., $A^* = A$, in this case, the endpoints of $W(A)$ coincide with the minimum and the maximum eigenvalues of A. Furthermore, $W(A)$ is a line segment in the complex plane if and only if the matrix A is normal and has collinear eigenvalues; or equivalently, if and only if $A = aH + bI$ for some $a, b \in \mathbb{C}$ and an* ***Hermitian*** *matrix H.*

3. Eigenvalues and Pseudo-spectrum of matrix A_a

Proposition 3.1. *The eigenvalues of the matrix A_a are:*

$$\begin{aligned}
&\lambda_1 = 1, \\
&\lambda_2 = \tfrac{2}{3}a + \tfrac{1}{3}ax_3^2 - \tfrac{1}{2}S + \tfrac{C}{2S} + \tfrac{1}{3} - \tfrac{\sqrt{3}}{2} i \left(S + \tfrac{C}{S}\right), \\
&\lambda_3 = \overline{\lambda_2}, \\
&\lambda_4 = \tfrac{2}{3}a + \tfrac{1}{3}ax_3^2 + S - \tfrac{C}{S} + \tfrac{1}{3},
\end{aligned}$$

with

$$S = \sqrt[3]{M + \sqrt{N} - \frac{8}{27}},$$

and

$$
\begin{aligned}
M &= \frac{2}{9}a+\frac{1}{9}a^2-\frac{1}{27}a^3+\frac{1}{6}x_3^2+\frac{11}{18}ax_3^2+\frac{1}{6}ax_3^4 \\
&\quad -\frac{1}{18}a^2x_3^2-\frac{1}{18}a^3x_3^2+\frac{1}{9}a^2x_3^4+\frac{1}{18}a^3x_3^4+\frac{1}{27}a^3x_3^6 \\
N &= \frac{4}{27}a^3-\frac{4}{27}a^2-\frac{1}{27}a^4-\frac{8}{27}x_3^2-\frac{13}{108}x_3^4-\frac{1}{27}x_3^6-\frac{2}{9}ax_3^2-\frac{1}{54}ax_3^4-\frac{1}{54}ax_3^6+\frac{7}{27}a^2x_3^2 \\
&\quad +\frac{4}{27}a^3x_3^2+\frac{7}{36}a^2x_3^4-\frac{1}{9}a^4x_3^2+\frac{1}{18}a^2x_3^6-\frac{11}{108}a^4x_3^4+\frac{1}{27}a^3x_3^6+\frac{1}{54}a^5x_3^4-\frac{1}{108}a^2x_3^8 \\
&\quad -\frac{1}{108}a^6x_3^4-\frac{1}{54}a^5x_3^6+\frac{1}{54}a^4x_3^8-\frac{1}{54}a^6x_3^6-\frac{1}{108}a^6x_3^8 \\
C &= \frac{2}{9}a-\frac{1}{9}a^2-\frac{1}{3}x_3^2-\frac{2}{9}ax_3^2-\frac{1}{9}a^2x_3^2-\frac{1}{9}a^2x_3^4-\frac{4}{9}
\end{aligned}
$$

Proof. We have

$$\det(A_a-\lambda I_4)=(1-\lambda)(-\lambda^3+L\lambda^2+K\lambda+(a^2-1)),$$

with

$$
\begin{aligned}
L &= \left(1+2a+ax_3^2\right), \\
K &= (-a^2-2a-a^2x_3^2+x_3^2+1),
\end{aligned}
$$

so, $\det(A-\lambda I_4)=0$, If and only if either $\lambda_1=1$ or

$$-\lambda^3+L\lambda^2+K\lambda+(a^2-1)=0.$$

According to the CARDAN method we find,

$$z^3+pz+q=0,$$

such as

$$z=\lambda-\frac{L}{3},\ z\in\mathbb{C}, \tag{3.1}$$

and

$$p = -(\tfrac{1}{3}L^2 + K) = -\tfrac{1}{3}(4 + a^2x_3^4 + ax_3^2 + a^2 - 2a + a^2x_3^2 + 3x_3^2) \ ,$$

$$q = -\tfrac{1}{27}(-16 + 2a^3x_3^6 + 6a^2x_3^4 + 33ax_3^2 - 2a^3 + 6a^2 - 3a^3x_3^2 + 12a + 3a^3x_3^4 - 3a^2x_3^2 + 9x_3^2) \ \cdot$$

Then the **CARDAN** method he says that the 3 solutions are:

$$z_k = j^k \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{\frac{-\Delta}{27}}\right)} + j^{-k} \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{\frac{-\Delta}{27}}\right)}, \quad 0 \le k \le 2$$

such as ,

$$\Delta = -4p^3 - 27q^2,$$

$$j = e^{i2\frac{\pi}{3}}.$$

So, according to (3.1) we find,

$$\lambda_k = z_k + \frac{L}{3}, \ 0 \le k \le 2$$

□

Pseudo-spectrum of A_a: since A is symmetrical therefore A_a is normal, therefore pseudo-spectrum noted by $\Lambda_\epsilon(A_a)$ given by:

$$\Lambda_\epsilon(A_a) = \{z \in \mathbb{C} : |z - \lambda_i| \le \epsilon\} \text{ with } i \in \{1, \ldots, 4\}.$$

3.1. **Numerical rang of matrix** A_a.

