



FUZZY ZAGREB INDICES AND SOME BOUNDS FOR FUZZY ZAGREB ENERGY

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ABSTRACT. Topological indices M_1/M_2 known as first/second Zagreb indices are defined as the sum of the sum/product of degrees of pairs of adjacent vertices of a simple graph. These indices and their properties have been studied in detail under chemical graph theory. In this paper we introduce the concepts of first, second and hyper Zagreb indices of fuzzy graphs. We also study the Zagreb matrices and the associated Zagreb energies of fuzzy graphs. Some bounds for these energies are also obtained.

1. INTRODUCTION

Topological indices are numerical quantities of structural molecular graphs. They are studied and applied in various fields by engineers, pharmacist, graph theorist and mathematicians. I. Gutman [1] in 1972, introduced the first Zagreb index and Randec in [2] introduced Randec index, which are oldest among the topological indices. I Gutman, Eliasi, Kulli, KC Das and many other experts have contributed in the developments of different Zagreb indices, Randic indices of simple graphs.

In case of classical graphs, both the vertices and edges have membership value one, but in case of fuzzy graphs both vertices and edges are equally important along with their fuzzy membership values. If the description of objects or their relationships or both are vague in nature, then we design a Fuzzy Graph model. In 1965, Zadeh [3] introduced the concept of fuzzy sets and fuzzy relations. Further Rosenfeld [4], Zimmerman [5], Thomson [6] and many experts in [7–12] have contributed significantly in the developments of fuzzy graphs.

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In [13], Anjali and Mathew introduced energy of fuzzy graphs and in [14] authors introduced Laplacian energy of fuzzy graphs. In [15, 16], authors discussed Wiener index of fuzzy graph and found relationships between connectivity index and Wiener index of a fuzzy graph. Recently, authors in [17, 18], discussed transitive blocks, Hamiltonian fuzzy graphs and their applications in fuzzy interconnection networks, human trafficking.

The paper is structured as follows: In section 2, we discuss the preliminary definitions required for the development of the content. In Section 3, we introduce the important definitions of fuzzy Zagreb indices, fuzzy Zagreb matrices and corresponding energies. Section 4 provides some bounds for the fuzzy Zagreb energies.

2. Preliminaries

In this section, we recall some definitions of Zagreb indices and some notions of fuzzy graphs which will play an important role in the subsequent sections of the paper. Basics of Zagreb indices can be referred in [19, 20]. Basics of graphs and fuzzy graphs can be referred in [4, 21].

Definition 2.1. Let $G=(V, E)$ be a simple graph. Degree of vertex u is denoted by d_u and $e = uv \in E$ is an edge in graph G . First Zagreb index $M_1(G)$ is defined as

$$M_1(G) = \sum_{uv \in E} [d_u + d_v] = \sum_{u \in V} d_u^2$$

Second Zagreb index $M_2(G)$ is defined as

$$M_2(G) = \sum_{uv \in E} d_u \cdot d_v$$

and Hyper Zagreb index $HM(G)$ is defined as

$$HM(G) = \sum_{uv \in E} [d_u + d_v]^2$$

Definition 2.2. A fuzzy graph $G=(V, \sigma, \mu)$; which also can simply be denoted by $G=(\sigma, \mu)$, is a graph with vertex-membership function $\sigma : V \rightarrow [0, 1]$ and edge-strength function $\mu : V \times V \rightarrow [0, 1]$ such that it satisfies the relation $\mu(x, y) \leq \min\{\sigma(x), \sigma(y)\}, \forall x, y \in V$. Corresponding crisp graph is denoted by $G=(\sigma^*, \mu^*)$.

Also, we denote strength of vertex u by $\mu(u)$, it represents the minimum of strengths of edges incident to the vertex u

$$\mu(u) = \bigwedge_{uv_i \in \mu^*} \mu(u, v_i)$$

3. Fuzzy Zagreb Indices

In this section, we introduce the definitions of fuzzy Zagreb first index, fuzzy Zagreb second index and fuzzy Zagreb hyper index along with associated fuzzy Zagreb matrices and the fuzzy Zagreb energies. These definitions are required to discuss the main results.

Definition 3.1. The fuzzy Zagreb first index of $G=(\sigma, \mu)$ is defined as

$$FM_1(G) = \sum_{uv \in \mu^*} [\sigma(u)\mu(u) + \sigma(v)\mu(v)]$$

Equivalently the index can also be defined as

$$FM_1(G) = \sum_{u \in \sigma^*} [\sigma(u)\mu(u)d_u]$$

Definition 3.2. The fuzzy Zagreb second index of $G=(\sigma, \mu)$ is defined as

$$FM_2(G) = \sum_{uv \in \mu^*} [\sigma(u)\mu(u).\sigma(v)\mu(v)]$$

Definition 3.3. The fuzzy hyper Zagreb index of $G=(\sigma, \mu)$ is defined as

$$FHM(G) = \sum_{uv \in \mu^*} [\sigma(u)\mu(u) + \sigma(v)\mu(v)]^2$$

Definition 3.4. If $G=(\sigma, \mu)$ is a fuzzy graph and $\sigma^* = \{u_1, u_2, \dots, u_n\}$ then first fuzzy Zagreb matrix is defined as $FZ^{(1)} = (fz^{(1)})_{i,j}$, where

