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DOWNHILL ZAGREB TOPOLOGICAL INDICES OF GRAPHS

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Abstract. Topological indices are graph invariants determined by the distance or degree of vertices of the molecular graph. Topological indices have been used effectively in chemical graph theory in explaining the structures and predicting certain physicochemical properties of chemical compounds. In this research, we introduce the first, second, and forgotten downhill Zagreb indices and calculate those topological indices for some standard families of graphs and the join of graphs. Also, the downhill topological indices for the firefly graph, book graph, and stacked book graph are established. Finally, the downhill indices of Graphene and honeycomb network are obtained.

1. Introduction

A graph is non empty set of vertices together with a number of edges connecting a subset of them. $V(G)$ and $E(G)$ denoted for vertex set and edge set respectively. If we consider the molecules as special chemical structures, and if we replace atoms and bonds with vertices and edges, respectively, the obtained graph is called a molecular graph. That means a molecular graph is a simple graph such that its vertices correspond to the atoms and its edges to the bonds. We note that hydrogen atoms are often omitted and the remaining part of the graph is sometimes called as the carbon graph of the corresponding molecule. Chemical graph theory which deals with the above mentioned relations between molecules and corresponding graphs is a branch of mathematical chemistry which has an important effect on the development of

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the molecular chemistry along with quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) studies.

In this research work, we are concerned with simple graphs which they are finite, undirected with no loops and multiple edges.

The open and closed neighborhoods of a vertex v in a graph G are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The degree of a vertex v in G is denoted by $deg_G(v)$ or briefly by $d_G(v)$, where $deg_G(v) = |N(v)|$. When there is no confusion, one can also omit G and use $d(v)$ instead of $d_G(v)$. The minimum degree and maximum degree of G are denoted by δ, and Δ respectively. In a graph G if $\delta = \Delta = k$, then the graph G is called regular graph of degree k. Also a graph with the property that $\Delta \leq 4$ is called a chemical graph.

The following notations and different types of graphs well known in the literature [9] and [2]. **A Double star** is the graph obtained from K_2 by joining s pendent edges to one end and r pendent edges to the other end of K_2. **A wheel** W_{n+1}, $n \geqslant 3$ is the join of C_n and K_1. **A helm graph**, denoted by H_n, is a graph obtained from W_{n+1} by attaching an end edge to each rim vertex of W_{n+1}, where the vertices corresponding to C_n are known as rim vertices. The **gear graph** is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges. The **sierpinski sieve graph** S_n is the graph obtained from the connectivity of the sierpinski sieve. The graph has $\frac{3(3^{n-1}+1)}{2}$ vertices and 3^n edges. The **tadpole graph** $T_{r,s}$ is obtained by joining a cycle C_r and a path P_s by a bridge, where $r \geqslant 3$ and $s \geqslant 1$. The **Cartesian product** G of two graphs G_1 and G_2, denoted $G_1 \times G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$, and two distinct vertices (a, b) and (c, d) of $G_1 \times G_2$ are adjacent if either $a = c$ and $bd \in E(G_2)$, or $b = d$ and $ac \in (G_1)$. The **join** $G = G_1 \vee G_2$ of two graphs G_1 and G_2 has vertex set $V(G) = V(G_1) \bigcup V(G_2)$ and edge set $E(G) = E(G_1) \bigcup E(G_2) \{uv : u \in V(G_1), v \in V(G_2)\}$. The **corona product** $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the ith-copy of H with the ith-vertex of G. **Book graph** is a Cartesian product of a star and single edge, denoted by B_m. The m-book graph is defined as the graph Cartesian product $S_{m+1} \times P_2$, where S_{m+1} is a star graph and P_2 is the path graph . The **stacked book graph** of order (m, n) is defined as the graph Cartesian product $S_{m+1} \times P_n$, where S_m is a star graph and P_n is the path graph on n nodes, and it is denoted by $B_{m,n}$.

As it is known, the path, cycle, and complete graphs with n vertices are denoted by P_n, C_n, K_n; and the complete bipartite graph is denoted by $K_{r,s}$.

A topological index of a graph is a real number associated with chemical constitution purporting for correlation of chemical structure with various physical properties, chemical reactivity or biological activity. In recent decades, a large number of topological indices have been defined and utilized for chemical documentation, isomer discrimination, study of molecular complexity, chirality, similarity/dissimilarity, QSAR/QSPR, drug design and database selection, lead optimization, etc.

As an example, the boiling point of a molecule is directly related to the forces between the atoms. When a solution is heated, the temperature is increased and as it is increased, the kinetic energy between molecules increases. This means that the molecular motion becomes so intense that the bonds between molecules break and become a gas. The moment the liquid turns to gas is labeled as the boiling point. The boiling point can give important clues about the physical properties of chemical structures. Molecules which strongly interact or bond with each other through a variety of inter-molecular forces cannot move easily or rapidly and therefore, do not achieve the kinetic energy necessary to escape the liquid state. That is why the boiling points of the alkanes increase with molecular size.

The most useful and famous topological indices of a graph are the first and second Zagreb indices which have been introduced by Gutman and Trinajstic in [7]. They are denoted by $M_1(G)$ and $M_2(G)$ and were defined as

$$M_1(G) = \sum_{u \in V(G)} \big[d(u)\big]^2$$

and

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v),$$

respectively. The forgotten topological index was introduced by Furtula and Gutman [4] and it is stated that the forgotten topological index is a special case of first Zagreb index. Shehnaz Akhter, Muhammad Imran extended the study of forgotten topological index and determined the close formula of F-index for some graph operations. The Zagreb indices have been studied extensively. Many new reformulated and extended versions of the Zagreb indices have been introduced. For more discussion on these indices, we encourage the readers to consult the articles ([1], [3], [5], [6], [11], [12], [13], [14], [16], [17], [18], [19]).

The downhill domination is introduced for the first time in [10]. For more details about the downhill and uphill domination, we refer to [8, 15].

In this research work, motivated by the downhill domination and the huge applications of topological indices, we define the downhill degree and introduce the first, second and forgotten downhill Zagreb indices and calculate these topological indices for, some standard families of graphs, join of two graph. Also, the downhill topological indices for firefly graph, book graph Stacked book graph are established. Finally, the downhill indices of Graphene and Honeycomb Network are obtained.

2. Downhill Zagreb Indices

In this section, we define the first, second and forgotten downhill Zagreb indices and calculate these topological indices for some standard families of graphs.

