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# PREDICTION INTERVALS FOR THE FIRST AND LAST POINT IN FUTURE SAMPLE HAVING FROM A NEW BATHTUB SHAPE FAILURE RATE LIFE TIME MODEL IN THE PRESENCE OF OUTLIERS

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Abstract. Acquiring Bayesian prediction intervals for the first and final points of observation with the bathtub-shape distribution of the failure rate life-time type under the conditions of available outliers is the focus of the research. These bounds of predication acquired on the basis of the right Type-II censored sample. The procedure is presented with the help of a wide range of illustrative example.

## 1. Introduction

The researchers have to apply the same type of distribution in the context of numerous statistical problems to use the previously obtained data in order to predict the future data. This need has been the subject of a number of academic researches and studies with the analysis of the corresponding practical application ( [1]; [2]; [3]). Simultaneously, the research which offered the most significant applications was conducted and further analyzed [4]. More particularly, the researcher devoted his work to the increasing function of failure rate or two parameters of the bathtub-shape life-time distribution. Chen states that the distribution has $\lambda$ and $\beta$ as parameters; in that case, the following equations denote the functions of cumulative distribution and probability density:

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$$f(x;\,\lambda,\,\beta)\,=\,\lambda\,\beta\,x^{\beta-1}e^{x^{\beta}\,-\,\lambda\,(e^{x^{\beta}}-1)},\quad x>0,\,\lambda>0,\,\beta>0, \tag{1.1}$$

and

$$F(x;\,\lambda,\,\beta)\,=\,1\,-\,e^{-\,\lambda\,(e^{x^{\beta}}-1)}. \tag{1.2}$$

According to the calculation of Selim [5], the Bayesian estimations are based on the bathbut-shape life-time distribution with two parameters founded on the record values. In the studies of Niazi and Abd-Elrahman [6], Bayesian prediction boundaries are acquired for bathbut-shape life-time distribution with two parameters and a failure rate function. Derivation of Bayesian prediction bounds is done for new model of life-time bathbut-shape failure rate with the doubly right Type-II censored samples [7]. The theoretical and practical value of the research was huge, considering the life-time distribution function based on the Bayesian prediction intervals [8]. There was also an investigation devoted to the Chen distribution in the e-Bayesian assessment on the basis of the censoring scheme type-I [9]. Th present research deals with model of Chen($\lambda$, $\beta$) in the framework of two different sampling plans to obtain Bayesian intervals for prediction needed in the future studies: Provided that $x_1,\, x_2,\, \cdots,\, x_n$ is an order random sample from model (1.2) with size $n$ and $x_r,\, x_{r+1},\, \cdots,\, x_n$, the sample has $(n-r)$ as the largest for the sample observations. The statistical analysis uses merely the ordered observations that remained, *i.e.* $x = (x_1,\, \cdots,\, x_r)$. It is clear that this sampling type includes a complete ordered sample $r = n$ as a special case.

In the first case, a censored sample Type-II is $x_1 < \cdots < x_r$ with $r < n$ defined as an observed sample, while the unobserved sample is identified with $x_{r+1} < x_{r+2} < \cdots x_n$ as the remaining values. The study of all the rest values of $n - r$ is done with the use of an observed sample.

In the second case, the sample of type-II, identical to that in the previous case, is presented with $x_1 < \cdots < x_r$ while $z_1 < z_2 < \cdots < z_m$ is viewed as a future sample of unobserved type with the use of the same population. In terms of using the selected configurations of sampling, it is essential to specify the prediction intervals, which apply previous observations with the aim to identify the future observations.

