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IMAGE RESTORATION USING A NOVEL MODEL COMBINING THE PERONA-MALIK EQUATION AND THE HEAT EQUATION

SAMIRA LECHEHEB*, MESSAOUD MAOUNI, HAKIM LAKHAL

Laboratory of Applied Mathematics and History and Didactics of Maths "LAMAHIS", Department of Mathematics, University of 20 August 1955, Skikda, Algeria

* *Corresponding author:* lecheheb.samira24@gmail.com; s.lecheheb@univ-skikda.dz

Abstract. This article is devoted to the mathematical study of a new proposed model based on a Perona-Malik equation combined with a heat equation. This study shows how system of partial differential equations can be used to restore a digital image. By using compactness method and the monotonicity arguments, with suitable assumptions on the nonlinearities, we prove the existence of the weak solution for the proposed model which its consistency is given in our work.

1. Introduction

In recent years, the application of partial differential equations in image processing have attracted attention of many authors in computer vision. Various techniques have been developed in image processing during the last decades. Now these techniques are used for all kinds of tasks in all kinds of domains: industrial inspection, medical visualization, human computer interfaces, artistic effects, etc. Texture extraction and image restoration are the two fundamental problems that have made a significant contribution to this discipline, as can be ascertained from recent survey papers [1,4,5,8,10,11,16,17,19,21]. The question here is: how to preserve the contours of an image while the elimination of the noise. In 1990, Perona and Malik [20] answered this question in their model which is one of the first attempts to derive a model that incorporates local information from an image within a PDE framework. The numerical results of this model represent

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an efficient and effective tool for image denoising. Nevertheless, two important phenomena have been observed. The first one proved by Kichenassamy [14] that the Perona-Malik model is an ill-posed problem in the sense of Hadamard, this paradoxical result is sometimes referred to as the Perona-Malik Paradox [14]. The second one is the phenomenon known as the staircase effect. To make this problem well posed, Catté et all [9] suggested introducing the regularization in space and time directly into the continuous equation. There is an extensive literature for the purpose of denoising in image processing. Let us mention the work of Aboulaich et al [2], concerning new diffusion models for the image processing. The proposed model is a combination of fast growth with respect to low gradient and slow growth when the gradient is large. In their valuable monograph [7], Atlas et al. have presented a new model for image restoration. The proposed model is an interpolation of two classical models, Perona-Malik and p-Laplacian. By using the monotonicity arguments, they proved the existence and the uniqueness of solutions for p large enough. They also studied the asymptotic behavior of the solution as $p \to \infty$ and they proved that the limit problem coincides with the Perona-Malik model in the some subregion. In 2016, Afraites et al [3] proposed a model removing noise while preserving the edges and reducing staircase effect where they combined a nonlinear regularization of total variation (TV) operator's with a decomposition approach of H^{-1} norm suggested by Guo et al. ([12, 13]). The generalization of this work was made by Atlas et al [6]. Until the work of Lecheheb et al. [15] combining the Perona-Malik equation with the heat equation, the authors were able to demonstrate the existence and consistency of the their proposed model. We build up on their works by providing a generalization to the case of systems. In this article, we present a novel model for image denoising, which combines the Perona-Malik equation and the heat equation. Our model is well-posed. By using the compactness method and the monotonicity arguments, we prove the existence of solutions for the following system

$$\begin{cases} -\mathrm{div}\Big(g_1(|\nabla v|)\nabla u\Big) - \dfrac{1}{\lambda_1^2}\Delta u = f(x,u,v) \quad \text{in } \Omega, \\ -\mathrm{div}\Big(g_2(|\nabla u|)\nabla v\Big) - \dfrac{1}{\lambda_2^2}\Delta v = h(x,u,v) \quad \text{in } \Omega, \\ \Big(g_1(|\nabla v|) + \dfrac{1}{\lambda_1^2}\Big)\nabla u \cdot \vec{n} = \Big(g_2(|\nabla u|) + \dfrac{1}{\lambda_2^2}\Big)\nabla v \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, with Neumann boundary conditions. $\vec{n}$ is the unit outward normal at the image boundaries $\partial\Omega$ and $0 < \lambda \leq 1$ suth that $\lambda = (\lambda_1, \lambda_2)$. The function $g(\cdot) = (g_1, g_2)$ is defined by one of the following expressions:

$$g(k) = \frac{1}{1 + (\frac{k}{\lambda})^2} \quad \text{or} \quad g(k) = \exp\big(-\frac{k^2}{2\lambda^2}\big),$$

It is clear that the function $g(k)$ is a decreasing non-negative function satisfying the following conditions

$$\begin{cases} \lim\limits_{k\to 0} g(k) = 1, \\ \\ \lim\limits_{k\to +\infty} g(k) = 0. \end{cases} \tag{1.2}$$

We remark that, if $g_i = 1$ for $i = 1, 2$ we recover the linear diffusion.

