International Journal of Analysis and Applications
Volume 19, Number 3 (2021), 389-404
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-19-2021-389

ON A NEW APPROACH BY MODIFIED $(p;q)$-SZÁSZ-MIRAKYAN OPERATORS

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Abstract. In this paper, we introduce a new type of $(p;q)$ exponential function with some properties and a modified $(p;q)$-Szász-Mirakyan operators by virtue of this function by investigating approximation properties. We obtain moments of generalized $(p;q)$-Szász-Mirakyan operators. Furthermore, we derive direct results, rate of convergence, weighted approximation result, statistical convergence and Voronovskaya type result of these operators with numerical examples. Graphical representations reveal that modified $(p;q)$-Szász-Mirakyan operators have a better approximation to continuous functions than pioneer one.

1. Introduction

Approximation theory is one of the oldest branches of mathematics. To approximate continuous functions with q-analogue of linear positive operators is significant application of q-calculus in approximation theory. Cieśliński [1] established alternative definition of q-exponential function. He defined q-exponential function

Received November 29th, 2020; accepted January 4th, 2021; published April 9th, 2021.

2010 *Mathematics Subject Classification.* 46B28.

Key words and phrases. $(p;q)$-Szász-Mirakyan operators; uniform convergence; Voronovskaya type result; statistical convergence.

using Cayley transformation. The main advantages of the new q-exponential function consist of better qualitative properties i.e., its properties are more similar to properties of $e^z, z \in \mathbb{C}$ [1]. Over the years, many research papers were developed on q-analogue of various linear positive operators and their approximation properties.

Recently, in [2] research of Bernstein-Stancu operators on $(p;q)$- integers were performed and discussed uniform convergence and direct result of the operators. Eventually, in [3] $(p;q)$-analogue of Bernstein operators was investigated and developed the same convergence. Acar [5] and Mursaleen et.al [4], [12] proposed $(p;q)$-generalization of Szász-Mirakyan operators and discussed uniform convergence, rate of convergence, Voronovskaya result in those papers.

The motivation of recent work is developing a new type of $(p;q)$ exponential function and utilizing this new exponential function to modify $(p;q)$-Szász-Mirakyan operators. We studied uniform convergence and statistical convergence of modified $(p;q)$-Szász-Mirakyan operators. In the first section, we discussed some sequences and rate of convergence of operators. We also proved Voronovskaya type result. In the last section, we present some graphical representations.

Consider, $0 < q < p \leq 1$. The definition of $(p;q)$-integer is,

$$\{m\}_{p,q} = \frac{p^m - q^m}{p-q}, \; m \in \mathbb{N}, \tag{1.1}$$

$$\{0\}_{p,q} = 0,$$

and $(p;q)-$factorial is

$$\{m\}_{p,q}! = \prod_{k=1}^{m} \{k\}_{p;q}, \quad m \in \mathbb{N}$$

$$\{0\}_{p;q}! = 1. \tag{1.2}$$

$(p;q)$-exponential function is defined as [6]

$$e_{p,q}(z) = \sum_{j=0}^{\infty} \frac{p^{\frac{j(j-1)}{2}} z^j}{\{j\}_{p,q}!}, \tag{1.3}$$

$$E_{p,q}(z) = \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}} z^j}{\{j\}_{p,q}!}. \tag{1.4}$$

The $(p;q)$-exponential functions have following property:

$$e_{p,q}(z)E_{p,q}(-z) = E_{p,q}(z)e_{p,q}(-z) = 1.$$

Another way of defining two $(p;q)$-exponentials as infinite products is

$$e_{p,q}(z)=\prod_{j=0}^{\infty}\frac{1}{(p^j-q^j(p-q)z)},\tag{1.5}$$

$$E_{p,q}(z)=\prod_{j=0}^{\infty}(p^j+q^j(p-q)z).\tag{1.6}$$

2. New type of $(p;q)$-exponential function

New $(p;q)$-exponential function is determined as

$$\mathcal{E}_{p,q}(z)=e_{p,q}(z/2)E_{p,q}(z/2)=\prod_{j=0}^{\infty}\frac{p^j+q^j(p-q)\frac{z}{2}}{p^j-q^j(p-q)\frac{z}{2}},\tag{2.1}$$

$e_{p,q}(z), E_{p,q}(z)$ are usual $(p;q)$-exponential functions.