Definition 2.1. *[10] Let $G = (V, E)$ be a graph. A $u-v$ path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, ...v_{k+1}$ in G is a downhill path if for every i,$1 \leqslant i \leqslant k$, $deg(v_i) \geqslant deg(v_{i+1})$.*

Definition 2.2. *In a graph $G = (V, E)$. A vertex v is downhill dominates a vertex u if there exists a downhill path originated from v to u. The downhill neighborhood of a vertex v is denoted by $N_{dn}(v)$ and defined as: $N_{dn}(v) = \{u : v \text{ downhill dominates} \, u\}$. The downhill degree of the vertex v, denotes by $d_{dn}(v)$, is the number of downhill neighbors of v, that means $d_{dn}(v) = |N_{dn}(v)|$.*
The downhill closed neighborhood, $N_{dn}[v]$, of a vertex v is the downhill open neighborhood of v taken together with v. It follows that $N_{dn}[v] = N_{dn}(v) \cup \{v\}$.

Definition 2.3. *Let $G = (V, E)$ be a graph. Then the first, second and forgotten downhill Zagreb indices are defined as:*

$$DWM_1(G) = \sum_{v \in V(G)} (d_{dn}(v))^2,$$

$$DWM_2(G) = \sum_{vu \in E(G)} d_{dn}(v) d_{dn}(u)$$

and

$$DWF(G) = \sum_{v \in V(G)} (d_{dn}(v))^3.$$

Example 2.1. *Let G be a house graph as in Figure 1. Then*

$$DWM_1(G) = \sum_{v \in V(G)} (d_{dn}(v))^2 = 0 + 25 + 25 + 25 + 1 + 1 = 77,$$

$$DWM_2(G) = \sum_{vu \in E(G)} d_{dn}(v) d_{dn}(u) = 0 \times 5 + 3 \times 5 \times 5 + 2 \times 5 \times 1 + 1 \times 1 = 86,$$

$$DWF(G) = \sum_{v \in V(G)} (d_{dn}(v))^3 = 0 + 125 + 125 + 125 + 1 + 1 = 377.$$

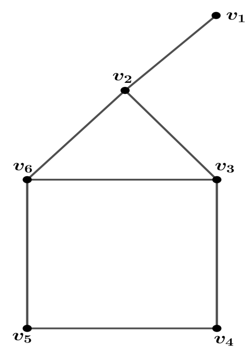

Figure 1. The House Graph.

Proposition 2.1. *Let G be the connected k-regular graph of n vertices. Then,*

$$\begin{aligned} DWM_1(G) &= n(n-1)^2, \\ DWM_2(G) &= \frac{nk(n-1)^2}{2}, \\ DWF(G) &= n(n-1)^3. \end{aligned}$$

Proof. Let G be the connected k-regular graph of n vertices. It is easy to see that for any vertex in G has downhill degree $n-1$. Hence, $DWM_1(G) = n(n-1)^2$ and $DWF(G) = n(n-1)^3$. Similarly, there are $\frac{nk}{2}$ edges in G, where each edge has endpoints of downhill degree $n-1$. Hence, $DWM_2(G) = \frac{nk(n-1)^2}{2}$. □

Corollary 2.1.

(1) *For any cycle C_n, $DWM_1(C_n) = DWM_2(C_n) = n(n-1)^2$ and $DWF(C_n) = n(n-1)^3$.*
(2) *For any complete graph K_n, $DWM_1(K_n) = n(n-1)^2$, $DWM_2(K_n) = \frac{n(n-1)^3}{2}$ and $DWF(K_n) = n(n-1)^3$.*

Proposition 2.2. *Let $G \cong P_n$ be a path of $n \geq 3$ vertices. Then*

$$\begin{aligned} DWM_1(G) &= (n-2)(n-1)^2, \\ DWM_2(G) &= (n-3)(n-1)^2, \\ DWF(G) &= (n-2)(n-1)^3. \end{aligned}$$

Proof. Let $G \cong P_n$ be a path of n vertices, where $n \geq 3$, the path has $n-2$ vertices of downhill degree $n-1$ and 2 vertices of downhill degree 0. Hence, $DWM_1(G) = (n-2)(n-1)^2$ and $DWF(G) = (n-2)(n-1)^3$. There are $n-1$ edges in G of which $n-3$ edges have endpoints of downhill degree $n-1$ and 2 edges have one endpoint of downhill degree $n-1$ and the another endpoint of downhill degree 0. Hence, $DWM_2(G) = (n-3)(n-1)^2$. □

Proposition 2.3. *Let $G \cong S_n$ be a star with $n+1$ vertices, where $n \geq 2$. Then,*

$$\begin{aligned} DWM_1(G) &= n^2, \\ DWM_2(G) &= 0, \\ DWF(G) &= n^3. \end{aligned}$$

Proof. Let $G \cong S_n$ be a star with $n+1$ vertices, where $n \geq 2$, clearly there are $n+1$ vertices of downhill degree n and n vertices of downhill degree 0. Hence,$DWM_1(G) = n^2$ and $DWF(G) = n^3$. The star graph G has n edges, where each edge has one endpoint of downhill degree n and the another endpoint of downhill degree 0. Hence, $DWM_2(G) = 0$. □

Proposition 2.4. *Let $G \cong S_{s,r}$ be a double star with $s+r+2$ vertices, where $s, r \geq 2$. Then,*

$$DWM_1(G) = \begin{cases} 2(s+r+1)^2 & if \quad s = r; \\ (s+r+1)^2 + r^2 & if \quad s > r. \end{cases}$$

Proof. Let $G \cong S_{s,r}$ be a double star with $s+r+2$ vertices, where $s, r \geq 2$. We have two cases:

Case 1. If $s = r$. Then the graph has two vertices of downhill degree $s+r+1$ and $s+r$ vertices of downhill degree 0. Hence, $DWM_1(G) = 2(s+r+1)^2$.

Case 2. If $s > r$, then there are one vertex of downhill degree $s+r+1$, one vertex of downhill degree r and $s+r$ vertices of downhill degree 0. Hence, $DWM_1(G) = (s+r+1)^2 + r^2$. □

Proposition 2.5. *Let $G \cong S_{s,r}$ be a double star with $s+r+2$ vertices, where $s, r \geq 2$. Then,*

$$DWM_2(G) = \begin{cases} (s+r+1)^2 & if \quad s = r; \\ r(s+r+1) & if \quad s > r. \end{cases}$$

Proof. Let $G \cong S_{s,r}$ be a double star with $s+r+2$ vertices and $s+r+1$ edges, where $s, r \geq 2$. We have two cases:

Case 1. If $s = r$. It has one edge has endpoints of downhill degree $s+r+1$. Also, there are $s+r$ edges in G, where each edge has one endpoint of downhill degree $s+r+1$ and the another endpoint of downhill degree 0. Hence, $DWM_2(G) = (s+r+1)^2$.

Case 2. If $s > r$. It has one edge has one endpoint of downhill degree $s+r+1$ and the another endpoint of downhill degree r. Also, there are $s+r$ edges where each edge has endpoint of downhill degree $s+r+1$ or r and the another endpoint of downhill degree 0. Hence, $DWM_2(G) = r(s+r+1)$. □

In the same way we can get the following result.