For the rest of this article, we assume that $f, h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are carathéodory functions satisfying the growth condition

$$\begin{cases} |f(x, s, z)| \le d_1(x) + \dfrac{1}{2\lambda_1^2}|s| + \dfrac{1}{2\lambda_2^2}|z|, \\ |h(x, s, z)| \le d_2(x) + \dfrac{1}{2\lambda_1^2}|s| + \dfrac{1}{2\lambda_2^2}|z|, \end{cases} \tag{1.3}$$

where $d = (d_1, d_2)$ is in $(L^2(\Omega))^2$ and $0 < \lambda \le 1$, $(\lambda = (\lambda_1, \lambda_2))$.

The content of this paper is arranged as follows. In the next section we will present the main results. In the section 3, we will study the existence of the solutions of the problem (1.1) under some different conditions on the nonlinear terms. This existence is obtained by using the compactness method [18] and the monotonicity arguments.

2. Main results

In this section, we are ready to present the definition of a weak solution for problem (1.1) and the main result. At beginning, let

$$\mathrm{V} = H_0^1(\Omega) \times H_0^1(\Omega),$$

which is a Banach space endowed with the norm

$$\|(u, v)\|_{\mathrm{V}}^2 = \|u\|_{H_0^1(\Omega)}^2 + \|v\|_{H_0^1(\Omega)}^2,$$

and let $\mathrm{W} = L^2(\Omega) \times L^2(\Omega)$. In the sequel, $\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ will denote the usual norms of $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively.

We give now the definition of a weak solution for problem (1.1)

Definition 2.1. *We say that $(u, v) \in \mathrm{V}$ is a weak solution for the system (1.1) if for any $(\phi, \psi) \in \mathrm{V}$ we have*

$$\begin{aligned} &\int_\Omega (g_1(|\nabla v|) + \frac{1}{\lambda_1^2})\nabla u \nabla \phi \,\mathrm{d}x + \int_\Omega (g_2(|\nabla u|) + \frac{1}{\lambda_2^2})\nabla v \nabla \psi \,\mathrm{d}x \\ = \;&\int_\Omega f(x, u, v)\phi \,\mathrm{d}x + \int_\Omega h(x, u, v)\psi \,\mathrm{d}x. \end{aligned} \tag{2.1}$$

The main result of this article read as follows.

Theorem 2.1. *Assume conditions* (1.2) *and* (1.3) *are fulfilled. Then problem* (1.1) *has at least one solution.*

3. Proof of main result

In the present section, we will use the compactness method to obtain existence results from finite-dimensional approximations.

Proof. Let X be a finite-dimensional subspace of V endowed with the V-norm, and X^* its dual. Define de mappings $L : \mathrm{X} \times [0,1] \longrightarrow \mathrm{X}^*$ by

$$\begin{aligned}
&\langle L(u,v,t),(\phi,\psi)\rangle_{\mathrm{V}} \\
&= \int_\Omega \Big(g_1(t|\nabla v|) + \frac{1}{\lambda_1^2}\Big)\nabla u \nabla \phi \,\mathrm{d}x + \int_\Omega \Big(g_2(t|\nabla u|) + \frac{1}{\lambda_2^2}\Big)\nabla v \nabla \psi \,\mathrm{d}x \\
&- \int_\Omega f(x,tu,tv)\phi \,\mathrm{d}x - \int_\Omega h(x,tu,tv)\psi \,\mathrm{d}x,
\end{aligned} \tag{3.1}$$

for all $(\phi,\psi) \in \mathrm{X}$, L is well defined. Next, we divide the proof into four steps:

Step 1:: In this step, we show

$$\Big\{(u,v) \in X : L(u,v,t) = 0 \text{ for some } t \in [0,1]\Big\} \subset \bar{B}\Big(\frac{2}{\min(k_1,k_2)}\|(d_1,d_2)\|_{\mathrm{W}}\Big)$$

is established.

