Int. J. Anal. Appl. (2023), 21:63

International Journal of Analysis and Applications

Optimality Conditions for Set-Valued Optimization Problems

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ABSTRACT. In this paper, we first prove that the generalized subconvexlikeness introduced by Yang, Yang and Chen [1] and the presubconvelikeness introduced by Zeng [2] are equivalent. We discuss set-valued nonconvex optimization problems and obtain some optimality conditions.

1. Introduction

Set-valued optimization is a vibrant and expanding branch of mathematics that deals with optimization problems where the objectives and/or the constraints are set-valued maps. Corley [3] pointed out that the dual problem of a multiobjective optimization involves the optimization of a set-valued map, while Klein and Thompson [4] gave some examples in Economics where it is necessary to use set-valued maps instead of single-valued maps. There are many recent developments about set-values optimization problems, e.g., [5-9].

Convex and generalized convex optimization is a rich branch of mathematics. Many interesting and useful definitions of generalized convexities were introduced. Borwein [10] proposed the definition of cone convexity, Fan [11] introduced the definition of convexlikeness. Yang, Yang and Chen [1] defined the generalized subconvexlike functions, while Zeng [2] introduced a presubconvexlikeness.

In this paper, we first prove that the generalized subconvexlikeness introduced by Yang, Yang, and Chen [1] and the presubconvexlikeness introduced by Zeng [2] are equivalent, in locally convex

Received: Oct. 31, 2020.

2020 Mathematics Subject Classification. 90C26, 90C29, 90C48.

Key words and phrases. generalized subconvexlike maps; presubconvexlike maps; set-valued optimization; optimality conditions.

https://doi.org/10.28924/2291-8639-21-2023-63

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ISSN: 2291-8639

topological spaces. And then, we deal with set-valued optimization problems and obtain some optimality conditions.

A subset Y_+ of a real linear topological space Y is a cone if $\lambda y \in Y_+$ for all $y \in Y_+$ and $\lambda \geq 0$. We denote by 0_Y the zero element in the linear topological space Y and simply by 0 if there is no confusion. A convex cone is one for which $\lambda_1 y_1 + \lambda_2 y_2 \in Y_+$ for all $y_1, y_2 \in Y_+$ and $\lambda_1, \lambda_2 \geq 0$. A pointed cone is one for which $Y_+ \bigcap (-Y_+) = \{0\}$. Let Y be a real linear topological space with pointed convex cone Y_+. We denote the partial order induced by Y_+ as follows:

$$y_1 \succ y_2 \text{ iff } y_1 - y_2 \in Y_+,$$

$$y_1 \succ\succ y_2 \text{ iff } y_1 - y_2 \in \operatorname{int} Y_+,$$

where $\operatorname{int} Y_+$ denotes the topological interior of a set Y_+. Let X, Z_i, W_j be real linear topological spaces and Y be an ordered linear topological space with the partial order induced by a pointed convex cone Y_+.

We recall some notions of generalized convexity of set-valued maps. First we recall the notion of cone-convexity of a set-valued map introduced by Borwein [10].

Definition 1.1 (Convexity) Let X, Y be real linear topological spaces, $D \subseteq X$ a nonempty convex set and Y_+ a convex cone in Y. A set-valued map $f: X \to Y$ is said to be Y_+-convex on D if and only if $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, there holds

$$\alpha f(x_1) + (1-\alpha) f(x_2) \subseteq f(\alpha x_1 + (1-\alpha) x_2) + Y_+.$$

The following notion of generalized convexity is a set-valued map version of Ky Fan convexity [11] (Ky Fan's definition was for vector-valued optimization problems).

Definition 1.2 (Convexlike) Let X, Y be real linear topological spaces, $D \subseteq X$ a nonempty set and Y_+ be a convex cone in Y. A set-valued map $f: X \to Y$ is said to be Y_+-convexlike on D if and only if $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$ such that

$$\alpha f(x_1) + (1-\alpha) f(x_2) \subseteq f(x_3) + Y_+.$$

The following concept of generalized subconvexlikeness was introduced by Yang, Yang and Chen [1] ([1] introduced subconvexlikeness for vector-valued optimization).

Definition 1.3 (Generalized subconvexlike) Let Y be a linear topological space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in Y. A set-valued map $f: D \to Y$ is said to be generalized Y_+-subconvexlike on D if $\exists u \in \operatorname{int} Y_+$ such that $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$, $\exists \tau > 0$ there holds

$$\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau f(x_3) + Y_+ .$$

The following Lemma 1.1 is from Chen and Rong [12, Proposition 3.1].

Lemma 1.1 A function $f: D \to Y$ is generalized Y_+-subconvexlike on D if $\forall\, u \in \operatorname{int} Y_+$, $\forall x_1, x_2 \in D$, $\forall\, \alpha \in [0,1]$, $\exists\, x_3 \in D$, $\exists\, \tau > 0$ such that

$$u + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau f(x_3) + Y_+ .$$

A bounded function in a real linear topological space can be fined as following Definition 1.4 (e.g,, see Yosida [13]).