$$(fz^{(1)})_{i,j} = \begin{cases} \sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j) & , \text{ if } i \neq j \text{ and } u_i.u_j \in \mu^* \\ 0 & , \text{ if } u_i.u_j \notin \mu^* \\ 0 & , \text{ if } i = j \end{cases}$$

and second fuzzy Zagreb matrix is defined as $FZ^{(2)} = (fz^{(2)})_{i,j}$, where

$$(fz^{(2)})_{i,j} = \begin{cases} \sigma(u_i)\mu(u_i).\sigma(u_j)\mu(u_j) & , \text{ if } i \neq j \text{ and } u_i.u_j \in \mu^* \\ 0 & , \text{ if } u_i.u_j \notin \mu^* \\ 0 & , \text{ if } i = j \end{cases}$$

Definition 3.5. If $G=(\sigma, \mu)$ is a fuzzy graph and $\sigma^* = \{u_1, u_2, \dots, u_n\}$, if $FZ^{(1)}$ is the first fuzzy Zagreb matrix with its eigen values $\xi_1^{(1)}, \xi_2^{(1)}, \dots, \xi_n^{(1)}$ then the first fuzzy Zagreb energy is defined as

$$FZE^{(1)} = \sum_{i=1}^n |\xi_i^{(1)}|$$

Definition 3.6. If $G=(\sigma, \mu)$ is a fuzzy graph and $\sigma^* = \{u_1, u_2, \dots, u_n\}$, if $FZ^{(2)}$ is the second fuzzy Zagreb matrix with $\xi_1^{(2)}, \xi_2^{(2)}, \dots, \xi_n^{(2)}$ as its eigen values then the second fuzzy Zagreb energy is defined as

$$FZE^{(2)} = \sum_{i=1}^n |\xi_i^{(2)}|$$

Example 3.1. Consider the fuzzy graphs $G=(\sigma, \mu)$ as shown in fig.1 Here $\sigma^* = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ with membership values $\sigma(u_1) = 0.4, \sigma(u_2) = 0.2, \sigma(u_3) = 0.6, \sigma(u_4) = 0.5, \sigma(u_5) = 0.7, \sigma(u_6) = 0.8$ and strengths of edges $\mu(u_1u_2) = 0.1, \mu(u_2u_3) = 0.2, \mu(u_2u_4) = 0.1, \mu(u_3u_4) = 0.4, \mu(u_4u_5) = 0.4, \mu(u_5u_6) = 0.5, \mu(u_5u_1) = 0.3, \mu(u_1u_6) = 0.3$ then we get $FM_1 = 1.68, FM_2 = 0.1107$ and $FHM = 0.0468$. The first and second fuzzy Zagreb matrices are given by

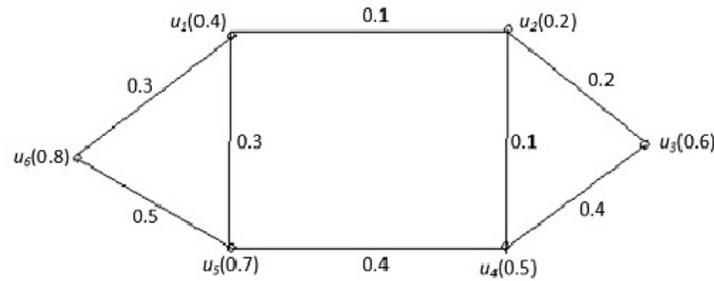


FIGURE 1.

$$FZ^{(1)} = \begin{matrix} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 0 & 0.06 & 0 & 0 & 0.25 & 0.28 \\ 0.06 & 0 & 0.14 & 0.07 & 0 & 0 \\ 0 & 0.14 & 0 & 0.17 & 0 & 0 \\ 0 & 0.07 & 0.17 & 0 & 0.26 & 0 \\ 0.25 & 0 & 0 & 0.26 & 0 & 0.45 \\ 0.28 & 0 & 0 & 0 & 0.45 & 0 \end{pmatrix} \end{matrix}$$

and

$$FZ^{(2)} = \begin{matrix} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 0 & 0.0008 & 0 & 0 & 0.0084 & 0.0096 \\ 0.0008 & 0 & 0.024 & 0.001 & 0 & 0 \\ 0 & 0.024 & 0 & 0.006 & 0 & 0 \\ 0 & 0.001 & 0.006 & 0 & 0.0105 & 0 \\ 0.0084 & 0 & 0 & 0.0105 & 0 & 0.0504 \\ 0.0096 & 0 & 0 & 0 & 0.0504 & 0 \end{pmatrix} \end{matrix}$$

Eigen values of $FZ^{(1)}$ are given by $\xi_1^{(1)} = 0.7067$, $\xi_2^{(1)} = -0.5276$, $\xi_3^{(1)} = -0.2595$, $\xi_4^{(1)} = 0.2487$, $\xi_5^{(1)} = -0.1704$, $\xi_6^{(1)} = 0.0021$ hence the first fuzzy Zagreb energy is $FZE^{(1)} = 1.915$.