Indeed, if $L(u,v,t) = 0$ for some $(u,v,t) \in \mathrm{X} \times [0,1]$, then

$$0 = \langle L(u,v,t),u,v\rangle \geq \min(k_1,k_2)\|(u,v)\|_{\mathrm{V}} - 2\|(d_1,d_2)\|_{\mathrm{W}},$$

which implies that $\|(u,v)\|_{\mathrm{V}} \leq \frac{2}{\min(k_1,k_2)}\|(d_1,d_2)\|_{\mathrm{W}}$. Consequently, for any $\mathrm{R} > \frac{2}{\min(k_1,k_2)}\|(d_1,d_2)\|_{\mathrm{W}}$, we have

$$L(u,v,t) \neq 0 \;\text{ if } (u,v,t) \in \partial B^{\mathrm{X}}(\mathrm{R}) \times [0,1]. \tag{3.2}$$

Step 2: In this step, we show $L\Big(\bar{B}^{\mathrm{X}}(\mathrm{R}) \times [0,1]\Big) \subset \bar{B}^{\mathrm{X}^*}\Big(\mathrm{MR} + 2\|(d_1,d_2)\|_{\mathrm{W}}\Big)$ is established.

If $(u,v,t) \in \bar{B}^{\mathrm{X}}(\mathrm{R}) \times [0,1]$, we have

$$\begin{aligned}
|\langle L(u,v,t),(\phi,\psi)\rangle| &\leq \bigg(\max\Big(\frac{2\lambda_1^2+3}{2\lambda_1^2},\frac{2\lambda_2^2+3}{2\lambda_2^2}\Big)\|(u,v)\|_{\mathrm{V}} + 2\|(d_1,d_2)\|_{\mathrm{W}}\bigg)\|(\varphi,\psi)\|_{\mathrm{V}} \\
&\leq \bigg(\underbrace{\max\Big(\frac{2\lambda_1^2+3}{2\lambda_1^2},\frac{2\lambda_2^2+3}{2\lambda_2^2}\Big)}_{\mathrm{M}}\mathrm{R} + 2\|(d_1,d_2)\|_{\mathrm{W}}\bigg)\|(\varphi,\psi)\|_V \\
&\leq \Big(\mathrm{MR} + 2\|(d_1,d_2)\|_{\mathrm{W}}\Big)\|(\varphi,\psi)\|_V,
\end{aligned}$$

for all $(\varphi,\psi) \in \mathrm{V}$, and hence

$$L\Big(\bar{B}^{\mathrm{X}}(R) \times [0,1]\Big) \subset \bar{B}^{\mathrm{X}^*}\Big(\mathrm{MR} + 2\|(d_1,d_2)\|_{\mathrm{W}}\Big). \tag{3.3}$$

Step 3: In this step, we prove that L is a continuous mapping on $\bar{B}^{\mathrm{X}}(R) \times [0,1]$.

Let $(u_n, v_n, t_n) \in \bar{B}^{\mathrm{X}}(R) \times [0,1]$ converge to (u, v, t) in $X \times [0,1]$, i.e in $V \times [0,1]$. Since $(L(u_n, v_n, t_n))$ is bounded because of (3.3), to prove that $L(u_n, v_n, t_n) \to L(u, v, t)$, it is sufficient to show that $L(u, v, t)$ is the unique cluster point of $(L(u_n, v_n, t_n))$. Let $l \in \mathrm{X}^*$ be such a cluster point, still we denote by $(t_n), (u_n)$ and (v_n) a subsequence of $(t_n), (u_n)$ and (v_n) respectively such that $L(u_n, v_n, t_n) \to l$ in X^*. Since $(u_n, v_n) \to (u, v)$ in V, it follows that $(u_n, v_n) \to (u, v)$ in W, and hence, going if necessary to a subsequence, we may assume that $(u_n, v_n) \to (u, v)$ a.e. in Ω . On the other hand, $(\partial_i u_n, \partial_i v_n) \to (\partial_i u, \partial_i v)$ in W, therefore $(\nabla u_n, \nabla v_n) \to (\nabla u, \nabla v)$ a.e in Ω. This implies that

$$g_1(t_n|\nabla v_n|) \to g_1(t|\nabla v|) \quad \text{a.e. in } \Omega,$$

$$g_2(t_n|\nabla u_n|) \to g_2(t|\nabla u|) \quad \text{a.e. in } \Omega,$$

and hence, for any $(\phi, \psi) \in X$,

$$g_1(t_n|\nabla v_n|)\nabla\phi \to g_1(t|\nabla v|)\nabla\phi \text{ in } L^2(\Omega),$$

$$g_2(t_n|\nabla u_n|)\nabla\psi \to g_2(t|\nabla u|)\nabla\psi \text{ in } L^2(\Omega).$$