Proposition 2.6. *Let $G \cong S_{s,r}$ be a double star with $s+r+2$ vertices, where $s, r \geq 2$. Then,*

$$DWF(G) = \begin{cases} 2(s+r+1)^3 & if \quad s = r; \\ (s+r+1)^3 + r^3 & if \quad s > r. \end{cases}$$

Proposition 2.7. *Let $G \cong K_{s,r}$ be the complete bipartite graph, where $s < r$. Then,*

$$\begin{aligned} DWM_1(G) &= sr^2, \\ DWM_2(G) &= 0, \\ DWF(G) &= sr^3. \end{aligned}$$

Proof. Let $G \cong K_{s,r}$ be the complete bipartite graph, where $s < r$. There are $s+r$ vertices of which s vertices of downhill degree r and r vertices of downhill degree 0. Hence, $DWM_1(G) = sr^2$ and $DWF(G) = sr^3$. There are sr edges in G, where each edge has one endpoint of downhill degree r and the another endpoint of downhill degree 0. Hence, $DWM_2(G) = 0$. □

Proposition 2.8. *Let $G \cong W_n$ be a wheel graph of $n+1$ vertices, where $n \geq 4$. Then,*

$$\begin{aligned} DWM_1(G) &= n(n^2 - n + 1), \\ DWM_2(G) &= n(2n^2 - 3n + 1), \\ DWF(G) &= n(n^3 - 2n^2 + 3n - 1). \end{aligned}$$

Proof. Let $G \cong W_n$ be a wheel graph of $n+1$ vertices, where $n \geq 4$. There is one vertex of downhill degree n and n vertices of downhill degree $n-1$. Hence, $DWM_1(G) = n(n^2 - n + 1)$ and $DWF(G) = n(n^3 - 2n^2 + 3n - 1)$. There are $2n$ edges in G of which n edges have one endpoint of downhill degree n and the another endpoint of downhill degree $n-1$ and n edges have endpoints of downhill degree $n-1$. Hence, $DWM_2(G) = n(2n^2 - 3n + 1)$. □

Proposition 2.9. *Let $G \cong G_n$ be a gear graph of $2n+1$ vertices, where $n \geq 4$. Then,*

$$\begin{aligned} DWM_1(G) &= 4n(n+1), \\ DWM_2(G) &= 4n^2, \\ DWF(G) &= 8n(n^2 + 1). \end{aligned}$$

Proof. Let $G \cong G_n$ be a gear of $2n+1$ vertices, where $n \geq 4$. Then the graph has one vertex of downhill degree $2n$, n vertices of downhill degree 2 and n vertices of downhill degree 0. Hence, $DWM_1(G) = 4n(n+1)$ and $DWF(G) = 8n(n^2+1)$. There are $3n$ edges in G of which n edges have one endpoint of downhill degree $2n$ and the another endpoint of downhill degree 2 and $2n$ edges have one endpoint of downhill degree 2 and the another endpoint of downhill degree 0. Hence, $DWM_2(G) = 4n^2$. □

Proposition 2.10. *Let $G \cong H_n$ be a helm graph of $2n+1$ vertices, where $n \geq 5$. Then,*

$$\begin{aligned} DWM_1(G) &= n(4n^2+1), \\ DWM_2(G) &= n(8n^2-6n+1), \\ DWF(G) &= n(8n^3-4n^2+6n-1). \end{aligned}$$

Proof. Let $G \cong H_n$ be a helm graph of $2n+1$ vertices, where $n \geq 5$. Then there are one vertex of downhill degree $2n$, n vertices of downhill degree $2n-1$ and n vertices of downhill degree 0. Hence, $DWM_1(G) = n(4n^2+1)$ and $DWF(G) = n(8n^3-4n^2+6n-1)$. There are $3n$ edges of which n edges have one endpoint of downhill degree $2n$ and the another endpoint of downhill degree $2n-1$, n edges have endpoints of downhill degree $2n-1$ and n edges have one endpoint of downhill degree $2n-1$ and the another one of downhill degree 0. Hence, $DWM_2(G) = n(8n^2-6n+1)$. □

Proposition 2.11. *Let $G \cong S_n$ be the sierpinski of m vertices, where $m = \frac{3(3^{n-1}+1)}{2}$. Then,*

$$\begin{aligned} DWM_1(G) &= (m-3)(m-1)^2, \\ DWM_2(G) &= (2m-9)(m-1)^2, \\ DWF(G) &= (m-3)(m-1)^3. \end{aligned}$$

Proof. Let $G \cong S_n$ be the sierpinski graph of m vertices, where $m = \frac{3(3^{n-1}+1)}{2}$. There are $m-3$ vertices of downhill degree $m-1$ and 3 vertices of downhill degree 0. Hence, $DWM_1 = (m-3)(m-1)^2$ and $DWF(G) = (m-3)(m-1)^3$. There are $2m-3$ edges of which $2m-9$ has endpoints of downhill degree $m-1$ and 6 edges have one endpoint of downhill degree $m-1$ and the another endpoint of downhill degree 0. Hence, $DWM_2(G) = (2m-9)(m-1)^2$. □

Theorem 2.1. *Let $G \cong T_{n,m}$ be the tadpole graph with $n+m$ vertices, where $n, m \geq 3$. Then,*

$$\begin{aligned} DWM_1(G) &= (n+m-1)^2 + (n-1)(n-2)^2 + (m-1)^3, \\ DWM_2(G) &= (n+m-1)(2n+m-5) + (n-2)^3 + (m-2)(m-1)^2, \\ DWF(G) &= (n+m-1)^3 + (n-1)(n-2)^3 + (m-1)^4. \end{aligned}$$

Proof. Let $G \cong T_{n,m}$ be the tadpole graph with $n+m$ vertices. Then the graph has one vertex of downhill degree $n+m-1$, $n-1$ vertices of downhill degree $n-2$, $m-1$ vertices of downhill degree $m-1$ and one vertex of downhill degree 0. Hence,

$$\begin{aligned} DWM_1(G) &= (n+m-1)^2 + (n-1)(n-2)^2 + (m-1)^3, \\ DWF(G) &= (n+m-1)^3 + (n-1)(n-2)^3 + (m-1)^4. \end{aligned}$$