Theorem 2.1. *$(p;q)$-exponential function $\mathcal{E}_{p,q}(z)$ is analytic in $|z|<R_{p,q}$*

$$\mathcal{E}_{p,q}(z)=\sum_{j=0}^{\infty}\frac{z^j}{[j]_{p,q}!},\quad |z|<R_{p,q}\tag{2.2}$$

where,

$$R_{p,q}=\begin{cases}\frac{2}{p-q}, & 0<q<p<1.\\ \frac{2q}{q-p}, & q>p.\\ \infty, & p=q=1.\end{cases}\tag{2.3}$$

$$[j]_{p,q}=\frac{p^j-q^j}{p-q}.\frac{2}{p^{j-1}+q^{j-1}}=\{j\}_{p,q}.\frac{2}{p^{j-1}+q^{j-1}},\tag{2.4}$$

$$\text{and}\quad [j]_{p,q}!=\prod_{m=1}^{j}[m]_{p,q}=\prod_{m=1}^{j}\{m\}_{p,q}\frac{2}{p^{j-1}+q^{j-1}}=\{j\}_{p,q}!\frac{2^j}{\prod_{m=0}^{j-1}(p^m+q^m)}.\tag{2.5}$$

Proof. Since (1.5) and (1.6) are absolutely convergent in $|z|<1$, multiplying (1.5) and (1.6), we obtain

$$\begin{aligned}e_{p,q}(z/2)E_{p,q}(z/2)&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{p^{\frac{n(n-1)}{2}}q^{\frac{k(k-1)}{2}}(\frac{z}{2})^{n+k}}{\{n\}_{p,q}!\{k\}_{p,q}!}\\&=\sum_{j=0}^{\infty}\frac{(\frac{z}{2})^j}{\{j\}_{p,q}!}\sum_{k=0}^{\infty}\frac{p^{\frac{(j-k)(j-k-1)}{2}}q^{\frac{k(k-1)}{2}}\{j\}_{p,q}!}{\{j-k\}_{p,q}!\{k\}_{p,q}!}.\end{aligned}\tag{2.6}$$

Using formula for the $(p;q)$-binomial coefficients [7], we have

$$\prod_{r=0}^{j-1}(p^r+xq^r)=\sum_{k=0}^{j}\frac{p^{\frac{(j-k)(j-k-1)}{2}}q^{\frac{k(k-1)}{2}}\{j\}_{p,q}!}{\{j-k\}_{p,q}!\{k\}_{p,q}!}x^k.\tag{2.7}$$

In particular,

$$\sum_{k=0}^{j}\frac{p^{\frac{(j-k)(j-k-1)}{2}}q^{\frac{k(k-1)}{2}}\{j\}_{p,q}!}{\{j-k\}_{p,q}!\{k\}_{p,q}!}=(1+1)(p+q)...(p^{j-1}+q^{j-1}).\tag{2.8}$$

Substituting (2.8) into (2.6), we obtain (2.2), where $[j]_{p,q}$ defined as in (2.4). To get the radius of convergence,

$$\begin{aligned} &\lim_{n\to\infty}\left|\frac{z^{j+1}}{[j+1]_{p,q}!}\right|\left|\frac{[j]_{p,q}!}{z^j}\right| \\ &= \lim_{n\to\infty}\left|\frac{z}{[j+1]_{p,q}}\right| \\ &= \begin{cases} \frac{(p-q)|z|}{2}, & \text{for } \ q<p, \\ \frac{(q-p)|z|}{2q}, & \text{for } \ q>p. \end{cases} \end{aligned} \tag{2.9}$$

Applying d'Alembert's test on (2.9), we obtain $(p, q \neq 1)$ the radius of convergence (2.3). For $p = q = 1$, $\mathcal{E}_{p,q}(z)$ is e^z, thus $R_1 = \infty$. □

Theorem 2.2. *The $\mathcal{E}_{p,q}(z)$ satisfies the following properties:*

$$1.\ \mathcal{E}_{p,q}(-z) = (\mathcal{E}_{p,q}(z))^{-1}, \qquad 2.\ |\mathcal{E}_{p,q}(ix)| = 1. \tag{2.10}$$

Proof. The first part of above equation (2.10) directly comes from the definition of $\mathcal{E}_{p,q}(z)$.
That implies, $\overline{\mathcal{E}_{p,q}(z)} = \mathcal{E}_{p,q}(\bar{z})$.

Then, $|\mathcal{E}_{p,q}(ix)|^2 \ = \ \overline{\mathcal{E}_{p,q}(ix)} \ \ \mathcal{E}_{p,q}(ix) = 1.$ □

The above $(p;q)$-exponential function (2.1) has more improved properties similar to function e^z.

The definition of (p,q)-Szász-Mirakyan operators in [5] is

$$A_{m,p,q}(f;x) = \sum_{j=0}^{\infty}\frac{1}{E_{p,q}(\{m\}_{p,q}x)}q^{\frac{j(j-1)}{2}}\frac{\{m\}_{p,q}^j x^j}{\{j\}_{p,q}!}f\left(\frac{\{j\}_{p,q}}{q^{j-2}\{m\}_{p,q}}\right). \tag{2.11}$$

Acar obtained moments, uniform convergence and Voronovskaya result of the above operators.