Definition 1.4 (Bounded set-valued map) A subset M of a real linear topological space Y is said to be a bounded subset if for any given neighbourhood U of 0, $\exists$ positive scalar β such that $\beta^{-1} M \subseteq U$, where $\beta^{-1} M = \{y \in Y; y = \beta^{-1} v; v \in M\}$. A set-valued map f : $D \to Y$ is said to bounded map if $f(Y)$ is a bounded subset of Y.

The following Definition 1.5 was introduced by Zeng [2] for single-valued functions.

Definition 1.5 (Presubconvexlike) Let Y be a linear topological space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in Y. A set-valued map f: $D \to Y$ is said to be Y_+-presubconvexlike on D if $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, $\forall \varepsilon > 0$, $\exists x_3 \in D$, $\exists \tau > 0$, $\exists$ bounded set-valued map u: $D \to Y$ such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau f(x_3) + Y_+ .$$

It is obvious that Y_+-convex $\Rightarrow Y_+$-convexlike $\Rightarrow$ generalized Y_+-subconvexlike $\Rightarrow Y_+$-presubconvexlike.

It is important to note that the concept of convexlike or any weaker concepts are only nontrivial if Y is not the one-dimensional Euclidean space since any real-valued function is R^+-convexlike.

2. The Equivalence of Generalized Subconvexlikeness and Presubconvexlikeness

In this section, we are going to prove that Definition 1.4 (Generalized subconvexlikeness) and Definition 1.5 (Presubconvexlikeness) are equivalent.

Definition 2.1 (1) A subset M of Y is said to be convex, if $y_1, y_2 \in M$ and $0 < \alpha < 1$ implies $\alpha y_1 + (1-\alpha) y_2 \in M$;

(2) M is said to be balanced if $y \in M$ and $|\alpha| \le 1$ implies $\alpha y \in M$;

(3) M is said to be absorbing if for any given neighbourhood U of 0, there exists a positive scalar β such that $\beta^{-1} M \subseteq U$, where $\beta^{-1} M = \{y \in Y; y = \beta^{-1} v; v \in M\}$.

Definition 2.2 A real linear topological space Y is called a locally convex, linear topological space (we call it a locally convex topological space, in the sequel) if any neighborhood of 0_Y contains a convex, balanced, and absorbing open set.

From [13, pp.26 Theorem, pp.33 Definition 1] one has Lemma 2.1.

Lemma 2.1 Banach spaces are locally convex topological spaces, so are finite dimensional Euclidean spaces.

Proposition 2.1 Let Y be a locally convex topological space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in Y. A set-valued map $f: D \to Y$ is generalized Y_+-subconvexlike on D if and only if $\bigcup_{t>0}(tf(D) + \operatorname{int} Y_+)$ is convex.

Proof. The necessity. See [1, Theorem 2.1].

The sufficiency. Assume that $\bigcup_{t>0}(tf(D) + \operatorname{int} Y_+)$ is convex, aim to show that $f: D \to Y$ is generalized Y_+-subconvexlike on D. From Lemma 1.1, we are going to show that, $\forall\, u \in \operatorname{int} Y_+$, $\forall x_1, x_2 \in D$, $\forall\, \alpha \in [0,1]$, $\exists\, x_3 \in D$, $\exists\, \tau > 0$ such that

$$u + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau f(x_3) + Y_+.$$

$\forall y \in \operatorname{int} Y_+,\ \forall t > 0,$ since $\operatorname{int} Y_+$ is a cone, one has

$$ty \in \operatorname{int} Y_+.$$

$\forall y_1 \in f(x_1), y_2 \in f(x_2),\ \forall \alpha \in R,$ one has

$$f(x_1) + ty,\ f(x_2) + ty \subseteq \bigcup_{t>0}(tf(D) + \operatorname{int} Y_+).$$

From the convexity of $\bigcup_{t>0}(tf(D) + \operatorname{int} Y_+)$, $\exists x_3 \in D, \exists y_3 \in \operatorname{int} Y_+, \exists\, \tau > 0$ such that

$$\begin{aligned}&\alpha(f(x_1)+ty)+(1-\alpha)(f(x_2)+ty)\\&\subseteq \alpha f(x_1)+(1-\alpha)f(x_2)+ty\\&\subseteq \tau f(x_3)+y_3\\&\subseteq \bigcup_{t>0}(tf(D)+\operatorname{int}Y_+).\end{aligned}$$

For the given $u\in\operatorname{int}Y_+$, From Definition 2.2, $\exists$ neighbourhood U of 0 such that U is convex, balanced, and absorbing, and $u+U\subseteq\operatorname{int}Y_+$, where $u+U$ is a neighbourhood of u. Therefore, we may take $t>0$ small enough, such that $-ty\in U$. Then,

$$-ty+u\in u+U\subseteq\operatorname{int}Y_+.$$

This and the convexity of $\operatorname{int}Y_+$ imply that

$$y_3-ty+u\in\operatorname{int}Y_+.$$

And so

$$u+\alpha f(x_1)+(1-\alpha)f(x_2)\subseteq\tau f(x_3)+y_3-ty\subseteq\tau f(x_3)+\operatorname{int}Y_+\subseteq\tau f(x_3)+Y_+.$$

Proposition 2.2 Let Y be a locally convex topological space and $D\subseteq X$ be a nonempty set and Y_+ be a convex cone in Y. A set-valued map $f:D\to Y$ is Y_+-presubconvexlike on D if and only if $\bigcup_{t>0}(tf(D)+\operatorname{int}Y_+)$ is convex.