## 2. Function of Likelihood

Provided that a random sample with the derivation from the population with specified probabilities in (1.1) and (1.2) is $x_1 < x_2 < \cdots < x_n$, it is time to assign $x_1, x_2, \cdots, x_n$ to the life test. Recording of the failure times is done merely beginning from the failure timeframe $r^{th}$ with $r < n$. The analyzed study outputs are identified in the research as the censored data Type-II with the estimated function of likelihood presented below:

$$L(\lambda,\, \beta;\, \underline{x}) \propto [1 - F_X(x_{(r)};\, \lambda,\, \beta)]^{n-r} \prod_{i=1}^{r} [f_X(x_{(i)};\, \lambda,\, \beta)]$$
$$= (\lambda\, \beta)^r \, \exp\Big\{ \sum_{i=1}^{r} (\beta \ln x_{(i)} + x_{(i)}^{\beta}) - \lambda\, T_1(\beta;\, \underline{x}) \Big\},\, x_{(k)} \, > \, 0$$

(2.1)

where

$$\underline{x} = (x_{(1)}, \cdots, x_{(r)}),$$
$$T_1(\beta;\, \underline{x}) = \sum_{i=1}^{r} (e^{x_{(i)}^{\beta}} - 1) + (n-r)(e^{x_{(r)}^{\beta}} - 1). \tag{2.2}$$

## 3. Density Functions of Prior and Posterior Types

In the present work, it is taken that the researcher applies a simple prior density function for making proper measurement. According to the previous assumptions for the $\lambda$ and $\beta$ parameters, they are presented as:

$$\pi(\lambda,\, \beta) = \pi_1(\lambda)\, \pi_2(\beta) \tag{3.1}$$

where $\pi_1(\lambda)$ is a conjugate prior given by

$$\pi_1(\lambda) \, = \, \frac{b_1^{a_1}}{\Gamma(a_1)} \, \lambda^{a_1 - 1} \, e^{-b_1\, \lambda},\, \lambda \, > \, 0 \, (a_1,\, b_1 \, > \, 0). \tag{3.2}$$

and

$$\pi_2(\beta) \, = \, \frac{b_2^{a_2}}{\Gamma(a_2)} \, \beta^{a_2 - 1} \, e^{-b_2\, \beta},\, \beta \, > \, 0 \, (a_2,\, b_2 \, > \, 0). \tag{3.3}$$

According to Sarhan et all [10], the function analyzed in the points (3.2) and (3.3) should be applied with the corresponding two parameters to identify the bathtub-shaped distribution. Further summarization of joint posterior density function of $\lambda$ and $\beta$ parameters is given below with the use of the joint prior density function estimated in (4.1) and likelihood function estimated in (5.4):

$$\pi_2^*(\lambda,\, \beta,\, |\underline{x}) \propto \lambda^{r + a_1 - 1} \beta^{r + a_2 - 1} \exp\Big\{ \sum_{i=1}^{r} (\beta \ln x_{(i)} + x_{(i)}^{\beta}) \, - \, b_2\, \beta - \, \lambda\, [T_1(\beta;\, \underline{x}) \, + \, b_1] \Big\}, \tag{3.4}$$

Therefor, the $\lambda$ and $\beta$ posterior density function can presented as

$$\pi^*(\lambda,\, \beta|\underline{x}) \;\; \propto \;\; g_1(\lambda|data)\, g_2(\beta|\, data) \; g_3(\lambda, \beta|\, data), \tag{3.5}$$

with $g_1(\lambda|\beta, data)$ as Gamma density under the constraints of $r$ shape and $T_1(\beta; \underline{x})$ scale, while a proper density function of $g_2(\beta|data)$ is presented below as:

$$g_2(\beta|data) \propto \frac{1}{[T_1(\beta;\, \underline{x})]^r}\, \beta^{r-1}\, \exp\left\{ \sum_{i=1}^{r} \beta(\ln x_{(i)} - b_2) \right\} \tag{3.6}$$

while $g_3(\lambda, \beta|\, data))$ can be presented as follows:

$$g_3(\lambda,\, \beta|data) = \lambda^{a_1}\, \beta^{a_2}\, e^{-\lambda b_1 + x^{\beta}}. \tag{3.7}$$

Thereby, taking into consideration the squared error loss function, all functions of $\lambda$ and $\beta$ can be interpreted through the Bayesian estimation as follows:

$$\hat{g}(\lambda,\, \beta) \quad = \frac{\int_0^\infty \int_0^\infty g(\lambda,\, \beta) g_1(\lambda|data)\, g_2(\beta|\lambda,\, data) g_3(\lambda,\, \beta|data) d\lambda d\beta}{\int_0^\infty \int_0^\infty g_1(\lambda|data)\, g_2(\beta|\lambda,\, data)\, g_3(\lambda,\, \beta|data) d\lambda d\beta}. \tag{3.8}$$

Analysis of the equation (5.6) makes it possible to claim that its conversion into a simple closed form is impossible. Thus, estimation of the Bayesian predications in terms of $\lambda$ and $\beta$ as the inputs denoted above is not possible in this form either. Consequently, one of the assumptions covers the potential efficiency of applying the technique of importance sampling. It should be done in accordance with the idea suggested by Chen and Shao [11], which implies approximation of (5.6) to find a solution to the restrictions related to simple closed forms.