We define a different sort of modified (p,q)-Szász-Mirakyan operators via new $(p;q)$-exponential function for $f \in C[0,\infty]$ in (2.1) is

$$S_{n,p,q}(f;x) = \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)}\sum_{k=0}^{\infty}\frac{[n]_{p,q}^k x^k}{[k]_{p,q}!}f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right), \tag{2.12}$$

where $0 < q < p \leq 1$, $n \in \mathbb{N}, 0 \leq x < \frac{2}{(p-q)[n]_{p,q}} = \frac{p^{n-1}+q^{n-1}}{p^n-q^n}$.

Remark 2.1. *We choose an x between 0 and $\frac{p^{n-1}+q^{n-1}}{p^n-q^n}$ because we want $\mathcal{E}_{p,q}([n]_{p,q}x)$ to be convergent.*

Remark 2.2. *From calculations for every $k \in \mathbb{N}$;*

$$\frac{[k]_{p,q}}{[n]_{p,q}} = \frac{(p^k-q^k)(p^{n-1}+q^{n-1})}{(p^n-q^n)(p^{k-1}+q^{k-1})},$$
$$0 \leq \frac{[k]_{p,q}}{[n]_{p,q}} < \frac{p^{n-1}+q^{n-1}}{p^n-q^n},$$

Then we consider

$$s_n(p,q;x) = \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \cdot \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!}. \tag{2.13}$$

Clearly, $s_n(p,q;x)$ is positive for $0 < q < p \leq 1$, $n \in \mathbb{N}$ and every $0 \leq x < \frac{2}{(p-q)[n]_{p,q}}$.

The operator $S_{n,p,q}$ is linear and positive.

3. Moments of $S_{n,p,q}$

Here, we determine approximation moments of operators (2.12).

Lemma 3.1. *For $n \in \mathbb{N}$ and $0 < q < p \leq 1$. Below equalities are verified:*

$$S_{n,p,q}(1;x) = 1, \tag{3.1}$$

$$S_{n,p,q}(t;x) = x, \tag{3.2}$$

$$S_{n,p,q}(t^2;x) = x^2 + \frac{x}{[n]_{p,q}}, \tag{3.3}$$

$$S_{n,p,q}(t^3;x) = x^3 + \frac{3x^2}{[n]_{p,q}} + \frac{x}{[n]_{p,q}^2}. \tag{3.4}$$

Proof. The result is obvious for $S_{n,p,q}(1;x)$.

Now

$$\begin{aligned} S_{n,p,q}(t;x) &= \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \sum_{k=0}^{\infty} \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!} \left(\frac{[k]_{p,q}}{[n]_{p,q}} \right) \\ &= \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \sum_{k=1}^{\infty} \frac{[n]_{p,q}^{k-1} x^k}{[k-1]_{p,q}!} \\ &= x. \end{aligned}$$

and

$$\begin{aligned} S_{n,p,q}(t^2;x) &= \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \sum_{k=0}^{\infty} \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!} \left(\frac{[k]_{p,q}}{[n]_{p,q}} \right)^2 \\ &= \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \sum_{k=1}^{\infty} \frac{[n]_{p,q}^{k-1} x^k}{[k-1]_{p,q}!} \left(\frac{1}{[n]_{p,q}} \right) + \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \sum_{k=2}^{\infty} \frac{[n]_{p,q}^{k-2} x^k}{[k-2]_{p,q}!} \\ &= \frac{x}{[n]_{p,q}} + x^2. \end{aligned}$$

Also

$$\begin{aligned} S_{n,p,q}(t^3;x) &= \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \sum_{k=0}^{\infty} \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!} \left(\frac{[k]_{p,q}}{[n]_{p,q}} \right)^3 \\ &= \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \sum_{k=0}^{\infty} \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!} \left(\frac{[k]_{p,q}^3 - 3[k]_{p,q}^2 + 2[k]_{p,q} + 3[k]_{p,q}^2 - 2[k]_{p,q}}{[n]_{p,q}^3} \right) \\ &= x^3 + \frac{3x^2}{[n]_{p,q}} + \frac{x}{[n]_{p,q}^2}. \end{aligned}$$

□

Central moments are:

$$S_{n,p,q}(t-x;x)=0, \tag{3.5}$$

$$S_{n,p,q}((t-x)^2;x)=\frac{x}{[n]_{p,q}}. \tag{3.6}$$

Remark 3.1. *From our choice of* p *and* q*, we know that* $\lim_{n\to\infty}[n]_{p,q}=\frac{1}{p-q}$*. But, to get the uniform convergence and other results of approximation for* $S_{n,p,q}$ *we suppose that sequences* $q_n\in(0,p_n);p_n\in(q_n,1]$ *such that* $q_n,\ p_n\to 1$ *and* $p_n^N\to a,\quad q_n^{N'}\to b$ *as* n *tending to infinity, i.e.,* $\lim_{n\to\infty}1/[n]_{p,q}=0$.