Proof. The necessity.

Suppose that f is Y_+-presubconvexlike on, aim to show that $\bigcup_{t>0}(tf(D)+\operatorname{int}Y_+)$ is convex.

$\forall v_1=t_1y_1+y_+^1, v_2=t_2y_2+y_+^2\in\bigcup_{t>0}(tf(D)+\operatorname{int}Y_+)$, $\exists x_1,x_2\in D$ such that $y_1\in f(x_1)$, $y_2\in f(x_2)$. Let

$$y_+^0=\alpha y_+^1+(1-\alpha)y_+^2,$$

then $y_+^0\in\operatorname{int}Y_+$. Therefore, $\exists$neighbourhood U of 0 such that y_+^0+U is a neighbourhood of y_+^0 and

$$y_+^0+U\subseteq\operatorname{int}Y_+.$$

By Definition 2.2, without loss of generality, we may assume that U is convex, balanced, and absorbing.

From the assumption of Y_+-presubconvexlikeness, $\forall\varepsilon>0$, $\exists x_3\in D$, $\exists$ bounded function u, and $\exists\tau>0$ such that

$$\frac{\alpha t_1}{\alpha t_1+(1-\alpha)t_2}f(x_1)+\frac{(1-\alpha)t_2}{\alpha t_1+(1-\alpha)t_2}f(x_2)\subseteq\tau f(x_3)-\varepsilon u+Y_+,$$

Therefore, $\forall y_3 \in f(x_3)$ such that

$$\begin{aligned}
&\alpha v_1 + (1-\alpha)v_2 \\
&= \alpha t_1 y_1 + (1-\alpha)t_2 y_2 + \alpha y_+^1 + (1-\alpha) y_+^2 \\
&= (\alpha t_1 + (1-\alpha)t_2)[\frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} y_1 + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} y_2] + y_+^0 \\
&\subseteq (\alpha t_1 + (1-\alpha)t_2)[\frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} f(x_1) + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} f(x_2)] + y_+^0 \\
&\subseteq (\alpha t_1 + (1-\alpha)t_2)[\tau f(x_3) - \varepsilon u + Y_+] + y_+^0 \\
&= (\alpha t_1 + (1-\alpha)t_2)\tau f(x_3) + (\alpha t_1 + (1-\alpha)t_2)(Y_+ - \varepsilon u) + y_+^0.
\end{aligned}$$

Since U is convex, balanced, and absorbing, by Definition 2.2, we may take $\varepsilon > 0$ small enough such that

$$-(\alpha t_1 + (1-\alpha)t_2)\varepsilon u \subseteq U .$$

Therefore

$$-(\alpha t_1 + (1-\alpha)t_2)\varepsilon u + y_+^0 \subseteq y_+^0 + U \subseteq \operatorname{int} Y_+ .$$

And then

$$(\alpha t_1 + (1-\alpha)t_2)Y_+ - (\alpha t_1 + (1-\alpha)t_2)\varepsilon u + y_+^0 \subseteq Y_+ + \operatorname{int} Y_+ \subseteq \operatorname{int} Y_+.$$

Therefore

$$\begin{aligned}
&\alpha v_1 + (1-\alpha)v_2 \\
&\subseteq (\alpha t_1 + (1-\alpha)t_2)\tau f(x_3) + (\alpha t_1 + (1-\alpha)t_2)(Y_+ - \varepsilon u) + y_+^0 \\
&\subseteq \bigcup_{t>0} (tf(D) + \operatorname{int} Y_+).
\end{aligned}$$

Hence $\bigcup_{t>0}(tf(D) + \operatorname{int} Y_+)$ is a convex set.

The sufficiency.

Assume that $\bigcup_{t>0}(tf(D) + \operatorname{int} Y_+)$ is convex. From Lemma 1.1 and Proposition 2.1, $\exists u \in \operatorname{int} Y_+$ such that for all $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, $\forall \varepsilon > 0$, $\exists x_3 \in D$, $\exists \tau > 0$ there holds

$$\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau f(x_3) + Y_+ .$$

The given $u \in \operatorname{int} Y_+$ can be consider as a bounded function.

By Propositions 1 and 2 one has Theorem 2.1.

Theorem 2.1 Let Y be a locally convex topological space and $D \subseteq X$ be a nonempty set, and Y_+ a convex cone in Y. A set-valued map $f: D \to Y$ is generalized Y_+-subconvexlike on D if and only if f is Y_+-presubconvexlike on D.