3.1. **Technique of Importance Sampling.** The methodology of importance sampling technique is applied to compute and validate the $\lambda$, $\beta$ Bayes estimates as well as a number of constructed relevant functions, for instance $g(\lambda,\, \beta)$. Algorithm presents the process of approximating the function of posterior density.

**Algorithm:**

(1) Using $g_1(\beta|data)$ for the estimation of $\beta$.
(2) Using $g_2(\lambda|\beta,\, data)$ for the estimation of $\lambda$.
(3) Repeating the stage 1 and 2 consecutively for generation of $(\lambda_1,\, \beta_1),\, (\lambda_2,\, \beta_2),\, \cdots,\, (\lambda_M,\, \beta_M)$.

The following equation presents the process of approximating in the context of the procedure of importance sampling under the restrictions of Bayesian estimates for $g(\lambda,\, \beta)$ along with the relevant control for squared error loss:

$$\hat{g}_{BS}(\lambda,\, \beta) \quad = \frac{\sum\limits_{i=1}^{M} g(\lambda_i,\, \beta_i)\, g_3(\lambda_i,\, \beta_i|data)}{\sum_{i=M_0}^{M}\, g_3(\lambda_i,\, \beta_i|data)}, \tag{3.9}$$

## 4. Prediction with Outliers Presence

In the analysis of the process of predicting the future observations with the outliers presence, it is appropriate to use the formal definitions given for (1.1). Besides, I apply the random sample of $x_1, x_2, \cdots, x_n$ created on the basis of Chen ($\lambda$, $\beta$) on the basis of the given function of population density. The following stage is using the independent unobserved sample as $y_1, y_2, \cdots, y_m$ as a result of using the same data to form a future sample. The next stage covers further testing of the boundaries of Bayesian prediction for $s^{th}$ with a single outlier in the range of future estimates for $y_s$, $s = 1, 2, \cdots, m$. The following equation presents the $y_s$ density function for a provided $\theta$ under the conditions described above:

$$\begin{aligned} h(y_s|\,\theta) \; &= D(s)\left[(s-1)F^{s-2}(1-F)^{m-s}F^{\star}f + F^{s-1}(1-F)^{m-s}f^{\star}\right. \\ &\left. +(m-s)F^{s-1}(1-F)^{m-s-1}(1-F^{\star})f\right], \end{aligned} \tag{4.1}$$

with

$$D(s) \; = \binom{m-1}{s-1} \tag{4.2}$$

The function of density is presented as $f = f(y|\,\theta)$ and the function of cumulative distribution is given as $F = F(y|\theta)$ for all $y^s$ that can't be defined as outliers. Balakrishnan and Ambagabpitiya [12] refer to $f^{*} = f^{*}(y|\theta)$ and $F^{*} = F^{*}(y|\theta)$ as to outliers. Acquiring the $f^{*}$ and $F^{*}$ functions is done for the Chen ($\lambda$, $\beta$) model via using a different parameter $\lambda$ by $\lambda\ \lambda_0$, or $\lambda + \lambda_0$ according to the classification of the outliers. The study of Alanzi and Niazi [8] is devoted the analysis of prediction interval on the basis of using doubly Type-II censored sample for the future to have $\lambda$ replaced by $\lambda\ \lambda_0$, or $\lambda + \lambda_0$. Furthermore, the research conducted by Alanzi [13] was devoted to study of a right Type-II censored sample to have $\lambda$ replaced with $\lambda\ \lambda_0$ with further calculation of the interval for prediction aimed at first and final future observation. The present study implies replacement of $\lambda$ with $\lambda + \lambda_0$ as well as calculating the interval for predication needed for the first and final observation that involve using the right Type-II censored samples.