Now, we have uniform convergence of new kind of operators for all $f\in C_\vartheta[0,\infty)$ where

$$C_\vartheta[0,\infty)=\{f\in C[0,\infty):|f(t)|\le A(1+t)^\vartheta\}\text{ for }A>0,\quad \vartheta>0$$

and

$$\|f\|=\sup_{x\ge0}\frac{|f(x)|}{1+x^2}.$$

Theorem 3.1. *Let* (p_n) *and* (q_n) *be the sequences such that* $p_n\to1,\quad q_n\to1$ *and* $p_n^N\to a,\quad q_n^{N'}\to b$ *as* n *tending to infinity then for each* $f\in C_\vartheta[0,\infty)$

$$\lim_{n\to\infty}\|S_{n,p,q}(f)-f\|_\vartheta=0. \tag{3.7}$$

Proof. From Korovkin's result, we put evidence that

$$\lim_{n\to\infty}\|S_{n,p,q}(t^i)-x^i\|_\vartheta=0,\qquad i=0,1,2.$$

Since $S_{n,p,q}(1;x)=1$, the result is clear for $i=0$.
For $i=1$

$$\lim_{n\to\infty}\|S_{n,p,q}(t)-x\|_\vartheta=\lim_{n\to\infty}\|x-x\|_\vartheta=0.$$

and for $i=2$

$$\begin{aligned}\lim_{n\to\infty}\|S_{n,p,q}(t^2)-x^2\|_\vartheta&=\lim_{n\to\infty}\|x^2+\frac{x}{[n]_{p,q}}-x^2\|_\vartheta\\&=0.\end{aligned}$$

Hence $S_{n,p,q}(f;x)$ is uniformly convergent to $f\in C_\vartheta[0,\infty]$. □

Example 3.1. *For* $p=0.99$ *and* $q=0.96$*, sequences of* $S_{n,p,q}$ *defined by (2.12) is convergent to* $f(x)=x^2-5x+10$ *(Fig. 1) and* $g(x)=x^3-x+1$ *(Fig. 2) with increasing values of* n $(n=10,20,30)$ *respectively.*

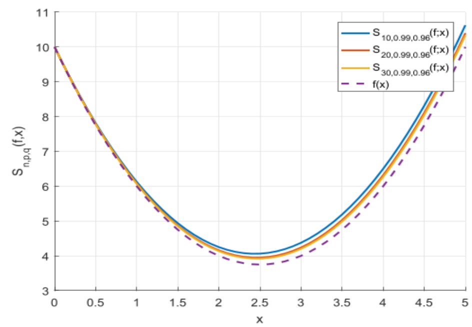

FIGURE 1. Approximation to f by $S_{n,p,q}$ for $n = 10, 20, 30$.

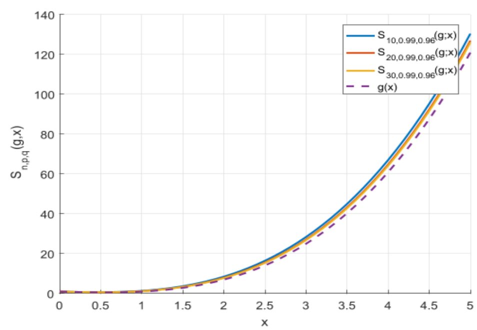

FIGURE 2. Approximation to g by $S_{n,p,q}$ for $n = 10, 20, 30$.

Example 3.2. *For different choices of p and q, the sequence of operators $S_{n,p,q}$ defined by (2.12) is convergent to $f(x) = x^2 - 5x + 10$ (Fig. 3) and $g(x) = x^3 - x + 1$ (Fig. 4) with $n = 50$ respectively.*

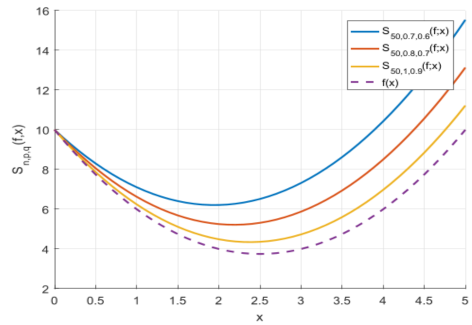

FIGURE 3. Approximation to f by $S_{n,p,q}$ for $n = 50$.