3. Optimal Conditions

We consider the following optimization problem with set-valued maps:

$$(VP) \qquad Y_+\text{-min} \qquad f(x)$$

$$s.t. \ g_i(x) \cap (-Z_{i+}) \neq 0, \ i = 1, 2, \cdots, m$$

$$0 \in h_j(x), \ j = 1, 2, \cdots, n$$

$$x \in D$$

where $f: X \to Y$, $g_i: X \to Z_i$, $h_j: X \to W_j$ are set-valued maps, Z_{i+} is a closed convex cone in Z_i and D is a nonempty subset of X.

For a set-valued map $f: X \to Y$, we denote by $f(D) = \bigcup_{x \in D} f(x)$.

We now explain the kind of optimality we consider here. Let F be the feasible set of (VP), i.e.

$$F := \{x \in D : g_i(x) \cap (-Z_{i+}) \neq \emptyset, i = 1, 2, \cdots, m; 0 \in h_j(x), j = 1, 2, \cdots, n\}.$$

We are looking for a weakly efficient solution of (VP) defined as follows.

Definition 3.1 (Weakly Efficient Solution) A point $\bar{x} \in F$ is said to be a weakly efficient solution of (VP) with a weakly efficient value $\bar{y} \in f(\bar{x})$ if for every $x \in F$, there exists no $y \in f(x)$ satisfying $\bar{y} \succ\succ y$.

Consider the set-valued optimization problem (VP). From now on we assume that Y_+, Z_{i+} are pointed convex cones with nonempty interior of $\operatorname{int} Y_+$, $\operatorname{int} Z_{i+}$, respectively. The following three assumptions will be used in this paper.

(*A1*) Generalized Convexity Assumption. There exist $u_0 \in \operatorname{int} Y_+$, $u_i \in \operatorname{int} Z_{i+}$ such that for all $x_1, x_2 \in D$, $\varepsilon > 0$, $\alpha \in [0,1]$,, there exist $x_3 \in D$, $\tau_i > 0 (i = 1, 2, \cdots, m), t_j > 0 (j = 1, 2, \cdots, n)$ such that

$$\begin{aligned} &\varepsilon u_0 + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau_0 f(x_3) + Y_+ \\ &\varepsilon u_i + \alpha g_i(x_1) + (1-\alpha) g_i(x_2) \subseteq \tau_i g_i(x_3) + Z_{i+} \\ &\alpha h_j(x_1) + (1-\alpha) h_j(x_2) \subseteq t_j h_j(x_3) \end{aligned}$$

(*A2*) Interior Point Assumption.

$$\operatorname{int} h_j(D) \neq \varnothing, \ (j = 1, 2, \ldots, n).$$

($A3$) Finite Dimension Assumption. W_j (j = 1, 2, …, n) are finite dimensional spaces.

Similar to the proof of Propositions 2.1 or 2.2, one has Proposition 3.1.

Proposition 3.1 Assumption (A1) is satisfied if and only if the following set is convex:

$$\begin{aligned} &B := \{(y, z, w) \in Y \times \Pi_{i=1}^m Z_i \times \Pi_{j=1}^n W_j : \\ &\exists x \in D, \tau_i, t_j > 0, s.t. y \in \tau_0 f(x) + \operatorname{int} Y_+, \\ &z_i \in \tau_i g_i(x) + \operatorname{int} Z_{i+}, w_j \in t_j h_j(x)\}. \end{aligned}$$

Proposition 3.2 (Alternative Theorem) Assume that the assumption (A1) and either (A2) or (A3) are satisfied. Consider the following generalized inequality-equality systems:

[*System 1*]

$$\exists x \in D, s.t. f(x) \bigcap (-\operatorname{int} Y_+) \neq \varnothing, g_i(x) \bigcap (-Z_{i+}) \neq \varnothing, 0 \in h_j(x).$$

[*System 2*]

$$\begin{aligned} &\exists (\xi, \eta, \varsigma) \in (Y_+^* \times \Pi_{i=1}^m Z_{i+}^* \times \Pi_{j=1}^n W_j^*) \setminus \{0\}, s.t. \forall x \in D \\ &\xi(f(x)) + \sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \varsigma_j(h_j(x)) \geq 0. \end{aligned}$$

Then if System 1 has no solution x, then System 2 has a solution (ξ, η, ς). If System 2 has a solution (ξ, η, ς) with $\xi \neq 0$, then System 1 has no solution.