Proposition 3.2. *The numerical rang of matrix* A_a *check the following relation:*

$$\left|\frac{x^*A_ax}{x^*x}\right| \le (1 + |a|)(1 + |x_3|) + \left|ax_3^2\right|$$

Proof. We have

$$W(A) = \left\{\frac{x^*Ax}{x^*x}, \quad x \in \mathbb{C}^4, \ x \ne 0\right\}$$

we put $x = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$, with $z_i = r_ie^{i\theta_i}$. We have

$$x^*A_ax = a\left|z_1\right|^2 + a\left|z_4\right|^2 + \left|z_2\right|^2 + \left|z_3\right|^2 + ax_3(z_1\overline{z_2} + z_2\overline{z_1}) + x_3(z_2\overline{z_4} + z_4\overline{z_2}) + (z_1\overline{z_4} + z_4\overline{z_1}) + a\left|z_2\right|^2 x_3^2,$$

so,

$$\frac{x^*Ax}{x^*x} = 1 + \frac{(a-1)(|z_1|^2+|z_4|^2)}{\sum\limits_{i=1}^{4}|z_i|^2} + ax_3\frac{z_1\overline{z_2}+z_2\overline{z_1}}{\sum\limits_{i=1}^{4}|z_i|^2} + x_3\frac{z_2\overline{z_4}+z_4\overline{z_2}}{\sum\limits_{i=1}^{4}|z_i|^2} + \frac{z_1\overline{z_4}+z_4\overline{z_1}}{\sum\limits_{i=1}^{4}|z_i|^2} + ax_3^2\frac{|z_2|^2}{\sum\limits_{i=1}^{4}|z_i|^2}.$$

We have

$$\frac{|z_j|^2}{\sum\limits_{i=1}^{4}|z_i|^2} \le 1,\ \forall j \in \{1,\dots,4\}. \tag{3.2}$$

and

$$\frac{z_i\overline{z_j}+z_j\overline{z_i}}{\sum\limits_{i=1}^{4}|z_i|^2} \le 1, \quad \forall i,j \in \{1,\dots,4\}, \tag{3.3}$$

So from (3.2) and (3.3) we find

$$\left|\frac{x^*A_ax}{x^*x}\right| \le 1 + |ax_3| + |x_3| + |a| + \left|ax_3^2\right|.$$

It had to be proven. □

Example 3.1. 1) *For $a=0$ and $x_3=0$,*

$$A_0^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

so

$$\frac{g_0^0(x^*,x)}{x^*x} = 1 - \frac{r_1^2+r_4^2-2r_1r_4\cos(\theta_1-\theta_4)}{r_1^2+r_2^2+r_3^2+r_4^2} \le 1,$$

moreover $1 \in W(A_0^0)$

On the other hand, we have

$$-\frac{r_1^2+r_4^2-2r_1r_4\cos(\theta_1-\theta_4)}{r_1^2+r_2^2+r_3^2+r_4^2} \ge -2,$$

therefore

$$\frac{g_0^0(x^*,x)}{x^*x} \ge -1,$$

moreover $-1 \in W(A_0^0)$. So $W(A_0^0) = [-1,1]$

2) *For $a=0$ and $x_3=0.5$,*

$$A_0^{0.5} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 1 & 0.5 & 0 & 0 \end{pmatrix},$$

so

$$\frac{g_0^{0.5}(x^*,x)}{x^*x} = \frac{g_0^0(x^*,x)}{x^*x} + \frac{r_2 r_4 \cos(\theta_2-\theta_4)}{r_1^2+r_2^2+r_3^2+r_4^2}.$$

We have,

$$\frac{r_2 r_4 \cos(\theta_2-\theta_4)}{r_1^2+r_2^2+r_3^2+r_4^2} \leq \frac{1}{2},$$

and

$$\frac{g_0^{0.5}(x^*,x)}{x^*x} \geq -\frac{5}{4}$$

so

$$-\frac{5}{4} \leq \frac{g_0^{0.5}(x^*,x)}{x^*x} \leq \frac{3}{2},$$

but $-\frac{5}{4}$ *and* $\frac{3}{2}$ *does not belong to* $W(A_0^{0.5})$*, so we get* $W(A_0^{0.5}) \subset \left]-\frac{5}{4}, \frac{3}{2}\right[$.

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