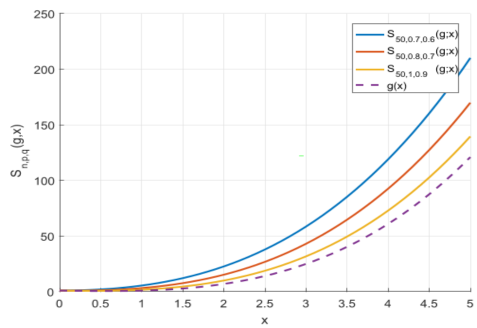

FIGURE 4. Approximation to g by $S_{n,p,q}$ for $n = 50$.

4. Some consequences

For this section, we provide several results on local approximation for $S_{n,p,q}(f;x)$. Here, $C_b[0,\infty)$ is the set of bounded, continuous functions f on $[0,\infty)$. Attached norm on $C_b[0,\infty)$ is defined by $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. Peetre's K-functional is given by

$$K_2(f,\delta) = \inf_{h\in W^2} \{\|f-h\| + \delta\|h''\|\}$$

where $W^2 = \{h \in C_b[0,\infty) : h', h'' \in C_b[0,\infty)\}$. From ([9], p.177), there exists $A > 0$ such that $K_2(f,\delta) \leq A\omega_2(f,\delta^{1/2}), \delta > 0$, where

$$\omega_2(f,\delta^{1/2}) = \sup_{0<\eta<\delta^{1/2}, x\in[0,\infty)} |f(x+2\eta) - 2f(x+\eta) + f(x)|$$

is the second order modulus of continuity of functions f in $C_b[0,\infty)$. The first order modulus of continuity of function $f \in C_b[0,\infty)$ is defined by

$$\omega(f,\delta^{1/2}) = \sup_{0<\eta<\delta^{1/2}, x\in[0,\infty)} |f(x+\eta) - f(x)|.$$

Theorem 4.1. *Let $0 < q < 1$ and $p \in (q, 1]$. The operators $S_{n,p,q}$ map from C_b into C_b. Also, the following inequality is satisfied.*

$$\|S_{n,p,q}(f;x)\|_{C_b} \leq \|f\|_{C_b}. \tag{4.1}$$

Proof. From the definition of $S_{n,p,q}(f;x)$,

$$|S_{n,p,q}(f;x)| \leq \frac{1}{\mathcal{E}_{p,q}([n]_{p,q}x)} \sum_{k=0}^{\infty} \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!} \left| f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right)\right|.$$

Applying supremum to both sides here

$$\sup_{x\geq 0} |S_{n,p,q}(f;x)| \leq \sup_{x\geq 0} |f(x)| \frac{1}{\mathcal{E}_{p,q}} S_{n,p,q}(1;x).$$

One has

$$\|S_{n,p,q}(f;x)\|_{C_b} \leq \|f\|_{C_b}.$$

□

Theorem 4.2. *Let (p_n) and (q_n) be the sequences such that $p_n \to 1$, $q_n \to 1$ and $p_n^N \to a$, $q_n^{N'} \to b$ as n tending to infinity. Then for $f \in C_b[0,\infty)$, there exists $A > 0$ such that*

$$|S_{n,p,q}(f;x) - f(x)| \leq A\omega_2\left(f, \sqrt{\frac{x}{[n]_{p,q}}}\right). \tag{4.2}$$

Proof. For $h \in W^2$, using Taylor's expansion

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-u)h''(u)du.$$

Now

$$|S_{n,p,q}h(t) - h(x)| \leq \tfrac{1}{2}\|h''\| S_{n,p,q}((t-x)^2;x).$$

Also

$$|S_{n,p,q}(f;x)| \leq \|f\|.$$

Hence

$$\begin{aligned}|S_{n,p,q}(f;x) - f(x)| &\leq |S_{n,p,q}((f-h)(x);x) - (f-h)(x)| + |S_{n,p,q}(h;x) - h(x)| \\ &\leq 2\|f-h\| + \frac{1}{2}\|h''\| S_{n,p,q}((t-x)^2;x).\end{aligned}$$

Taking infimum of the right hand side of above inequality for all $h \in W^2$,

$$|S_{n,p,q}(f;x) - f(x)| \leq 2K_2\left(f; \tfrac{1}{4}\tfrac{x}{[n]_{p,q}}\right).$$

Since $\omega_2(f,\lambda\delta) \leq (\lambda+1)^2\omega_2(f;\delta)$,

$$|S_{n,p,q}(f;x) - f(x)| \leq A\omega_2\left(f, \sqrt{\tfrac{x}{[n]_{p,q}}}\right).$$

□

5. Rate of convergence

Suppose that $C[0,\infty)$ is set of all continuous functions on $[0,\infty)$ and consider following sets:

$C_\vartheta[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq A(1+t)^\vartheta\}$ for $A > 0, \quad \vartheta > 0$

and

$C^*_\vartheta[0,\infty) = \{f \in C_\vartheta[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} < \infty\}$.