There are $m+n$ edges of which one edge has one endpoint of downhill degree $n+m-1$ and the another endpoint of downhill degree $m-1$, 2 edges have one endpoint of downhill degree $n+m-1$ and the another endpoint $n-2$, $n-2$ edges have endpoints of downhill degree $n-2$, $m-2$ edges have endpoints of downhill degree $m-1$ and one edge has one endpoint of downhill degree $m-1$ and another endpoint of downhill degree 0. Hence,

$$DWM_2(G) = (n+m-1)(2n+m-5) + (n-2)^3 + (m-2)(m-1)^2.$$

□

3. The Downhill Zagreb Indices of Graphs under Some Binary Operations

Theorem 3.1. *Let $G \cong C_n \vee P_m$ be the join graph with $n+m$ vertices, where $n, m \geq 3$. Then,*

$$DWM_1(G) = \begin{cases} 2(n-1)(2n-1)^2 & if \quad n = m; \\ (m-2)(n+m-1)^2 + (n+2)(n+1)^2 & if \quad n = m+1; \\ (m-2)(n+m-1)^2 + 2n^2 + n(n-1)^2 & if \quad n > m+1; \\ n(n+m-1)^2 + (m-2)(m-1)^2 & if \quad n < m. \end{cases}$$

Proof. Let $G \cong C_n \vee P_m$ be the join graph with $n+m$ vertices, where $m \geq 3$. We have four cases:

Case 1. If $n = m$. In this case, there are $2n-2$ vertices of downhill degree $2n-1$ and 2 vertices of downhill degree 0. Hence,

$$DWM_1(G) = 2(n-1)(2n-1)^2.$$

Case 2. If $n = m+1$. In this case, there are $m-2$ vertices of downhill degree $n+m-1$ and $n+2$ vertices of downhill degree $n+1$. Hence,

$$DWM_1(G) = (m-2)(n+m-1)^2 + (n+2)(n+1)^2.$$

Case 3. If $n > m+1$. In this case, there are $m-2$ vertices of downhill degree $n+m-1$, 2 vertices of downhill degree n and n vertices of downhill degree $n-1$. Hence,

$$DWM_1(G) = (m-2)(n+m-1)^2 + 2n^2 + n(n-1)^2.$$

Case 4. If $n < m$. In this case, there are n vertices of downhill degree $n+m-1$, $m-2$ vertices of downhill degree $m-1$ and 2 vertices of downhill degree 0. Hence,

$$DWM_1(G) = n(n+m-1)^2 + (m-2)(m-1)^2.$$

□

Theorem 3.2. *Let $G \cong C_n \vee P_m$ be the join graph with $n+m$ vertices, where $n, m \geq 3$. Then,*

$$DWM_2(G) = \begin{cases} (n^2-3)(2n-1)^2 & if \quad n = m; \\ A & if \quad n = m+1; \\ B & if \quad n > m+1; \\ C & if \quad n < m, \end{cases}$$

where, $A = (m-3)(n+m-1)^2 + 3n(n+1)^2 + (nm-2n+2)(n+m-1)(n+1)$,
$B = (m-3)(n+m-1)^2 + n(n+m-1)(nm-m-2n+4) + n(n-1)(3n-1)$
and $C = n(n+m-1)^2 + (m-3)(m-1)^2 + n(m-2)(m-1)(n+m-1)$.

Proof. Let $G \cong C_n \vee P_m$ be the join graph with $n+m$ vertices, where $n, m \geq 3$. Clearly the graph G has $nm+n+m-1$ edges and we have four cases:

Case 1. If $n = m$. In this case, there are n^2-3 edges have endpoints of downhill degree $2n-1$ and $2(n+1)$ edges have one endpoint of downhill degree $2n-1$ and the another endpoint of downhil degree 0. Hence,

$$DWM_2(G) = (n^2-3)(2n-1)^2.$$

Case 2. If $n = m+1$. In this case, there are $m-3$ edges have endpoints of downhill degree $n+m-1$, $3n$ edges have endpoints of downhill degree $n+1$ and $nm-2n+2$ edges have one endpoint of downhill degree $n+m-1$ and the another endpoint of downhill degree $n+1$. Hence,

$$DWM_2(G) = (m-3)(n+m-1)^2 + 3n(n+1)^2 + (nm-2n+2)(n+m-1)(n+1).$$

Case 3. If $n > m+1$. In this case, there are $m-3$ edges have endpoints of downhill degree $n+m-1$, n edges have endpoints of downhill degree $n-1$, $nm-2n$ edges have one endpoint of downhill degree $n+m-1$ and the another endpoint of downhill degree $n-1$, $2n$ edges have one endpoint of downhill degree n and the another endpoint of downhill degree $n-1$ and 2 edges have one endpoint of downhill degree $n+m-1$ and the another endpoint of downhill degree n. Hence,

$$DWM_2(G) = (m-3)(n+m-1)^2 + n(n+m-1)(nm-m-2n+4) + n(n-1)(3n-1).$$

Case 4. If $n < m$. In this case, there are n edges have endpoints of downhill degree $n+m-1$, $m-3$ have endpoints downhill degree $m-1$, $n(m-2)$ edges have one endpoint of downhill degree $n+m-1$ and the another endpoint of downhill degree $m-1$ and $2(n+1)$ edges have one endpoint of downhill degree $n+m-1$ or $m-1$ and the another endpoint of downhill degree 0. Hence,

$$DWM_2(G) = n(n+m-1)^2 + (m-3)(m-1)^2 + n(m-2)(m-1)(n+m-1).$$

□

Theorem 3.3. *Let $G \cong C_n \vee P_m$ be the join graph with $n+m$ vertices, where $n, m \geq 3$. Then,*

$$DWF(G) = \begin{cases} 2(n-1)(2n-1)^3 & if \quad n = m; \\ (m-2)(n+m-1)^3 + (n+2)(n+1)^3 & if \quad n = m+1; \\ (m-2)(n+m-1)^3 + 2n^3 + n(n-1)^3 & if \quad n > m+1; \\ n(n+m-1)^3 + (m-2)(m-1)^3 & if \quad n < m. \end{cases}$$

Proof. The proof similar to the proof of Theorem 3.1. □

Proposition 3.1. *Let $G \cong B_m$ be a book graph of $2(m+1)$ vertices, where $m \geq 2$. Then,*

$$\begin{aligned} DWM_1(G) &= 2(4m^2 + 5m + 1), \\ DWM_2(G) &= m(8m + 7) + 1, \\ DWF(G) &= 16m^3 + 24m^2 + 14m + 2. \end{aligned}$$

Proof. Let $G \cong B_m$ be a book of $2(m+1)$ vertices, where $m \geq 2$. Then there are 2 vertices of downhill degree $2m+1$ and $2m$ vertices of downhill degree 1. Hence,

$$\begin{aligned} DWM_1(G) &= 2(4m^2 + 5m + 1), \\ DWF(G) &= 16m^3 + 24m^2 + 14m + 2. \end{aligned}$$