## 5. Prediction of The First Observation

The case of prediction of the first implies having distribution in the first $y_1$ in the $m$-size future sample via adding $s = 1$ in (4.1) with only one outlier presence (type $\lambda + \lambda_0$); it is done as follow:

$$h(y_1|\,\theta) \; = (1-F)^{m-1}f^{\star} + (m-1)(1-F)^{m-2}(1-F^{\star})f, \tag{5.1}$$

It is possible to acquire $Y_1$ density function with a single type $\lambda + \lambda_0$ outlier presence in the case of $Chen(\lambda,\,\beta)$ via changing of (1.1) for $f$ and (1.2) for $F$ in (5.1). On replacement of $\lambda$ for $\lambda + \lambda_0$. $f^{*}$ and $F^{*}$ have the same values as they do in (1.1) and (1.2). It is possible to present a density function in the following simplified form:

$$h_1(y_1|\,\lambda,\,\beta) \;=\; f(y_1;\,(\lambda\,m + \lambda_0),\,\beta), \tag{5.2}$$

with cdf of $y_1$ presented as follows:

$$H_1(y_1|\,\lambda,\,\beta) = \;F(y_1;\,(\lambda\,m + \lambda_0),\,\beta). \tag{5.3}$$

Estimation of the predictive density of $Y\;=\;y_1$, with $x$, $(\lambda\,m\,+\,\lambda_0)$ and $\beta$ is as follows:

$$h_1^*(y|\underline{x}) = \int_0^\infty \int_0^\infty h_1(y_1|\,\lambda,\,\beta)\,\pi^*(\lambda,\,\beta|\,\underline{x})\,d\lambda\,d\beta, \tag{5.4}$$

Estimation of the $Y\;=\;y_1$, predictive distribution function with $x$, $\lambda$ and $\beta$ is as follows:

$$H_1^*(y\,|\underline{x}) = \int_0^\infty \int_0^\infty H_1(y_1|\underline{x},\,\lambda,\,\beta)\,\pi^*(\lambda,\,\beta|\,\underline{x})\,d\lambda\,d\beta, \tag{5.5}$$

$\{(\lambda_i,\,\beta_i);\,i = 1,\,2,\,\cdots,M\}$ are assumed to be MCMC samples obtained after generation from $\pi^*(\lambda,\,\beta|\,\underline{x})$ and corresponding parameters of estimation to ensure consistency of $h_1^*(y_1|\underline{x},\,\lambda,\,\beta)$ and $H^*(y_1|\underline{x},\,\lambda,\,\beta)$. Thus,

$$\hat{h}_1^*(y\,|\underline{x}) \;=\; \sum_{i=1}^{M} h_1(y_1|\,\lambda_i,\,\beta_i)\,h_i \tag{5.6}$$

and

$$\hat{H}_1^*(y\,|\underline{x}) \;=\; \sum_{i=1}^{M} H_1(y_1|\,\lambda_i,\,\beta_i)\,h_i \tag{5.7}$$

with

$$g_i \;=\; \frac{g_3(\lambda_i,\,\beta_i|\,data)}{\sum_{i=1}^{M} g_3(\lambda_i,\,\beta_i|\,data)}; \quad i = 1,\,2,\,\cdots,\,M. \tag{5.8}$$

On the basis of the above-mentioned analysis, the Bayesian estimation for $Y_1$,$(1-\tau)\,100\,\%$ implies having $P[L(\underline{x})\;\leq\;Y_1\;\leq\;U(\underline{x})]\;=\;1-\tau$, with $L(\underline{x})$ as the highest limit for $y_1$ and $U(\underline{x})$ as the lowest one. The following estimation is made on the basis of prior estimates for (5.7), $1-\frac{\tau}{2}$ and $\frac{\tau}{2}$, thus:

$$P[Y\;\geq\;L(\underline{x})|\,\underline{x}] = 1-\frac{\tau}{2}\;\Rightarrow\;\hat{H}_1^*(L(\underline{x})|\,\underline{x})\;=\;\frac{\tau}{2} \tag{5.9}$$

and

$$P[Y\;\leq\;U(\underline{x})|\,\underline{x}] = \frac{\tau}{2}\;\Rightarrow\;\hat{H}_1^*(U(\underline{x})|\,\underline{x})\;=\;1-\frac{\tau}{2}. \tag{5.10}$$

Calculation of the prediction limits of $y_1$ is done using the equations (5.9) and (5.10).