Proof. Suppose that System 1 has no solution, then $0 \notin B$. Since (A1) holds, the set B is convex. By assumption, $\Pi_{i=1}^m Z_i$ is infinite dimensional and (A2) holds (which is equivalent to saying that $\operatorname{int} B \neq \varnothing$) or $\Pi_{j=1}^n W_j$ is finite dimensional. Therefore by the separation theorem, $\exists$ nonzero vector $(\xi, \eta, \varsigma) \in Y^* \times \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ such that

$$\xi(\tau_0 y + y_0) + \sum_{i=1}^m \eta_i(\tau_i z_i + z_i^0) + \sum_{j=1}^n \varsigma_j(t_j w_j) \geq 0$$

for all

$$x \in D, y \in f(x), z_i \in g_i(x), w_j \in h_j(x), y_0 \in \operatorname{int} Y_+, z_i^0 \in \operatorname{int} Z_{i+}, \tau_i > 0, t_j > 0.$$

Since int Y_+, intZ_{i+} are convex cones, we have

$$\xi(\tau_0 y + s_0 y_0) + \sum_{i=1}^m \eta_i(\tau_i z_i + s_i z_i^0) + \sum_{j=1}^n \varsigma_j(t_j w_j) \geq 0$$

For all

$$x \in D, y \in f(x), z_i \in g_i(x), w_j \in h_j(x), y_0 \in \operatorname{int} Y_+, z_i^0 \in \operatorname{int} Z_{i+}, \tau_i > 0, t_j > 0, s_i > 0 (i = 0, 1, 2, \cdots m).$$

Taking $\tau_i \to 0, t_j \to 0, s_i \to 0 (i = 0,1,2,\cdots,m)$, we obtain

$$\xi(y_0) > 0, \forall y_0 \in \operatorname{int} Y_+,$$

and consequently

$$\xi(y_0) \geq 0, \forall y_0 \in Y_+ \subseteq clY_+ = cl \operatorname{int} Y_+,$$

where clY_+ is the topological closure of the set Y_+. Similarly, we have

$$\eta_i(z_i) \geq 0, \forall z_i \in Z_{i+},$$

and hence $\xi \in Y_+^*$, $\eta_i \in Z_{i+}^*$.

Let $\tau_i = 1 (i = 1,2,\cdots,m), t_j = 1 (j = 1,2,\cdots,n)$ and take $s_i \to 0 (i = 0,1,2,\cdots m)$, we have

$$\xi(y) + \sum_{i=1}^m \eta_i(z_i) + \sum_{j=1}^n \varsigma_j(w_j) \geq 0$$

For $x \in D, y \in f(x), z_i \in g_i(x), w_j \in h_j(x)$. Hence, System 2 has a solution (ξ, η, ς).

Conversely, suppose that System 2 has a solution (ξ, η, ς) with $\xi \neq 0$. If System 1 has a solution $\forall x \in D$, there would exist $y \in f(x), z_i \in g_i(x), w_j \in h_j(x)$ such that

$$y \in -\operatorname{int} Y_+, z_i \in -Z_{i+}, w_j = 0.$$

Thus,

$$\xi(y) < 0, \eta_i(z_i) \leq 0, \varsigma_j w_j = 0,$$

i.e.,

$$\xi(y) + \sum_{i=1}^m \eta_i(z_i) + \sum_{j=1}^n \varsigma_j(w_j) < 0.$$

which is a contradiction and hence System 1 does not have a solution.

Theorem 3.1 [Fritz John Type Necessary Optimality Condition] Assume that the generalized convexity assumption (A1) is satisfied and either (A2) or (A3) holds. If $\bar{x} \in F$ is a weakly efficient solution of (*VP*) with $\bar{y} \in f(\bar{x})$, $\exists$ nonzero vector $(\xi, \eta, \varsigma) \in Y^* \times \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ such that

$$\begin{aligned} &\xi(\bar{y}) = \min_{x \in D} [\xi(f(x)) + \sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \varsigma_j(h_j(x))] \\ &\min \sum_{i=1}^m \eta_i(g_i(\bar{x})) = 0, \end{aligned}$$

where $\min \sum_{i=1}^m \eta_i(g_i(\bar{x})) := \min_{z_i \in g_i(\bar{x})} \sum_{i=1}^m \eta_i(z_i)$.

Proof. Since $\bar{x} \in F$ is a weakly efficient solution of (*VP*) with $\bar{y} \in f(\bar{x})$, by definition the following system

$$x \in D, (f(x) - y) \bigcap (-\mathrm{int}\, Y_+) \neq \emptyset, g_i(x) \bigcap (-Z_{i+}) \neq \emptyset, 0 \in h_j(x)$$

has no solution. By Proposition 2.2, there exists a nonzero vector $(\xi, \eta, \varsigma) \in Y^* \times \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ such that $\forall x \in D$ there holds

$$\xi(f(x) - \bar{y}) + \sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \varsigma_j(h_j(x)) \geq 0.$$

Since $\bar{x} \in F$, there exists $z_i \in g_i(\bar{x})$ such that $\bar{z}_i \in -Z_{i+}$. For such $\bar{z}_i$, it follows $\eta_i \in Z_{i+}^*$ that $\eta_i(\bar{z}_i) \leq 0$. On the other hand, taking $x = \bar{x}$ we get

$$\xi(f(x) - \bar{y}) + \sum_{i=1}^m \eta_i(\bar{z}_i) + \sum_{j=1}^n \varsigma_j(h_j(\bar{x})) \geq 0,$$

and noticing that $\bar{y} \in f(\bar{x})$ and $0 \in h_j(\bar{x})$ we obtain

$$\sum_{i=1}^m \eta_i(\bar{z}_i) \geq 0,$$

and hence $\eta_i(\bar{z}_i) = 0$.