There are $3m+1$ edges in G of which one edges has endpoints of downhill degree $2m+1$, m edges have endpoints of downhill degree 1 and $2m$ edges have one endpoint of downhill degree $2m+1$ and the another endpoint of downhill degree 1. Hence,

$$DWM_2(G) = m(8m + 7) + 1.$$

□

Theorem 3.4. *Let $G \cong B_{m,t}$ be a stacked book graph with $t(m+1)$ vertices, where $m \geq 2$ and $t \geq 3$. Then,*

$$\begin{aligned} DWM_1(G) &= (t-2)(t(m+1)-1)^2 + m(t-2)(t-1)^2 + 2m^2, \\ DWM_2(G) &= (t(m+1)-1)((t-3)(t(m+1)-1) + m(t^2-3t+4)) + m(t-3)(t-1)^2, \\ DWF(G) &= (t-2)(t(m+1)-1)^3 + m(t-2)(t-1)^3 + 2m^3. \end{aligned}$$

Proof. Let $G \cong B_{m,t}$ be a stacked book graph with $t(m+1)$ vertices, where $m \geq 2$ and $t \geq 3$. There are $t-2$ vertices of downhill degree $t(m+1)-1$, $m(t-2)$ vertices of downhill degree $t-1$, 2 vertices of downhill degree m and $2m$ vertices of downhill degree 0. Hence,

$$\begin{aligned} DWM_1(G) &= (t-2)(t(m+1)-1)^2 + m(t-2)(t-1)^2 + 2m^2, \\ DWF(G) &= (t-2)(t(m+1)-1)^3 + m(t-2)(t-1)^3 + 2m^3. \end{aligned}$$

Suppose that $E_{a,b} = \{uv \in E(G) : d_{dn}(u) = a \text{ and } d_{dn}(v) = b\}$. The stacked book graph contains 6 types of edges $E_{0,t-1}, E_{0,m}, E_{t(m+1)-1,m}, E_{t(m+1)-1,t(m+1)-1}, E_{t-1,t-1}$ and $E_{t-1,t(m+1)-1}$ edges. In the Figure 2, the types of edges, $E_{0,t-1}, E_{0,m}, E_{t(m+1)-1,m}$, $E_{t(m+1)-1,t(m+1)-1}, E_{t-1,t-1}$ and $E_{t-1,t(m+1)-1}$ are colored in red, blue, green, yellow, pink and black, respectively.

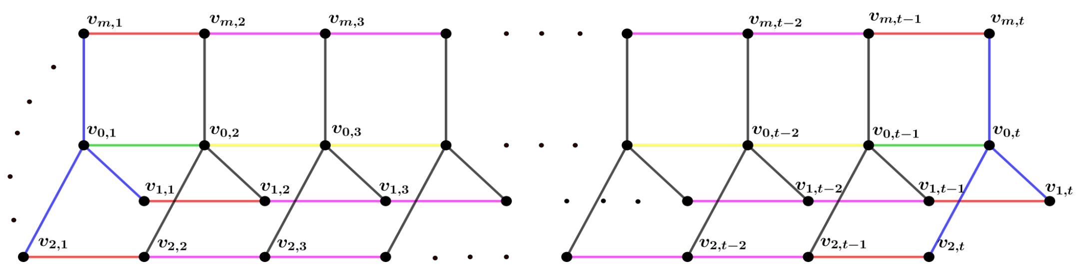

FIGURE 2. Stacked book graph $B_{m,t}$.

Table 1 gives the number of edges in each type.

Type	Number of edges
$E_{0,t-1}$	$2m$
$E_{0,m}$	$2m$
$E_{m,t(m+1)-1}$	2
$E_{t(m+1)-1,t(m+1)-1}$	$t-3$
$E_{t-1,t-1}$	$m(t-3)$
$E_{t-1,t(m+1)-1}$	$m(t-2)$

TABLE 1. The number of edges in the different types of edges of stacked book graph.

Thus, we get

$$\begin{aligned} DWM_2(G) &= m(t-3)(t-1)^2 + (t-3)(t(m+1)-1)^2 + m(t-2)(t-1)(t(m+1)-1) \\ &+ 2m(t(m+1)-1) \\ &= (t(m+1)-1)((t-3) + m(t-2)(t-1) + 2m) + m(t-3)(t-1)^2. \end{aligned}$$

Hence,

$$DWM_2(G) = (t(m+1)-1)((t-3)(t(m+1)-1)+m(t^2-3t+4))+m(t-3)(t-1)^2.$$

□

Proposition 3.2. *Let $G \cong F_{a,b,c}$ be the firefly graph with $2a+2b+c+1$ vertices. Then,*

$$\begin{aligned} DWM_1(G) &= (2a+2b+c)^2+2a+b, \\ DWM_2(G) &= (2a+2b+c)(2a+b)+a, \\ DWF(G) &= (2a+2b+c)^3+2a+b. \end{aligned}$$

Proof. Let $G \cong F_{a,b,c}$ be the firefly graph with $2a+2b+c+1$ vertices. It has one vertex of downhill degree $2a+2b+c$, $2a+b$ vertices of downhill degree 1 and $b+c$ vertices of downhill degree 0. Hence,

$$\begin{aligned} DWM_1(G) &= (2a+2b+c)^2+2a+b, \\ DWF(G) &= (2a+2b+c)^3+2a+b. \end{aligned}$$

The firefly graph contains 4 types of edges $E_{0,1}, E_{1,1}, E_{2a+2b+c,0}$ and $E_{2a+2b+c,1}$ edges. In the Figure 3, the different types of edges $E_{0,1}, E_{1,1}, E_{2a+2b+c,0}$ and $E_{2a+2b+c,1}$ are colored in blue, red, green and black, respectively.

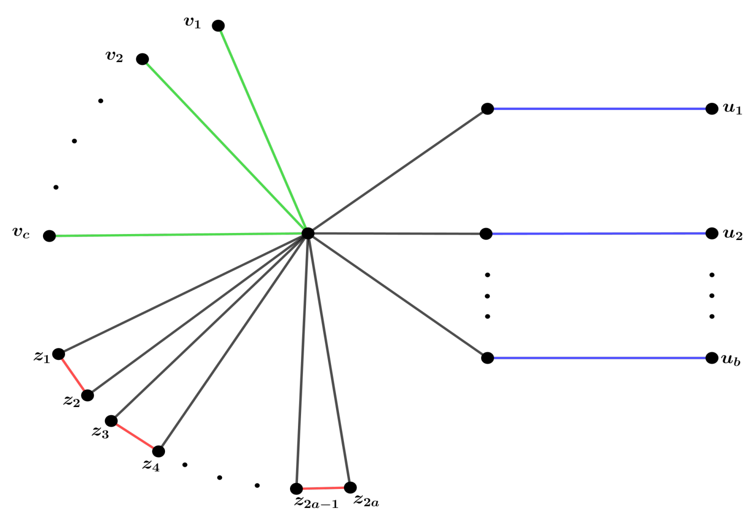

FIGURE 3. Firefly graph $F_{a,b,c}$.