## 6. Prediction of The Last Observation

Distribution of the last in a $m$-size sample with only a single outlier presence is ensured when $s = m$ is added in (4.1). It is possible to present the $Y_m$ density function for a provided $\theta$ with a single outlier presence as follows:

$$h_2(y_m|\,\theta) \;=\; (m-1)F^{m-2}F^{\star}f + F^{m-1}f^{\star}, \tag{6.1}$$

Provided that a single outlier of type $\lambda + \lambda_0$ is presence, it is possible to obtain the $Y_m$ density function in the case Chen $(\lambda,\, \beta)$ via replacing (1.1) for $f$ and (1.2) for $F$ in (6.1). The research uses the $f^{*}$ value from (1.1) and $F^{*}$ value from (1.2) after $\lambda$ is replaced with $\lambda + \lambda_0$. It is possible to present the mentioned density function in the following simplified form:

$$h_2(y_m|\,\lambda,\,\beta) = \Big[(\lambda + \lambda_0)\sum_{j=0}^{m-1} B_{1j}(y_m) + \lambda(m-1)\sum_{j=0}^{m-2} B_{2j}(y_m)\Big],\; y_m > 0, \tag{6.2}$$

with

$$\begin{aligned} B_{1j}(y_m) &= a_{1j}(m) f(y_m;\, \lambda\,(j+1) + \lambda_0,\, \beta),\\ B_{2j}(y_m) &= a_{2j}(m)\Big[f(y_m;\, \lambda\,(j+1),\, \beta) - f(y_m;\, \lambda\,(j+2) + \lambda_0,\, \beta), \end{aligned} \tag{6.3}$$

with $\ell = 1,\, 2$,

$$a_{\ell\, j}(m) \,=\, (-1)^j \binom{m-\ell}{j}, \tag{6.4}$$

The cdf that is related to pdf $h_2(y_m|\,\lambda,\,\beta)$ is as follows:

$$H_2(y_m|\,\lambda,\,\beta) = D(s)\,\Big[(\lambda + \lambda_0)\sum_{j=0}^{m-1} B^{*}_{1j}(y_m) + \beta(m-1)\sum_{j=0}^{m-2} B^{*}_{2j}(y_m)\Big], \tag{6.5}$$

with

$$\begin{aligned} B^{*}_{1j}(y_m) \;&=\; \frac{a_{1j}(m)}{\lambda\,(j+1) + \lambda_0} F(y_m;\, \lambda\,(j+1) + \lambda_0,\, \beta),\\ B^{*}_{2j}(y_m) \;&=\; \frac{a_{2j}(m)}{\lambda\,(j+1)} F(y_m; \lambda\,(j+1),\, \beta)\\ &\quad - \frac{a_{2j}(m)}{\lambda\,(j+2) + \lambda_0} F(y_m;\, \lambda\,(j+2) + \lambda_0,\, \beta), \end{aligned} \tag{6.6}$$

with $F(y_m;\, \lambda\,(j{+}1){+}\lambda_0,\, \beta)$ presented via (1.2). The $y_m$ predictive density provided that there are constraints for $x$ from the outlier $\lambda + \lambda_0$ can be estimated via using the equations (6.2) in (5.4) and the algorithm.

$$h_2^*(y_m|\underline{x}) = \int_0^\infty \int_0^\infty h_2(y_m|\,\lambda,\,\beta)\,\pi^*(\lambda,\,\beta|\,\underline{x})\,d\lambda\,d\beta, \tag{6.7}$$