For the last term,

$$f(x, t_n u_n, t_n v_n) \to f(x, tu, tv) \text{ a.e..}$$

Using Lebesgue dominated convergence theorem and (1.3), we arrive at

$$f(x, t_n u_n, t_n v_n) \to f(x, tu, tv) \text{ in } L^2(\Omega).$$

Consequently

$$\int_\Omega f(x, t_n u_n, t_n v_n)\phi \,\mathrm{d}x \to \int_\Omega f(x, tu, tv)\phi \,\mathrm{d}x.$$

Similarly we have

$$\int_\Omega h(x, t_n u_n, t_n v_n)\phi \,\mathrm{d}x \to \int_\Omega h(x, tu, tv)\phi \,\mathrm{d}x.$$

We conclude that

$$\begin{aligned}
&\langle L(u_n, v_n, t_n), (\phi, \psi)\rangle_V \\
&= \int_\Omega \Big(g_1(t_n|\nabla v_n|) + \frac{1}{\lambda_1^2}\Big)\nabla u_n \nabla\phi \,\mathrm{d}x + \int_\Omega \Big(g_2(t_n|\nabla u_n|) + \frac{1}{\lambda_2^2}\Big)\nabla v_n \nabla\psi \,\mathrm{d}x \\
&- \int_\Omega f(x, t_n u_n, t_n v_n)\phi \,\mathrm{d}x - \int_\Omega h(x, t_n u_n, t_n v_n)\psi \,\mathrm{d}x \\
&\to \int_\Omega \Big(g_1(t|\nabla v|) + \frac{1}{\lambda_1^2}\Big)\nabla u \nabla\phi \,\mathrm{d}x + \int_\Omega \Big(g_2(t|\nabla u|) + \frac{1}{\lambda_2^2}\Big)\nabla v \nabla\psi \,\mathrm{d}x \\
&- \int_\Omega f(x, tu, tv)\phi \,\mathrm{d}x - \int_\Omega h(x, tu, tv)\psi \,\mathrm{d}x = \langle L(u, v, t), (\varphi, \psi)\rangle_V.
\end{aligned}$$

Thus $l = L(u, v, t)$. All those properties allow us to apply the homotopy invariance property to

$$\deg_B\Big(L(\cdot,\cdot,1), B(R), 0\Big) = \deg_B\Big(L(\cdot,\cdot,0), B(R), 0\Big). \tag{3.4}$$

But $L(u, v, 0) = 0$ is equivalant to the problem

$$\begin{aligned}&(1+\frac{1}{\lambda_1^2})\int_\Omega \nabla u\nabla\phi\,\mathrm{d}x + (1+\frac{1}{\lambda_2^2})\int_\Omega \nabla v\nabla\psi\,\mathrm{d}x\\ &= \int_\Omega f(x)\phi\,\mathrm{d}x + \int_\Omega h(x)\psi\,\mathrm{d}x,\end{aligned}$$

for all $(\phi, \psi) \in \mathrm{X}$, whose solution is unique because of the boundedness of the set of its possible solutions. Consequently,

$$\deg_B\Big(L(\cdot,\cdot,0), B(R), 0\Big) = \pm 1,$$

and from (3.4) and the existence property of degree, there exists $(u, v) \in B^{\mathrm{X}}(R)$ which satisfies

$$\begin{aligned}&\int_\Omega\Big(g_1(|\nabla v|)+\frac{1}{\lambda_1^2}\Big)\nabla u\nabla\phi\,\mathrm{d}x + \int_\Omega\Big(g_2(|\nabla u|)+\frac{1}{\lambda_2^2}\Big)\nabla v\nabla\psi\,\mathrm{d}x\\ &= \int_\Omega f(x,u,v)\phi\,\mathrm{d}x + \int_\Omega h(x,u,v)\psi\,\mathrm{d}x,\\ &|(u,v)\|_{\mathrm{V}} \le \frac{2}{\min(k_1,k_2)}\|(d_1,d_2)\|_{\mathrm{W}},\end{aligned} \tag{3.5}$$

for all $(\phi, \psi) \in X$.