Table 2, gives the number of edges in each type.

Type	Number of edges
$E_{0,1}$	b
$E_{1,1}$	a
$E_{0,2a+2b+c}$	c
$E_{1,2a+2b+c}$	$2a+b$

Table 2. The number of edges in the different types of edges of firefly graph.

Hence,

$$DWM_2(G) = (2a + 2b + c)(2a + b) + a.$$

□

4. The Downhill Zagreb Indices of Honeycomb Network and Graphene

In this section, we obtain exact values for first, second and forgotten downhill indices for honeycomb network and Graphene.

Honeycomb Network

The honeycomb network very much important in computer graphics, cellular base stations, image processing and representation of benzene hydrocarbons in chemistry. The recursive use hexagonal tiling in a particular pattern, honeycomb networks are formed.

Definition 4.1. *The honeycomb network $HC(1)$ is a hexagon and $HC(2)$ is obtain by dding 6 hexagons to the boundary edges of $HC(1)$. The honeycomb network $HC(n)$ is obtained from $HC(n-1)$ by adding a layer of hexagons around the boundary of $HC(n-1)$. The number of vertices and edges of $HC(n)$ are $6n^2$ and $9n^2 - 3n$ respectively.*

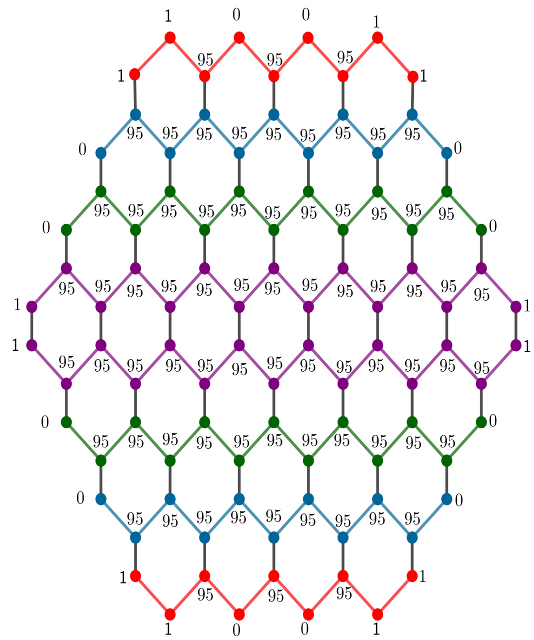

Figure 4. Honeycomb network $HC(4)$ with downhill degree of the vertices.

Theorem 4.1. *Let $G \cong HC(n)$ be a honeycomb network of dimension n, where $n \geq 3$. Then,*

$$\begin{aligned} DWM_1(G) &= (6n^2-6n)(6n^2-1)^2+12, \\ DWM_2(G) &= (9n^2-15n+6)(6n^2-1)^2+72n^2-6, \\ DWF(G) &= (6n^2-6n)(6n^2-1)^3+12. \end{aligned}$$

Proof. Let $G \cong HC(n)$ be a honeycomb network of dimension n, where $n \geq 3$. There are $2n$ lines in G as in Figure 4. By labeling the lines from up to down $L_1, L_2, ..., L_{2n}$, it is clear to see that, L_1 symmetric with L_{2n}. L_2 is symmetric with L_{2n-1} ... L_n is symmetric with L_{n+1}. The first line L_1 has $2n+1$ vertices in which 4 vertices are of downhill degree 1, $n-2$ vertices of downhill degree 0 and $n-1$ vertices of downhill degree $6n^2-1$. The line L_n has $4n-1$ vertices in which 2 vertices of downhill degree 1 and $4n-3$ vertices of downhill degree $6n^n-1$. Any line between L_1 and L_n has two vertices of downhill degree 0 and the others of downhill degree $6n^2-1$. The number of vertices of downhill degree 1 is 12, the number of vertices of downhill degree 0 is $6(n-2)$. Therefore, the number of vertices of downhill degree $6n^2-1$ is $6n^2-6n$. Hence,

$$\begin{aligned} DWM_1(G) &= (6n^2-6n)(6n^2-1)^2+12, \\ DWF(G) &= (6n^2-6n)(6n^2-1)^3+12. \end{aligned}$$

In a honeycomb network there are four types of edges based on the downhill degree of the vertices of each edge. The following table gives the four types and gives the number of edges in each type.

Type	Number of edges
$E_{1,1}$	6
$E_{1,6n^2-1}$	12
$E_{0,6n^2-1}$	$12(n-2)$
$E_{6n^2-1,6n^2-1}$	$9n^2-15n+6$

Thus, we get

$$\begin{aligned}
DWM_2(G) &= \sum_{uv\in E(G)} d_{dn}(u)d_{dn}(v)\\
&= |E_{1,1}|(1)(1)+|E_{1,6n^2-1}|(1)(6n^2-1)+|E_{0,6n^2-1}|(0)(6n^2-1)\\
&+ |E_{6n^2-1,6n^2-1}|(6n^2-1)(6n^2-1)\\
&= 6+12(6n^2-1)+(9n^2-15n+6)(6n^2-1)^2\\
&= (9n^2-15n+6)(6n^2-1)^2+72n^2-12+6.
\end{aligned}$$

Hence,

$$DWM_2(G) = (9n^2-15n+6)(6n^2-1)^2+72n^2-6.$$

□

Graphene

Graphene is a single layer of carbon atoms which are tightly bound in a hexagonal honeycomb lattice. Graphene is 200 times stronger than steel, one million times thinner than a human hair and word's most conductive material and such properties attracted researchers and scientists to study more about Graphene. Graphene has unique properties which unlocks various applications from electronics to optics, sensors, and bio-devices.

Theorem 4.2. *Let $G \cong G_{t,s}$ be the graph of Graphene with t rows of benzene rings and s benzene rings in each row. The first downhill Zagreb index is given by*

$$DWM_1(G) = \begin{cases} 150 & if \quad t=1, s=1;\\ 2(t-1)(4t+1)^2+2(t+34) & if \quad t>1, s=1;\\ 2(s-1)(4s+1)^2+72 & if \quad t=1, s>1;\\ 2(st-1)(2st+2s+2t-1)^2+2(t+12) & if \quad t>1, s>1. \end{cases}$$

Proof. Let $G \cong G_{t,s}$ be the graph of Graphene with t rows of benzene rings and s benzene rings in each row. Let $v_1, v_2, ..., v_{2s+1}$ be the vertices of the first line L_1. There are $t+1$ lines in which two lines with $2s+1$ vertices and the others lines have $2s+2$ vertices as in Figure 5. We have four cases:

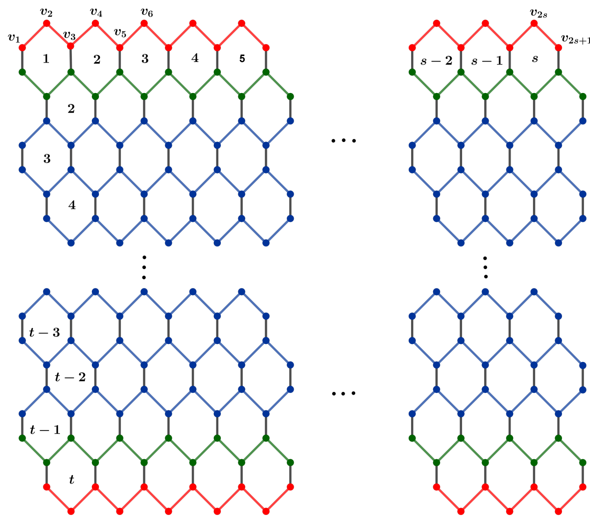

FIGURE 5. Graphene $G_{t,s}$.