With the predictive cdf of $y_m$, $G_2^*(y_m|\underline{x})$ can be defined as follows:

$$H_2^*(y_m|\underline{x}) = \int_0^\infty \int_0^\infty H_2(y_m|\,\lambda,\,\beta)\,\pi^*(\lambda,\,\beta|\,\underline{x})\,d\lambda\,d\beta, \tag{6.8}$$

with $\pi^*(\lambda,\,\beta|\,\underline{x})$ presented in (6.5) and $H_2(y_m|\,\lambda,\,\beta)$ presented in (3.5). Clearly, it is not possible to present either (6.7) or (6.8) in a closed form. Consequently, evaluation can't be done with the use of analytical approch. On the basis of the MCMC samples $\{(\lambda_i,\,\beta_i),\ i = 1,\,2,\,\cdots,\,M\}$, along with a consistent estimator used for $G_2^*(y_m|\underline{x})$ and $g_2^*(y_m|\underline{x})$ simulation of data, it is reasonable to state:

$$\hat{h}_2^*(y_m|\underline{x}) = \sum_{i=1}^{M} h_2(y_m|\,\lambda_i,\,\beta_i)\,h_i, \tag{6.9}$$

and

$$\hat{H}_2^*(y_m|\underline{x}) = \sum_{i=1}^{M} H_2(y_m|\,\lambda_i,\,\beta_i)\,h_i, \tag{6.10}$$

with estimation of $h_i$ from (5.8). Moreover, it is possible to estimate $\hat{G}_2^*(y_m|\underline{x})$ and $\hat{g}_2^*(y_m|\underline{x})$ for all $y_m$ with the use of MCMC samples $\{(\lambda_i,\,\beta_i),\ i = 1,\,2,\,\cdots,\,M\}$ that bear certain similarity to them. Furthermore, the Bayesian prediction boundaries of a $(1-\tau)100\%$ type for $y_m$ implies having $P[L(\underline{x}) \le y_m \le U(\underline{x})] = 1-\tau$ with $L(\underline{x})$ as the lowest Bayesian prediction limit for $y_m$ and $U(\underline{x})$ is the highest. It is possible to acquire them and identify as $L(\underline{x})$ and $U(\underline{x})$, which can be show via non-linear equations solutions as follows for $y_m$ by non-linear equations solutions.

$$P[Y \ge L(\underline{x})|\,\underline{x}] = 1 - \frac{\tau}{2} \Rightarrow \hat{H}_2^*(L(\underline{x})|\,\underline{x}) = \frac{\tau}{2} \tag{6.11}$$

and

$$P[Y \le U(\underline{x})|\,\underline{x}] = \frac{\tau}{2} \Rightarrow \hat{H}_2^*(U(\underline{x})|\,\underline{x}) = 1 - \frac{\tau}{2}. \tag{6.12}$$

Iterative statistical methodologies could apply the equations (6.11) and (6.12) mentioned above to ensure advanced regression that implies also implemented control for $y_m$ multicellularity or backward analysis and other factors in other cases.

For instance, generating $\lambda = 0.983884$ for the prior parameters $a_1 = 1.3$ and $b_1 = 2.1$ using the equation (3.2) for the prior density. Also, generating $\beta = 3.90261$ for the prior parameters $a_2 = 3.2$ and $b_2 = 1.4$ using the equation (3.3) for the prior density. Further, Chen distribution with $\lambda = 0.983884$ and $\beta = 1, 3.90261$ on the basis of using a different value of $r$ allows generating a random $n = 30$ size sample.

For an illustration of the example, I can make an assumption that there is different $m = 10$ size sample with a single outlier $\lambda + \lambda_0$ presence. There is a set goal to obtain prediction limits $Y_1$ and $Y_15$ estimated as 95% for the provided $\lambda_0$ value in terms of the first and last of future sample. Table 1,2,3 demonstrate the limits with the provided $\lambda + \lambda_0$ values.