The first order modulus of continuity on $[0,a]$ is defined as

$$\omega_a(f,\delta) = \sup_{|t-x|\leq\delta} \sup_{0\leq x\leq a} |f(t) - f(x)|.$$

Theorem 5.1. *Suppose that $f \in C^*_\vartheta[0,\infty)$. Let (p_n) and (q_n) be the sequences such that $p_n \to 1, \quad q_n \to 1$ and $p_n^N \to a, \quad q_n^{N'} \to b$ as n tending to infinity and $\omega_{a+1}(f,\delta)$ be the modulus of continuity on $[0,a+1] \subset [0,\infty)$. Then*

$$\|S_{n,p,q}(f;x) - f(x)\|_\vartheta \leq 6A_f(1+a^2)\frac{a}{[n]_{p,q}} + 2\omega_{a+1}\left(f; \sqrt{\frac{a}{[n]_{p,q}}}\right). \tag{5.1}$$

Proof. For $0 \leq x \leq a;\ 0 \leq t, \infty$; From [8]

$$|f(t) - f(x)| \leq 6A_f(1+a^2)(t-x)^2 + \omega_{a+1}(f;\delta)\left(\frac{|t-x|}{\delta} + 1\right).$$

Combining above inequality and Cauchy-Schwarz inequality,

$$\begin{aligned}\|S_{n,p,q}(f;x) - f(x)\|_\vartheta &\leq S_{n,p,q}(|(f;x) - f(x)|;x) \\ &\leq 6A_f(1+a^2)S_{n,p,q}((t-x)^2;x) \\ &+ \omega_{a+1}(f;\delta)\left(1 + \frac{1}{\delta^2}S_{n,p,q}((t-x)^2;x)\right).\end{aligned}$$

From the central moments of operators and for $0 \leq x \leq a$

$$S_{n,p,q}((t-x)^2;x) = \tfrac{x}{[n]_{p,q}} \leq \tfrac{a}{[n]_{p,q}} = \zeta_n.$$

Taking $\delta = \sqrt{\zeta}$

$$\|S_{n,p,q}(f;x) - f(x)\|_\vartheta \leq 6A_f(1+a^2)\tfrac{a}{[n]_{p,q}} + 2\omega_{a+1}\left(f; \sqrt{\tfrac{a}{[n]_{p,q}}}\right).$$

□

6. Weighted approximation result

Here, we discuss weighted approximation of $S_{n,p,q}$ through polynomial weight over the space C_M defined below.

Consider that

$$\mathfrak{w}_0(x) = 1, \quad \mathfrak{w}_M(x) = (1+x^M)^{-1}, \quad (x \geq 0, M \in \mathbb{N}),$$

$$C_M = \{f \in C[0,\infty) : \mathfrak{w}_M(f) \ \ is \ \ continuous, \ \ bounded \ \ and \ \ uniformly \ \ convergent.\}.$$

The norm is defined by

$$\|f\|_M = \sup_{x\geq 0} \mathfrak{w}_M(x)|f(x)|.$$

Also, we refer some results associated to Steklov means. For $h > 0$ it is defined in [11]

$$f_h(x) = \left(\frac{2}{h}\right)^2 \int_0^{h/2}\int_0^{h/2} [2f(x+s+t) - f(x+2(s+t))]dsdt.$$

We have

$$f(x) - f_h(x) = \left(\frac{2}{h}\right)^2 \int_0^{h/2}\int_0^{h/2} \Delta^2_{s+t} f(x) dsdt,$$
$$f''_h(x) = h^{-2}[8\Delta^2_{h/2} f(x) - \Delta^2_h f(x)],$$

and hence

$$\|f - f_h\|_M \leq \mathfrak{w}^2_M(f,h), \quad \|f''_h\|_M \leq 9h^{-2}\mathfrak{w}^2_M(f,h) \tag{6.1}$$

Theorem 6.1. *Suppose that $S_{n,p,q}(f;x)$ is defined as in (2.12), where (p_n) and (q_n) are the sequences such that $p_n \to 1, \ \ q_n \to 1$ and $p_n^N \to a, \ \ q_n^{N'} \to b$ as n tending to infinity. Let $M \in \mathbb{N}^*$, then for $f \in C_M$,*

$$\mathfrak{w}_M|S_{n,p,q}(f;x) - f(x)| \leq N_m \mathfrak{w}^2_M\left(f; \sqrt{\frac{x}{[n]_{p,q}}}\right), N_m > 0. \tag{6.2}$$

Proof. For $M = 0$, the result comes from Theorem 4.2. For $f \in C_M, \ \ M \in \mathbb{N}$,

$$\mathfrak{w}_M|S_{n,p,q}(f;x) - f(x)| \leq \mathfrak{w}_M|S_{n,p,q}(|f - f_h|;x)| + \mathfrak{w}_M|S_{n,p,q}(f_h;x) - f_h(x)|$$
$$+ \mathfrak{w}_M|f_h(x) - f(x)|.$$