Step 4: We now show the passage to the limit.

Consider the function $b_i : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$b_i(\zeta_i) = \Big(g_i(\zeta_i) + \frac{1}{\lambda_i^2}\Big)\zeta_i \quad \text{for any } \zeta_i \in \mathbb{R}^N \text{ and } i = 1, 2.$$

To prove the passage to the limit, we need the following lemma:

Lemma 3.1. *[16] Let $0 < \lambda_i \le 1$, for any $\zeta_i, \zeta_i' \in \mathbb{R}^N$ such that $\zeta_i \neq \zeta_i'$ we have*

$$(b_i(\zeta_i) - b_i(\zeta_i'))(\zeta_i - \zeta_i') > 0 \textit{ for } i = 1, 2.$$

The proof of the above lemma can be found in [16].

Lemma 3.2. *If $b \in C(\mathbb{R}^N, \mathbb{R}^N)$, $b(\zeta) \le (1 + \frac{1}{\lambda^2})\zeta$ for all $\zeta \in \mathbb{R}^N$ and if $u_n \to u$ in $H_0^1(\Omega)$ then $b(\nabla u_n) \to b(\nabla u)$ in $L^2(\Omega)$.*

Lemma (3.2) is proved by the dominated convergence theorem of Lebesgue.

Now, it is well known that one can write $V = \overline{\bigcup_{n\ge1} X_n}$ where $X_n \subset X_{n+1} (n \ge 1)$ and X_n has dimension n. Consequently, given any $(\phi, \psi) \in V$, there exists a sequence (ϕ_n, ψ_n) with $(\phi_n, \psi_n) \in X_n$

which converges to (ϕ, ψ). On the other hand, by (3.5) applied to $X = X_n$, there exists, for each $n \geq 1$, some $(u_n, v_n) \in X_n$ such that

$$\begin{aligned}
&\int_\Omega b_1(\nabla u_n)\nabla\varphi_1 \,\mathrm{d}x + \int_\Omega b_2(\nabla v_n)\nabla\varphi_2 \,\mathrm{d}x \\
&= \int_\Omega f(x, u_n, v_n)\varphi_1 \,\mathrm{d}x + \int_\Omega h(x, u_n, v_n)\varphi_2 \,\mathrm{d}x, \\
&\|(u_n, v_n)\|_{\mathrm{V}} \leq \frac{2}{\min(k_1, k_2)}\|(d_1, d_2)\|_{\mathrm{W}},
\end{aligned}$$

for all $(\varphi_1, \varphi_2) \in X_n$. In particular, taking $(\varphi_1, \varphi_2) = (\phi_n, \psi_n)$ introduced above,

$$\begin{aligned}
&\int_\Omega b_1(\nabla u_n)\nabla\phi_n \,\mathrm{d}x + \int_\Omega b_2(\nabla v_n)\nabla\psi_n \,\mathrm{d}x \\
&= \int_\Omega f(x, u_n, v_n)\phi_n \,\mathrm{d}x + \int_\Omega h(x, u_n, v_n)\psi_n \,\mathrm{d}x, \\
&\|(u_n, v_n)\|_{\mathrm{V}} \leq \frac{2}{\min(k_1, k_2)}\|(d_1, d_2)\|_{\mathrm{W}},
\end{aligned} \tag{3.6}$$

for all $n \geq 1$. The estimate in (3.6) implies that, going if necessary to subsequences, we can assume that there exists $(u, v) \in \mathrm{V}$ such that $(u_n, v_n) \to (u, v)$ weakly in V, $(u_n, v_n) \to (u, v)$ strongly in W and $(u_n, v_n) \to (u, v)$ a.e. in Ω. As $\big(b_1(\nabla u_n)\big)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, then there exists $\xi_1 \in L^2(\Omega)$ such that