Since

$$\xi(\bar{y}) + \sum_{i=1}^m \eta_i(\bar{z}_i) + \sum_{j=1}^n \varsigma_j(0) = \xi(y),$$

taking $x = \bar{x}$ again we get

$$\xi(f(x) - \bar{y}) + \sum_{i=1}^m \eta_i(g_i(\bar{x})) + \sum_{j=1}^n \varsigma_j(h_j(\bar{x})) \geq 0.$$

Noticing that $\bar{y} \in f(\bar{x})$ and $0 \in h_j(\bar{x})$, we obtain

$$\sum_{i=1}^m \eta_i(g_i(\bar{x})) \geq 0.$$

We have shown previously that there exists $z_i \in g_i(\bar{x})$ such that $\eta_i(\bar{z}_i) = 0$. Therefore

$$\min \sum_{i=1}^m \eta_i(g_i(\bar{x})) = 0.$$

Theorem 3.2 (Sufficient Optimality Condition) Let $\bar{x} \in F$ and $\bar{y} \in f(\bar{x})$. If there exists a $(\xi, \eta, \varsigma) \in Y^* \times \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ with $\xi \neq 0$ such that

$$\xi(\bar{y}) \leq \min_{x \in D} [\xi(f(x)) + \sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \varsigma_j(h_j(x))],$$

then $\bar{x}$ is a weakly efficient solution of (*VP*) with $\bar{y} \in f(\bar{x})$.

Proof. By contradiction, we assume that $\bar{x} \in F$ is not a weakly efficient solution of (*VP*) with $\bar{y} \in f(\bar{x})$. Then by definition, $\exists\ x^0 \in F$ and $\exists\ y^0 \in f(x^0)$ such that $y - y^0 \in \mathrm{int}\, Y_+$, which implies that $\xi(y-y^0)<0$. Since $x^0 \in F$, $0 \in h_j(x^0)$ and $\exists\ z_i^0 \in g_i(x^0)$ such that $z_i^0 \in -Z_{i+}$, and hence $\eta_i(z_i^0) \le 0$. Consequently,

$$\xi(y^0-\bar{y})+\sum_{i=1}^{m}\eta_i(z_i^0)+\sum_{j=1}^{n}\varsigma_j(0)<0.$$

Hence $\bar{x}$ is a weakly efficient solution of (*VP*) with $\bar{y} \in f(\bar{x})$.

From Theorem 3.2 and 3.3 one has Theorem 3.3.

Theorem 3.3 (Strong Duality) Suppose all assumptions in Theorem 3.1 hold and there is no nonzero vector $(\eta,\varsigma) \in R_+^m \times R^n$ satisfying the system:

$$\min_{x\in D}[\sum_{i=1}^{m}\eta_i g_i(x)+\sum_{j=1}^{n}\varsigma_j h_j(x)]=0$$
$$\eta_i g_i(\bar{x})=0.$$

Let $\bar{x}$ be a solution of problem (*P*). Then the strong duality holds. That is,

$$f(\bar{x})=\min_{g(x)\le 0,h(x)=0,x\in D} f(x)=\max_{\eta\ge 0}\min_{x\in D}[f(x)+\sum_{i=1}^{m}\eta_i g_i(x)+\sum_{j=1}^{n}\varsigma_j h_j(x)].$$

4. Applications to Single-Valued Optimization Problems

Consider the optimization problem:

$$(P)\qquad \min f(x)$$
$$s.t.\ \ g_i(x)\le 0\ \ (i=1,2,\ldots,m)$$
$$h_j(x)=0,(j=1,2,\cdots,n)$$
$$x\in D$$

where *f*, g_i, h_j: $X \to R$ are functions and *D* is a nonempty subset of *X*.

Applying Theorem 3.1 to the above single-valued optimization problem we have the following Fritz John type necessary optimality condition.

Theorem 4.1 Let $\bar{x}$ be an optimal solution of (*P*). Suppose the following generalized convexity assumption holds: $\exists\ u_i>0,(i=0,1,2,\cdots,m)$ such that $\forall x_1,x_2\in D$, $\forall\varepsilon>0$, $\forall\alpha\in[0,1]$, $\exists$ $x_3\in D$, $\exists\ \tau_i>0,(i=1,2,\cdots,m)$, $\exists\ t_j>0$, $(j=1,2,\cdots,n)$ there holds

$$\varepsilon u_0+\alpha f(x_1)+(1-\alpha)f(x_2)\subseteq\tau_0 f(x_3)+R_+$$
$$\varepsilon u_i+\alpha g_i(x_1)+(1-\alpha)g_i(x_2)\subseteq\tau_i g_i(x_3)+R_+$$
$$\alpha h_j(x_1)+(1-\alpha)h_j(x_2)=t_j h_j(x_3)$$
$$(i=1,2,\cdots,m;\ j=1,2,\cdots,n).$$