From Theorem 4.1 and the first property of Steklov means,

$$\mathfrak{w}_M|S_{n,p,q}(|f - f_h|;x)| \leq \|S_{n,p,q}(f - f_h)\|_M$$
$$\leq \|f - f_h\|_M \leq \mathfrak{w}^2_M(f,h). \tag{6.3}$$

Also, by Taylor's expansion

$$\mathfrak{w}_M|S_{n,p,q}(f_h;x) - f_h(x)| \leq \|f'_h\|_M S_{n,p,q}((t-x);x)$$
$$+ \frac{1}{2}\|f''_h\|_M S_{n,p,q}((t-x)^2;x).$$

From moments of operators $S_{n,p,q}$ and the second property of Steklov means

$$\mathfrak{w}_M|S_{n,p,q}(f_h;x) - f_h(x)| \leq \frac{9}{2h^2}\mathfrak{w}^2_M(f,h)S_{n,p,q}((t-x)^2;x)$$
$$\leq \frac{9}{2h^2}\mathfrak{w}^2_M(f,h)S_{n,p,q}((t-x)^2;x)$$
$$\leq \frac{9}{2h^2}\mathfrak{w}^2_M(f,h)\frac{x}{[n]_{p,q}}. \tag{6.4}$$

Setting $h = \sqrt{\frac{x}{[n]_{p,q}}}$ and from (6.1), (6.3), (6.4), we get

$$\mathfrak{w}_M |S_{n,p,q}(f;x) - f(x)| \leq N_m \mathfrak{w}_M^2 \left(f; \sqrt{\frac{x}{[n]_{p,q}}} \right).$$

□

7. Statistical convergence

In this section, we obtain statistical convergence for new modified $(p;q)$-Szász-Mirakyan operators. We need the following theorem [10] to prove statistical convergence of the operators on H' and we set all real valued functions on real-valued functions on $[0,\infty)$ with condition $|f(x) - f(y)| \leq \omega(|x-y|)$.

Theorem 7.1. *Let M_n be the sequence of positive linear operators from H' into $C_b[0,\infty)$ with three conditions*

$$st - \lim_{n\to\infty} \|M_n(t^j;x) - x^j\|_{C_b} = 0, \text{ j=0,1,2.}$$

Then

$$st - \lim_{n\to\infty} \|M_n(f;x) - f\|_{C_b} = 0.$$

Now, the result on statistical convergence of the operators defined in (2.12).

Theorem 7.2. *Suppose that $S_{n,p,q}(f;x)$ is defined as in (2.12), where (p_n) and (q_n) are the sequences such that $p_n \to 1$, $q_n \to 1$ and $p_n^N \to a$, $q_n^{N'} \to b$ as n tending to infinity. Then*

$$st - \lim_{n\to\infty} \|S_{n,p,q}(f;x) - f\|_{C_b} = 0. \tag{7.1}$$

Proof. From above theorem, we only have to prove that

$$st - \lim_{n\to\infty} \|S_{n,p,q}(t^j;x) - x^j\|_{C_b} = 0, \ j = 0,1,2.$$

From the moments of $S_{n,p,q}(f;x)$, it is obvious that the result is true for $j = 0, 1$.

For $j = 2$,

$$\|S_{n,p,q}(t^2;x) - x^2\|_{C_b} \leq \frac{1}{[n]_{p,q}}.$$

But

$$st - \lim_{n\to\infty} \frac{1}{[n]_{p,q}} = 0.$$

We define

$$U = \{n : \|S_{n,p,q}(t^2;x) - x^2\|_{C_b} \geq \epsilon\}$$
$$U_1 = \{n : \frac{1}{[n]_{p,q}} \geq \epsilon\}.$$

Clearly, $U \subseteq U_1$.

Then

$$\delta\{k \leq n : \|S_{n,p,q}(t^2;x) - x^2\|_{C_b} \geq \epsilon\} \leq \delta\{k \leq n : \frac{1}{[n]_{p,q}} \geq \epsilon\}$$

But the right hand side of the above inequality is zero because $st - \lim\limits_{n\to\infty} \frac{1}{[n]_{p,q}} = 0$. Hence,

$$st - \lim_{n\to\infty} \|S_{n,p,q}(t^2;x) - x^2\|_{C_b} = 0.$$

The theorem is proved. □

8. Voronovskaya type result

Theorem 8.1. *Let (p_n) and (q_n) be the sequences such that $p_n \to 1$, $q_n \to 1$ and $p_n^N \to a$, $q_n^{N'} \to b$ as n tending to infinity then for each function f, f', $f'' \in C_\vartheta^*[0,\infty)$*

$$\lim_{n\to\infty} [n]_{p,q}[S_{n,p,q}(f;x) - f(x)] = \frac{x}{2} f''(x), \tag{8.1}$$

is uniformly convergent on $[0,a]$, $a > 0$.