$$b_1(\nabla u_n) \to \xi_1 \text{ weakly in } L^2(\Omega).$$

Similarly, we obtain

$$b_2(\nabla v_n) \to \xi_2 \text{ weakly in } L^2(\Omega),$$

and $(\nabla\phi_n, \nabla\psi_n) \to (\nabla\phi, \nabla\psi)$ strongly in W. On the other hand, as $f(x, u_n, v_n) \to f(x, u, v)$ in $L^2(\Omega)$ and $h(x, u_n, v_n) \to h(x, u, v)$ in $L^2(\Omega)$, one can let $n \to \infty$ in (3.6) to obtain

$$\int_\Omega \xi_1\nabla\phi \,\mathrm{d}x + \int_\Omega \xi_2\nabla\psi \,\mathrm{d}x = \int_\Omega f(x, u, v)\phi \,\mathrm{d}x + \int_\Omega h(x, u, v)\psi \,\mathrm{d}x. \tag{3.7}$$

It remains to show that

$$\int_\Omega \xi_1\nabla\phi \,\mathrm{d}x = \int_\Omega b_1(\nabla u)\nabla\phi \,\mathrm{d}x, \tag{3.8}$$

and

$$\int_\Omega \xi_2\nabla\psi \,\mathrm{d}x = \int_\Omega b_2(\nabla v)\nabla\psi \,\mathrm{d}x. \tag{3.9}$$

To prove the two equalities, we use the trick of Minty [16]; we begin by studying the limit of

$$\int_\Omega b_1(\nabla u_n)\nabla u_n \,\mathrm{d}x,$$

and

$$\int_\Omega b_2(\nabla v_n)\nabla v_n \,\mathrm{d}x.$$

Indeed

$$\int_\Omega b_1(\nabla u_n)\nabla u_n \,\mathrm{d}x = \int_\Omega f(x,u_n,v_n)u_n \,\mathrm{d}x \to \int_\Omega f(x,u,v)u \,\mathrm{d}x,$$
$$\int_\Omega b_2(\nabla v_n)\nabla v_n \,\mathrm{d}x = \int_\Omega h(x,u_n,v_n)v_n \,\mathrm{d}x \to \int_\Omega h(x,u,v)v \,\mathrm{d}x,$$

because $(u_n, v_n) \to (u, v)$ weakly in V. But we know that (u, v) satisfies (3.7), and hence

$$\int_\Omega f(x,u,v)u \,\mathrm{d}x = \int_\Omega \xi_1 \nabla u \,\mathrm{d}x,$$

and

$$\int_\Omega h(x,u,v)v \,\mathrm{d}x = \int_\Omega \xi_2 \nabla v \,\mathrm{d}x.$$

Therefore

$$\begin{aligned} \lim_{n\to+\infty} \int_\Omega b_1(\nabla u_n)\nabla u_n \,\mathrm{d}x &= \int_\Omega f(x,u,v)u \,\mathrm{d}x \\ &= \int_\Omega \xi_1 \nabla u \,\mathrm{d}x, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \lim_{n\to+\infty} \int_\Omega b_2(\nabla v_n)\nabla v_n \,\mathrm{d}x &= \int_\Omega h(x,u,v)v \,\mathrm{d}x \\ &= \int_\Omega \xi_2 \nabla v \,\mathrm{d}x. \end{aligned} \tag{3.11}$$

Let $(\phi, \psi) \in \mathrm{V}$, it exists $(\phi_n, \psi_n)_{n\in\mathbb{N}}$ such that $(\phi_n, \psi_n) \in \mathrm{X}_n$ for all $n \in \mathbb{N}$ and $(\phi_n, \psi_n) \to (\varphi, \psi)$ in V when $n \to +\infty$. Thanks to Lemma 3.1, we will pass to the limit in the two terms

$$\int_\Omega b_1(\nabla u_n)\nabla \phi_n \,\mathrm{d}x,$$

and

$$\int_\Omega b_2(\nabla v_n)\nabla \psi_n \,\mathrm{d}x.$$

Indeed, for the first equation

$$\begin{aligned} &0 \le \int_\Omega (b_1(\nabla u_n) - b_1(\nabla \phi_n))(\nabla u_n - \nabla \phi_n) \,\mathrm{d}x = \\ &\int_\Omega b_1(\nabla u_n)\nabla u_n \,\mathrm{d}x - \int_\Omega b_1(\nabla u_n)\nabla \phi_n \,\mathrm{d}x - \int_\Omega b_1(\nabla \phi_n)\nabla u_n \,\mathrm{d}x + \int_\Omega b_1(\nabla \phi_n)\nabla \phi_n \,\mathrm{d}x \\ &= \quad T_{1,n} \quad - \quad T_{2,n} \quad - \quad T_{3,n} \quad + \quad T_{4,n}, \end{aligned}$$