TABLE 1. Bayesian prediction intervals 95 % for $y_1$ and $y_{15}$ with a single $\lambda + \lambda_0$ outlier presence, $n = 30, r = 20$.

| $\lambda_0$ | Observations | $y_1$ | $y_{15}$ |
|---|---|---|---|
| 0 | Lower and Upper limits | (0.199913, 0.690383) | 0.991503, 1.20199) |
| | Length | 0.490471 | 0.210491 |
| | Percentage of Coverage | 95.09 % | 94.84 % |
| 1 | Lower and Upper limits | (0.196435, 0.679618) | (0.986197, 1.20059) |
| | Length | 0.483183 | 0.214388 |
| | Percentage of Coverage | 94.57 % | 95.24 % |
| 2 | Lower and Upper limits | (0.193236, 0.669636) | (0.985184, 1.20058) |
| | Length | 0.4764 | 0.2154 |
| | Percentage of Coverage | 94.06 % | 95.33 % |
| 3 | Lower and Upper limits | (0.190279 , 0.660341) | (0.984974 , 1.20058) |
| | Length | 0.470062 | 0.21561 |
| | Percentage of Coverage | 93.41% | 95.33% |
| 4 | Lower and Upper limits | (0.187532 , 0.651651) | (0.98493, 1.20058) |
| | Length | 0.46412 | 0.215654 |
| | Percentage of Coverage | 92.93% | 95.33% |

TABLE 2. Bayesian prediction intervals 95 % for $y_1$ and $y_{15}$ with a single $\lambda + \lambda_0$, outlier presence $n = 30$, $r = 25$.

| $\lambda_0$ | Observations | $y_1$ | $y_{15}$ |
|---|---|---|---|
| 0 | Lower and Upper limits | $(0.193992, 0.679774)$ | $(0.983196, 1.19774)$ |
| | Length | 0.485782 | 0.214544 |
| | Percentage of Coverage | 94.72 % | 95.29% |
| 1 | Lower and Upper limits | $(0.190741, 0.669518)$ | $(0.977914, 1.1963)$ |
| | Length | 0.478776 | 0.218387 |
| | Percentage of Coverage | 94.18 % | 95.62% |
| 2 | Lower and Upper limits | $(0.187741, 0.659979)$ | $(0.976825, 1.1963)$ |
| | Length | 0.472238 | 0.219473 |
| | Percentage of Coverage | 93.57% | 95.74% |
| 3 | Lower and Upper limits | $(0.184957, 0.651072)$ | $(0.976582, 1.1963)$ |
| | Length | 0.466115 | 0.219716 |
| | Percentage of Coverage | 93.01 % | 95.7 % |
| 4 | Lower and Upper limits | $(0.182364, 0.642726)$ | $(0.976527, 1.1963)$ |
| | Length | 0.460362 | 0.219771 |
| | Percentage of Coverage | 92.02 % | 95.7% |

Table 3. Bayesian prediction intervals 95 % for $y_1$ and $y_{15}$ with a single $\lambda + \lambda_0$ outlier presence, $n = r = 30$.

| $\lambda_0$ | Observations | $y_1$ | $y_{15}$ |
|---|---|---|---|
| 0 | Lower and Upper limits | (0.197556, 0.685817) | (0.987722, 1.1998) |
| | Length | 0.488261 | 0.212081 |
| | Percentage of Coverage | 94.99 % | 95.06% |
| 1 | Lower and Upper limits | (0.194182, 0.675296) | (0.982433, 1.19838) |
| | Length | 0.481114 | 0.521755 |
| | Percentage of Coverage | 94.43% | 95.44% |
| 2 | Lower and Upper limits | (0.191073, 0.665526) | 0.981386, 1.19838) |
| | Length | 0.474453 | 0.21595 |
| | Percentage of Coverage | 93.9% | 95.59% |
| 3 | Lower and Upper limits | (0.188195, 0.656417) | (0.981161, 1.19838) |
| | Length | 0.468223 | 0.216995 |
| | Percentage of Coverage | 93.27 % | 95.59% |
| 4 | Lower and Upper limits | (0.185517, 0.647892) | (0.981112, 1.19838) |
| | Length | 0.462375 | 0.21722 |
| | Percentage of Coverage | 92.57% | 95.6% |

## 7. Conclusion on The Findings

The research analyzes a single $\lambda + \lambda_0$ outlier as multiple outliers can be studied only on the basis of a more profound analysis. It is possible to acquire the Bayesian prediction limits of the first $y_1$ and last $y_{15}$ in the homogeneous case for the future observation with no outliers by $\lambda + \lambda_0$ setting in (5.7) and (6.2) equation. Table 1, 2 and 3 show the potential impact of $\lambda$ value on the future observation boundaries and restrictions.

**Conflicts of Interest:** The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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