Case 1. If $t = 1$ and $s = 1$. The graph G become cycle with 6 vertices and by Corollary 2.1, $DWM_1(G) = 6(5)^2 = 150$.

Case 2. If $t > 1$ and $s = 1$.The number of vertices in this case is $4t + 2$. The first line L_1 has 3 vertices of downhill degree 3. By symmetry, it can be seen that the line L_{t+1} has the same vertices as L_1. The second line L_2 has one vertex of downhill degree 3, one vertex of downhill degree 1 and 2 vertices of downhill degree $4t + 1$. By symmetry, the line L_t has the same vertices as L_2. Also, there are L_k lines, where $k = t - 3$, having 2 vertices of downhill degree 1 and 2 vertices of downhill degree $4t + 1$. Thus, we get

$$DWM_1(G) = 2(t-1)(4t+1)^2 + 2(t+34).$$

Case 3. If $t = 1$ and $s > 1$. In this case, there are only two lines and each line has $2s + 1$ vertices. The first line L_1 has 4 vertices of downhill degree 3, $s - 2$ vertices of downhill degree 0 and $s - 1$ vertices of downhill degree $4s + 1$. By symmetry, it can be seen that the line L_2 has the same vertices as L_1. Thus, we get

$$DWM_1(G) = 2(s-1)(4s+1)^2 + 72.$$

Case 4. If $t > 1$ and $s > 1$. The number of vertices in this case is $2st + 2s + 2t$. Let $n = 2st + 2s + 2t$. The first line L_1 has two vertices of downhill degree 2, two vertices of downhill degree 1, $s-2$ vertices of downhill degree 0 and $s - 1$ vertices of downhill degree $n - 1$. By symmetry, it can be seen that the line L_{t+1} has the same vertices as L_1. The second line L_2 has one vertex of downhill degree 2, one vertex of downhill degree 1 and $2s$ vertices of downhill degree $n-1$. By symmetry, the line L_t has the same vertices as L_2. Also, there are $k = t-3$ lines, each line have 2 vertices of downhill degree 1 and $2s$ vertices of downhill degree $n-1$. Thus, the first downhill Zagreb index of a Graphene is given by,

$$DWM_1(G) = 2(st-1)(2st+2s+2t-1)^2 + 2(t+12).$$

□

Theorem 4.3. *Let $G \cong G_{t,s}$ be a graph of Graphene with t rows of benzene rings and s benzene rings in each row. The second downhill Zagreb index is given by*

$$DWM_2(G) = \begin{cases} 150 & if \quad t=1, s=1; \\ (4t+1)(8t^2-8t+5)+t+52 & if \quad t>1, s=1; \\ (4s+1)(4s^2-3s+11)+54 & if \quad t=1, s>1; \\ \alpha & if \quad t>1, s>1, \end{cases}$$

where $\alpha = (2st+2s+2t-1)\left((s(3t-2)-(1+t))(2st+2s+2t-1)+2(t+4)\right)+t+16.$

Proof. Let $G \cong G_{t,s}$ be the graph of Graphene with t rows of benzene rings and s benzene rings in each row. So the graph has $2st+2s+2t$ vertices and $3st+2s+2t-1$ edges. We have four cases:

Case 1. If $t=1$ and $s=1$. The graph G become cycle with 6 vertices and by Corollary 2.1, $DWM_2(G) = 6(5)^2 = 150$.

Case 2. If $t>1$ and $s=1$. The number of vertices in this case is $4t+2$ and the number of edges is $5t+1$. Two dimensional structure of Graphene contains the following types of edges $E_{1,1}, E_{3,3}, E_{1,4t+1}, E_{3,4t+1}$ and $E_{4t+1,4t+1}$. In Figure 6 the edge types, $E_{1,1}, E_{3,3}, E_{1,4t+1}, E_{3,4t+1}$ and $E_{4t+1,4t+1}$ are colored in red, blue, green, yellow and black, respectively.

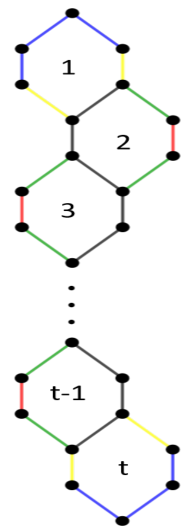

FIGURE 6. Graphene $G_{t,s}$ with $t > 1, s = 1$.

The number of edges in $E_{1,1}, E_{3,3}, E_{1,4t+1}, E_{3,4t+1}$ and $E_{4t+1,4t+1}$ in each row is mention in the following table.

Row	$\|E_{1,1}\|$	$\|E_{3,3}\|$	$\|E_{1,4t+1}\|$	$\|E_{3,4t+1}\|$	$\|E_{4t+1,4t+1}\|$
1	0	3	1	2	1
2	1	0	2	0	2
3	1	0	2	0	2
4	1	0	2	0	2
. . .	. . .	. . .	. . .	. . .	. . .
$t-1$	1	0	1	1	2
t	0	3	0	1	0
Total	$t-2$	6	$2+2(t-3)$	4	$1+2(t-2)$

Therefore, we have $|E_{1,1}| = t-2$ edges, $|E_{3,3}| = 6$ edges, $|E_{1,4t+1}| = 2t-4$ edges, $|E_{3,4t+1}| = 4$ edges and $|E_{4t+1,4t+1}| = 2t-3$ edges.

$$
\begin{aligned}
DWM_2(G) &= \sum_{uv\in E(G)} d_{dn}(u)d_{dn}(v)\\
&= |E_{1,1}|(1)(1)+|E_{3,3}|(3)(3)+|E_{1,4t+1}|(1)(4t+1)+|E_{3,4t+1}|(3)(4t+1)\\
&+ |E_{4t+1,4t+1}|(4t+1)(4t+1)\\
&= t-2+6\times 9+(2t-4)(4t+1)+4\times 3(4t+1)+(2t-3)(4t+1)(4t+1)\\
&= (4t+1)(2t-4+12+8t^2+2t-12t-3)+t+52.
\end{aligned}
$$

Hence,

$$DWM_2(G) = (4t+1)(8t^2-8t+5)+t+52.$$

Case 3. If $t=1$ and $s>1$. The number of vertices in this case is $4s+2$ and the number of edges is $5s+1$. Two dimensional structure of Graphene, we have $E_{3,3}, E_{3,4s+1}, E_{0,4s+1}$ and $E_{4s+1,4s+1}$ edges. In Figure 7, we have colored $E_{3,3}, E_{3,4s+1}, E_{0,4s+1}$ and $E_{4s+1,4s+1}$ edges in red, blue, green and black, respectively.