Proof. Consider Taylor's formula on $f \in C_\vartheta^*[0,\infty)$

$$f(t) = f(x) + (t-x)f'x + \tfrac{1}{2}(t-x)^2 f''(x) + P(t,x)(t-x)^2,$$

where $P(t,x)$ is Peano's remainder, $P(t,x) \to 0$ as $t \to x$.

Now

$$\begin{aligned}[n]_{p,q}[S_{n,p,q}(f;x) - f(x)] &= [n]_{p,q} f'(x) S_{n,p,q}((t-x);x) \\ &\quad + [n]_{p,q} \frac{f''(x)}{2} S_{n,p,q}((t-x)^2;x) + [n]_{p,q} S_{n,p,q}(P(t,x)(t-x)^2;x).\end{aligned}$$

From Cauchy-Schwarz inequality,

$$S_{n,p,q}(P(t,x)(t-x)^2;x) \leq \sqrt{S_{n,p,q}((P^2(t,x);x)} \sqrt{S_{n,p,q}((t-x)^4;x)}$$

is satisfied. Since $P(t,x) \in C_\vartheta^*[0,\infty)$ and $P(x,x) = 0$,

$$\lim_{n\to\infty} S_{n,p,q}((P^2(t,x);x) = P^2(x,x) = 0,$$

uniformly convergent for $x \in [0,a]$. So

$$\lim_{n\to\infty} [n]_{p,q} S_{n,p,q}(P(t,x)(t-x)^2;x) = 0$$

is obtained. Also

$$\lim_{n\to\infty} [n]_{p,q} S_{n,p,q}((t-x);x) = 0.$$

and

$$\lim_{n\to\infty}[n]_{p,q}S_{n,p,q}((t-x)^2;x)=x.$$

Hence,

$$\lim_{n\to\infty}[n]_{p,q}[S_{n,p,q}(f;x)-f(x)]=\frac{x}{2}f''(x).$$

□

9. Numerical Examples

Example 9.1. *We compute the absolute error (A.E.) of $S_{n,p,q}$ with the function $f(x)=x^2-5x+10$ and $g(x)=x^3-x+1$ for the different values of n taking $x=1$ and $x=2$ in Table 1 and Table 2, respectively. Also, the absolute error can be seen graphically in the Figure 5 and 6.*

Table 1. A.E. of operators and function at $x=1$

n	$\lvert S_{n,p,q}f-f\rvert$	$\lvert S_{n,p,q}g-g\rvert$
10	0.1252	0.3914
20	0.0798	0.2458
30	0.0673	0.2064
40	0.0633	0.1940
50	0.0631	0.1934

Table 2. A.E. of operators and function at $x=2$

n	$\lvert S_{n,p,q}f-f\rvert$	$\lvert S_{n,p,q}g-g\rvert$
10	0.2505	1.5342
20	0.1596	0.9704
30	0.1346	0.8165
40	0.1267	0.7682
50	0.1263	0.7657

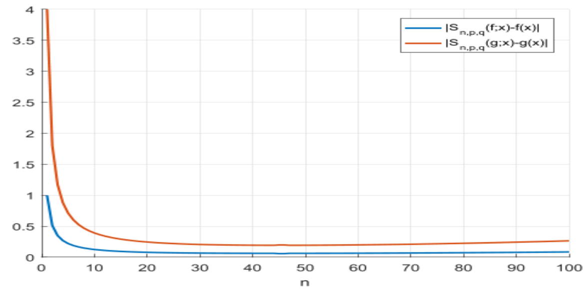

Figure 5. Absolute error of $S_{n,p,q}$ at $x = 1$.

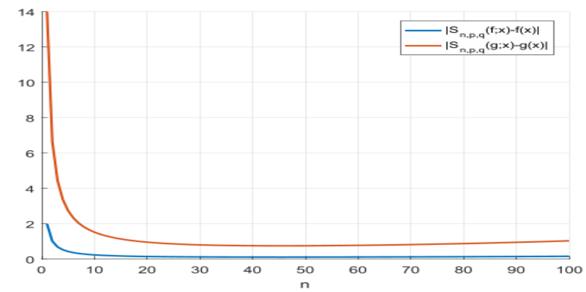

Figure 6. Absolute error of $S_{n,p,q}$ at $x = 2$.

10. Acknowledgements

The second author would like to express his gratitude to King Khalid University, Abha, Saudi Arabia, for providing administrative and technical support.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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