we saw in (3.10) that $T_{1,n} \to \int_\Omega \xi_1 \nabla u \,\mathrm{d}x$ when $n \to \infty$. We have

$$\lim_{n\to+\infty} T_{2,n} = \int_\Omega \xi_1 \nabla \phi \,\mathrm{d}x.$$

Similarly

$$\lim_{n\to+\infty} T_{3,n} = \int_\Omega b_1(\nabla \phi)\nabla u \,\mathrm{d}x.$$

Finally, we also have

$$\lim_{n\to+\infty} T_{4,n} = \int_\Omega b_1(\nabla\phi)\nabla\phi\,\mathrm{d}x,$$

when $n \to +\infty$. The passage to the limit therefore gives:

$$\int_\Omega (\xi_1 - b_1(\nabla\phi))(\nabla u - \nabla\phi)\,\mathrm{d}x \geq 0 \text{ for all } \phi \in H_0^1(\Omega).$$

Similarly, we obtain

$$\int_\Omega (\xi_2 - b_2(\nabla\psi))(\nabla u - \nabla\psi)\,\mathrm{d}x \geq 0 \text{ for all } \psi \in H_0^1(\Omega).$$

We now choose judicious test functions ϕ and ψ. We take

$$\phi = u + \frac{1}{n}w, \text{ with } w \in H_0^1(\Omega) \text{ and } n \in \mathbb{N}^*,$$

and

$$\psi = v + \frac{1}{n}\widetilde{w}, \text{ with } \widetilde{w} \in H_0^1(\Omega) \text{ and } n \in \mathbb{N}^*.$$

We thus obtain:

$$-\frac{1}{n}\int_\Omega \big(\xi_1 - b_1(\nabla u + \frac{1}{n}\nabla w)\big)\nabla w\,\mathrm{d}x \geq 0,$$

and

$$-\frac{1}{n}\int_\Omega \big(\xi_2 - b_2(\nabla v + \frac{1}{n}\nabla \widetilde{w})\big)\nabla \widetilde{w}\,\mathrm{d}x \geq 0,$$

then

$$\int_\Omega \big(\xi_1 - b_1(\nabla u + \frac{1}{n}\nabla w)\big)\nabla w\,\mathrm{d}x \leq 0,$$

and

$$\int_\Omega \big(\xi_2 - b_2(\nabla v + \frac{1}{n}\nabla \widetilde{w})\big)\nabla \widetilde{w}\,\mathrm{d}x \leq 0.$$

But

$$u + \frac{1}{n}w \to u \text{ in } H_0^1(\Omega),$$
$$v + \frac{1}{n}\widetilde{w} \to v \text{ in } H_0^1(\Omega),$$

from Lemma 3.2, we get

$$b_1\big(\nabla u + \frac{1}{n}\nabla w\big) \to b_1(\nabla u) \text{ in } L^2(\Omega),$$

and

$$b_2\big(\nabla v + \frac{1}{n}\nabla \widetilde{w}\big) \to b_2(\nabla v) \text{ in } L^2(\Omega).$$

Passing to the limit when $n \to +\infty$, we then obtain

$$\int_\Omega (\xi_1 - b_1(\nabla u))\nabla w\,\mathrm{d}x \leq 0, \quad \forall w \in H_0^1(\Omega),$$

and

$$\int_\Omega (\xi_2 - b_2(\nabla v))\nabla \widetilde{w}\,\mathrm{d}x \leq 0, \quad \forall \widetilde{w} \in H_0^1(\Omega).$$

By linearity (can change w into $-w$ and $\widetilde{w}$ into $-\widetilde{w}$), we have

$$\int_\Omega (\xi_1 - b_1(\nabla u))\nabla w\,\mathrm{d}x = 0, \quad \forall w \in H_0^1(\Omega),$$

and

$$\int_\Omega (\xi_2 - b_2(\nabla v))\nabla \widetilde{w}\,\mathrm{d}x = 0, \quad \forall \widetilde{w} \in H_0^1(\Omega).$$

We deduce that

$$\int_\Omega \xi_1 \nabla w\,\mathrm{d}x = \int_\Omega b_1(\nabla u)\nabla w\,\mathrm{d}x, \quad \forall w \in H_0^1(\Omega),$$
$$\int_\Omega \xi_2 \nabla \widetilde{w}\,\mathrm{d}x = \int_\Omega b_2(\nabla v)\nabla \widetilde{w}\,\mathrm{d}x, \quad \forall \widetilde{w} \in H_0^1(\Omega).$$

Hence we have showed that (u, v) is a solution of (1.1).