Then, $\exists$ nonzero vector $(\lambda,\eta,\varsigma)\in R_{+}\times R_{+}^{m}\times R^{n}$ such that

$$\lambda f(\bar{x})=\min_{x\in D}[\lambda f(x)+\sum\nolimits_{i=1}^{m}\eta_i g_i(x)+\sum\nolimits_{j=1}^{n}\varsigma_j h_j(x)]$$
$$\min\sum\nolimits_{i=1}^{m}\eta_i g_i(\bar{x})=0.$$

We now study some cases where the generalized convexity holds and consequently the Fritz John condition in the above theorem holds.

Theorem 4.2 Let $\bar{x}$ be an optimal solution of (*P*). Suppose one of the following set of assumptions hold.

(I) All functions g_i are nonnegative on the set D and $n = 0$ (i.e. there is no equality constraints).

(II) All functions f, g_i are nonnegative on the set D and $n = 1$. Then, $\exists$ non-zero vector

$$(\lambda,\eta,\varsigma)\in R_{+}\times R_{+}^{m}\times R^{n}$$

such that

$$\lambda f(\bar{x})=\min_{x\in D}[\lambda f(x)+\sum\nolimits_{i=1}^{m}\eta_i g_i(x)+\sum\nolimits_{j=1}^{n}\varsigma_j h_j(x)]$$
$$\min\sum\nolimits_{i=1}^{m}\eta_i g_i(\bar{x})=0.$$

Proof. From Theorem 4.1, it suffices to prove that the generalized convexity assumption holds.

First assume that assumption (I) holds. Let $x_1,x_2\in D$, $\alpha\in[0,1]$.

Case 1: $f(x_1)>f(x_2)$. Then

$$\alpha f(x_1)+(1-\alpha)f(x_2)-f(x_1)=(1-\alpha)(f(x_2)-f(x_1))\geq 0.$$

Let $x_3=x_1$. Then

$$\alpha f(x_1)+(1-\alpha)f(x_2)\subseteq f(x_3)+R_{+}. \tag{1}$$

Since g is nonnegative on set D, for small enough $\tau\in(0,\alpha]$ one has

$$\alpha g(x_1)+(1-\alpha)g(x_2)-\tau g(x_1)=(\alpha-\tau)g(x_1)+(1-\alpha)g(x_2)\geq 0.$$

That is,

$$\alpha g(x_1)+(1-\alpha)g(x_2)\in\tau g(x_3)+R_{+}^{m}. \tag{2}$$

Case 2: $f(x_1)\leq f(x_2)$. In this case by choosing $x_3=x_2$ similarly as in case 1 we can prove (1) and (2). Hence the generalized convexity assumption holds.

Now assume that assumption (II) holds. Let $x_1,x_2\in D$ and $\alpha\in[0,1]$. If $h(x_2)$ = 0 then

$$\alpha h(x_1)+(1-\alpha)h(x_2)=\alpha h(x_1).$$

Let $x_3 = x_1$. Then since *f*, g_i are nonnegative, similarly as in (I) one can find a small enough $\tau_0 > 0$ and $\tau_1 > 0$ such that

$$\alpha f(x_1) + (1-\alpha) f(x_2) \in \tau_0 f(x_3) + R_+. \quad (3)$$

$$\alpha g(x_1) + (1-\alpha) g(x_2) \in \tau_1 g(x_3) + R_+^m. \quad (4)$$

Otherwise if $h(x_2) \neq 0$, then one can find $\tau_2 > 0$ such that

$$\alpha h(x_1) + (1-\alpha) h(x_2) = \tau_2 h(x_2).$$

Let $x_3 = x_2$. Then since *f*, g_i are nonnegative, similarly as in (I) one can find a small enough $\tau_0 > 0$ and $\tau_1 > 0$ such that (3) and (4) hold. Hence the generalized convexity assumption holds.

Theorem 4.3 (Kuhn-Tucker Type Necessary Optimality Condition) Let $\bar{x}$ be an optimal solution of (*P*). Suppose all assumptions in Theorem 4.2 hold and there is no nonzero vector $(\eta, \varsigma) \in R_+^m \times R^n$ satisfying the system:

$$\min_{x \in D \cap U(\bar{x})} [\textstyle\sum_{i=1}^m \eta_i g_i(x) + \sum_{j=1}^n \varsigma_j h_j(x)] = 0$$
$$\eta_i g_i(\bar{x}) = 0.$$

where $U(\bar{x})$ is a neighbourhood of $\bar{x}$, then, $\exists\ (\eta, \varsigma) \in R_+^m \times R^n$ such that

$$f(\bar{x}) = \min_{x \in D \cap U(\bar{x})} [f(x) + \textstyle\sum_{i=1}^m \eta_i g_i(x) + \sum_{j=1}^n \varsigma_j h_j(x)]$$
$$\eta_i g_i(\bar{x}) = 0.$$

5. Conclusion Remark

Yang, Yang and Chen [1] defined the following generalized subconvexlike functions. ([1] introduced subconvexlikeness for vector-valued optimization).