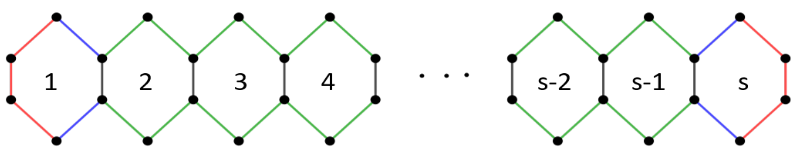

FIGURE 7. Graphene $G_{t,s}$ with $t=1, s>1$.

Therefore, we have $|E_{3,3}|=6$ edges, $|E_{3,4s+1}|=4$ edges, $|E_{0,4s+1}|=4(s-2)$ edges and $|E_{4s+1,4s+1}|=s-1$ edges.

$$
\begin{aligned}
DWM_2(G) &= \sum_{uv\in E(G)} d_{dn}(u)d_{dn}(v)\\
&= |E_{3,3}|(3)(3)+|E_{3,4s+1}|(3)(4s+1)+|E_{0,4s+1}|(0)(4s+1)\\
&+ |E_{4s+1,4s+1}|(4s+1)(4s+1)\\
&= 6\times 9+4\times 3(4s+1)+4(s-2)(0)(4s+1)+(s-1)(4s+1)(4s+1)\\
&= (4s+1)(12+4s^2+s-4s-1)+54.
\end{aligned}
$$

Hence,

$$DWM_2(G) = (4s+1)(4s^2-3s+11)+54.$$

Case 4. If $t>1$ and $s>1$. Let $n=2st+2s+2t$. Two dimensional structure of Graphene contains the following types of edges $E_{1,1}, E_{2,2}, E_{0,n-1}, E_{1,n-1}, E_{2,n-1}$ and $E_{n-1,n-1}$. In Figure 8, the edges

$E_{1,1}, E_{2,2}, E_{0,n-1}, E_{1,n-1}, E_{2,n-1}$ and $E_{n-1,n-1}$ are colored in red, blue, green, yellow, pink and black, respectively.

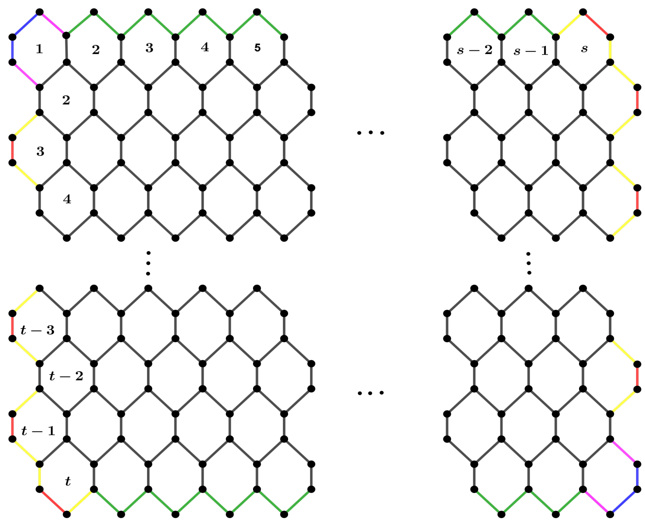

FIGURE 8. Graphene $G_{t,s}$ with $t > 1, s > 1$.

The number of edges in $E_{1,1}, E_{2,2}, E_{0,n-1}, E_{1,n-1}, E_{2,n-1}$ and $E_{n-1,n-1}$ in each row is mention in the following table.

Row	$\|E_{1,1}\|$	$\|E_{2,2}\|$	$\|E_{0,n-1}\|$	$\|E_{1,n-1}\|$	$\|E_{2,n-1}\|$	$\|E_{n-1,n-1}\|$
1	1	2	$2(s-2)$	3	2	$3s-2$
2	1	0	0	2	0	$3s-1$
3	1	0	0	2	0	$3s-1$
. . .	. . .	. . .	. . .	. . .	. . .	. . .
$t-1$	1	0	0	1	1	$3s-1$
t	1	2	$2(s-2)$	2	1	$s-1$
Total	t	4	$4(s-2)$	$2t$	4	$s(3t-2)-(1+t)$

Therefore, we have $|E_{1,1}| = t$ edges, $|E_{2,2}| = 4$ edges, $|E_{0,n-1}| = 4(s-2)$ edges, $|E_{1,n-1}| = 2t$ edges, $|E_{2,n-1}| = 4$ edges and $|E_{n-1,n-1}| = s(3t-2)-(1+t)$edges.

$$
\begin{aligned}
DWM_2(G) &= \sum_{uv\in E(G)} d_{dn}(u)d_{dn}(v) \\
&= |E_{1,1}|(1)(1) + |E_{2,2}|(2)(2) + |E_{0,n-1}|(0)(n-1) + |E_{1,n-1}|(1)(n-1) \\
&+ |E_{2,n-1}|(2)(n-1) + |E_{n-1,n-1}|(n-1)(n-1) \\
&= t + 4\times 9 + 2t(n-1) + 4\times 2(n-1) + s(3t-2) - (1+t))(n-1)(n-1) \\
&= (n-1)(2t+8+(s(3t-2)-(1+t))(n-1)) + t + 16.
\end{aligned}
$$

Hence,

$$DWM_2(G) = (2st+2s+2t-1)((s(3t-2)-(1+t))(2st+2s+2t-1)+2(t+4)) + t + 16.$$

□

Proposition 4.1. *Let $G \cong G_{t,s}$ be the graph of Graphene with t rows of benzene rings and s benzene rings in each row. The forgotten downhill Zagreb index is given by*

$$
DWF(G) = \begin{cases}
750 & if \quad t=1, s=1; \\
2(t-1)(4t+1)^3 + 2(t+106) & if \quad t>1, s=1; \\
2(s-1)(4s+1)^3 + 216 & if \quad t=1, s>1; \\
2(st-1)(2st+2s+2t-1)^3 + 2(t+24) & if \quad t>1, s>1.
\end{cases}
$$

Proof. The proof similar to the proof Theorem 4.2. □

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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