□

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References

[1] A. Aarab, N.E. Alaa, H. Khalfi, Generic reaction-diffusion model with application to image restoration and enhancement. Electron. J. Differ. Equ. 2018 (2018), 125.

[2] R. Aboulaich, D. Meskine, A. Souissi, New diffusion models in image processing, Computers Math. Appl. 56(4) (2008), 874-882.

[3] L. Afraites, A. Atlas, F. Karami, D. Meskine, Some class of parabolic systems applied to image processing. Discr. Contin. Dyn. Syst. Ser. B. 21(6) (2016), 1671-1687.

[4] N.E. Alaa, M. Aitoussous, W. Bouarifi, D. Bensikaddour, Image restoration unisg a reaction-diffusion process. Electron. J. Differ. Equ. 2014 (2014), 197.

[5] N.E. Alaa, M. Zirhem, Bio-inspired reaction diffusion system applied to image restoration. Int. J. Bio-Inspired Comput. 12(2) (2018), 128-137.

[6] A. Atlas, M. Bendahmane, F. Karami, D. Meskine, O. Oubbih, A nonlinear fractional reaction-diffusion system applied to image denoising and decomposition. Discr. Contin. Dyn. Syst. Ser. B. http://dx.doi.org/10.3934/dcdsb.2020321.

[7] A. Atlas, F. Karami, D. Meskine, The Perona-Malik inequality and application to image denoising. Nonlinear Anal., Real World Appl. 18 (2014), 57-68.

[8] G. Aubert, P. Kornprobst, Mathematical problems in image processing. Springer-Verlag, New-York. (2002).

[9] H.F. Catte, P.L. Lions, J.M. Morel, T. Coll, Image selective smoothing and edge detection by nonlinear diffusion. SIAM J. Numer. Anal. 29 (1) (1992), 182-193.

[10] A. Chambolle, P.L. Lions, Image recovery via total variation minimization and related problems. Numer. Math. 74 (2) (1997), 147-188.

[11] Y. Chen, M. Levine Rao, Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. 44 (4) (2004), 1383-1404.

[12] Z. Guo, Q. Liu, J. Sun, B. Wu, Reaction-diffusion systems with p(x)-growth for image denoising. Nonlinear Anal., Real World Appl. 12 (5) (2011), 2904-2918.

[13] Z. Guo, Q. Liu, On a reaction-diffusion system applied to image decomposition and restoration. Math. Computer Model. 53 (2011), 1336-1350.

[14] S. Kichenassamy, The Perona–Malik paradox. SIAM J. Appl. Math. 57 (5) (1997), 1328-1342.

[15] S. Lecheheb, M. Maouni, H. Lakhal, Existence of solutions of a quasilinear problem with Neumann boundary conditions. Bol. Soc. Parana. Mat. In press.

[16] S. Lecheheb, M. Maouni, H. Lakhal, Existence of the solution of a quasilinear equation and its application to image denoising. International Journal of Computer Science, Int. J. Computer Sci. Commun. Inform. Technol. 7 (2) (2019), 1-6.

[17] S. Lecheheb, M. Maouni, H. Lakhal, Image restoration using nonlinear elliptic Equation. Int. J. Computer Sci. Commun. Inform. Technol. 6 (2) (2019), 32-37.

[18] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Paris, (1969).

[19] S. Osher, L. Rudin, E. Fatemi, Nonlinear total variation based noise removal algorithms. Physica D, 40 (1992), 259-268.

[20] P. Perona, J. Malik, Scale-espace and edge detection using anisotropic diffusion. IEEE Trans. Pattern Anal. Mach. Intell. 12 (1990), 429-439.

[21] M. Zirhem, N.E. Alaa, Existence and uniqueness of an entropy solution for a nonlinear reaction-diffusion system applied to texture analysis. J. Math. Anal. Appl. 484 (1) (2020), 123-719.