Let *Y* be a topological vector space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in *Y*. A set-valued map $f : D \to Y$ is said to be generalized Y_+-subconvexlike on *D* if $\exists u \in \operatorname{int} Y_+$, such that $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$, $\exists \tau > 0$ there holds

$$\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau f(x_3) + Y_+. \quad (5)$$

And Zeng [2] introduced the presubconvexlikeness as follows.

Let Y be a topological vector space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in Y. A set-valued map $f: D \to Y$ is said to be Y_+-presubconvexlike on D if $\exists$ bounded set-valued map $u: D \to Y$ such that $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$, $\exists \tau > 0$,

$$\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \subseteq \tau f(x_3) + Y_+ . \tag{6}$$

The inclusions (5) and (6) may be written as

$$\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \prec \tau f(x_3),$$

by the partial order induced by the convex cone Y_+.

In this paper, we proved that the above two generalized convexities are equivalent.

And then, we worked with nonconvex set-valued optimization problems and attained some optimality conditions. Our Fritz John Type Necessary Optimality Condition (Theorem 3.1) and Kuhn-Tucker Type Necessary Optimality Condition (Theorem 4.3) extend the classic results in Clarke [14]. Our Proposition 3.1 are modifications of the alternative theorems in [15, 16]. Our Theorem 3.2 (sufficient optimality condition) extends Theorem 23 in [9]. Our Strong Duality Theorem (Theorem 3.3) extends Theorem 7 in Li and Chen [8].

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] X.M. Yang, X.Q. Yang, G.Y. Chen, Theorems of the Alternative and Optimization with Set-Valued Maps, J. Optim. Theory Appl. 107 (2000), 627–640. https://doi.org/10.1023/a:1026407517917.

[2] R. Zeng, A General Gordan Alternative Theorem with Weakened Convexity and Its Application, Optimization. 51 (2002), 709–717. https://doi.org/10.1080/0233193021000031615.

[3] H.W. Corley, Existence and Lagrangian Duality for Maximizations of Set-Valued Functions, J. Optim. Theory Appl. 54 (1987) 489–501. https://doi.org/10.1007/bf00940198.

[4] E. Klein, A.C. Thompson, Theory of Correspondences, Wiley, New York, 1984.

[5] Z. Li, A Theorem of the Alternative and Its Application to the Optimization of Set-Valued Maps, J. Optim. Theory Appl. 100 (1999), 365–375. https://doi.org/10.1023/a:1021786303883.

[6] Y.W. Huang, A Farkas-Minkowski Type Alternative Theorem and Its Applications to Set-Valued Equilibrium Problems, J. Nonlinear Convex Anal. 3 (2002), 17-24.

[7] O. Ferrero, Theorems of the Alternative for Set-Valued Functions in Infinite-Dimensional Spaces, Optimization. 20 (1989), 167–175. https://doi.org/10.1080/02331938908843428.

[8] Z.F. Li, G.Y. Chen, Lagrangian Multipliers, Saddle Points, and Duality in Vector Optimization of Set-Valued Maps, J. Math. Anal. Appl. 215 (1997), 297–316. https://doi.org/10.1006/jmaa.1997.5568.

[9] A. Oussarhan, I. Daidai, Necessary and Sufficient Conditions for Set-Valued Maps with Set Optimization, Abstr. Appl. Anal. 2018 (2018), 5962049. https://doi.org/10.1155/2018/5962049.

[10] J. Borwein, Multivalued Convexity and Optimization: A Unified Approach to Inequality and Equality Constraints, Math. Program. 13 (1977), 183–199. https://doi.org/10.1007/bf01584336.

[11] K. Fan, Minimax Theorems, Proc. Natl. Acad. Sci. U.S.A. 39 (1953), 42–47. https://doi.org/10.1073/pnas.39.1.42.

[12] G.Y. Chen, W.D. Rong, Characterizations of the Benson Proper Efficiency for Nonconvex Vector Optimization, J. Optim. Theory Appl. 98 (1998), 365–384. https://doi.org/10.1023/a:1022689517921.

[13] K. Yosida, Functional Analysis, 6th Edition, Springer, 1980.

[14] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley-Interscience, New York, 1983.

[15] T.D. Chuong, Robust Alternative Theorem for Linear Inequalities With Applications to Robust Multiobjective Optimization, Oper. Res. Lett. 45 (2017), 575–580. https://doi.org/10.1016/j.orl.2017.09.002.

[16] M. Ruiz Galán, A Theorem of The Alternative With an Arbitrary Number of Inequalities and Quadratic Programming, J. Glob. Optim. 69 (2017), 427–442. https://doi.org/10.1007/s10898-017-0525-x.