Eigen values of $FZ^{(2)}$ are given by $\xi_1^{(2)} = 0.0544$, $\xi_2^{(2)} = -0.0515$, $\xi_3^{(2)} = 0.0249$, $\xi_4^{(2)} = -0.0245$, $\xi_5^{(2)} = -0.00404$, $\xi_6^{(2)} = 0.00069$ hence the second fuzzy Zagreb energy is $FZE^{(2)} = 0.16$.

4. Main Results

In this paper we will discuss fuzzy Zagreb first index and the corresponding first fuzzy Zagreb matrix and first fuzzy Zagreb energy. The analogous study of second fuzzy Zagreb quantities along with first and second fuzzy Eztrada Zagreb energies will be communicated in forthcoming paper.

For simplicity of notations, fuzzy Zagreb first matrix will be denoted by FZ , its $(i, j)^{th}$ element by $(fz)_{i,j}$ and the corresponding fuzzy Zagreb energy by $FZE = \sum_{i=1}^n |\xi_i|$, where $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n$ are eigen values of FZ . Here FZ is a real symmetric matrix and hence all its eigen values are real. For non-negative integer k , the k^{th} spectral moment of FZ is given by $N_k = \sum_{i=1}^n (\xi_i)^k = tr(FZ^k)$

Although elementary, the following result deserves to be stated as:

Theorem 4.1. *If $G=(V, \sigma, \mu)$ and $H=(W, \sigma', \mu')$ are fuzzy isomorphic graphs then $FM_1(G) = FM_1(H)$ and $FM_2(G) = FM_2(H)$*

Proof. If $G=(V, \sigma, \mu)$ and $H=(W, \sigma', \mu')$ are fuzzy isomorphic graphs then there exists a bijective function $f : V \rightarrow W$ such that for every $u \in V$ and $uv \in \mu^*$, $\sigma(u) = \sigma'(f(u))$ and $\mu(uv) = \mu'(f(u)f(v))$. Hence we can easily conclude that

$$\begin{aligned} FM_1(G) &= \sum_{uv \in \mu^*} [\sigma(u)\mu(u) + \sigma(v)\mu(v)] \\ &= \sum_{f(u)f(v) \in \mu'^*} [\sigma'(f(u))\mu'(f(u)) + \sigma'(f(v))\mu'(f(v))] \\ &= FM_1(H) \end{aligned}$$

and also

$$\begin{aligned} FM_2(G) &= \sum_{uv \in \mu^*} [\sigma(u)\mu(u).\sigma(v)\mu(v)] \\ &= \sum_{f(u)f(v) \in \mu'^*} [\sigma'(f(u))\mu'(f(u)).\sigma'(f(v))\mu'(f(v))] \\ &= FM_2(H) \end{aligned}$$

□

Theorem 4.2. *If $G=(V, \sigma, \mu)$ with $V = \{u_1, u_2, \dots, u_n\}$ is a fuzzy graph and FZ is its fuzzy Zagreb matrix then*

(1) $N_1 = tr(FZ) = 0$

(2) $N_2 = tr(FZ^2) = 2.FHM$

Proof. (1) By definition of FZ , all its diagonal elements are zero, hence trace of FZ is zero.

(2) The diagonal elements of FZ^2 are given by

$$\begin{aligned} (fz^2)_{i,i} &= \sum_{j=1}^n (fz)_{i,j} \cdot (fz)_{j,i} \\ &= \sum_{j=1}^n (fz)_{i,j}^2 \\ &= \sum_{\substack{u_i, u_j \in \mu^* \\ i, j \in \{1, 2, \dots, n\}}} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)]^2 \end{aligned}$$

hence

$$\begin{aligned} N_2 &= tr(FZ^2) \\ &= \sum_{i=1}^n \sum_{\substack{u_i, u_j \in \mu^* \\ i, j \in \{1, 2, \dots, n\}}} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)]^2 \\ &= 2 \cdot \sum_{\substack{u_i, u_j \in \mu^* \\ i, j \in \{1, 2, \dots, n\}}} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)]^2 \\ &= 2.FHM \end{aligned}$$

□

Theorem 4.3. Let $G=(V, \sigma, \mu)$ be a fuzzy graph with $|V| = n$ vertices. Let FZ be the corresponding fuzzy Zagreb matrix with eigen values $\xi_1, \xi_2, \dots, \xi_n$ and FZE is its fuzzy Zagreb energy then

$$\sqrt{2(FHM) + n(n - 1)|det(FZ)|^{\frac{2}{n}}} \leq FZE \leq \sqrt{2n.(FHM)}$$

Proof. Upper Bound:

Applying Cauchy-Schwartz inequality to the n numbers $(1, 1, \dots, 1)$ and $(\xi_1, \xi_2, \dots, \xi_n)$ we get,

$$\sum_{i=1}^n (1 \cdot |\xi_i|) \leq \left(\sum_{i=1}^n 1\right)^{1/2} \cdot \left(\sum_{i=1}^n |\xi_i|^2\right)^{1/2}$$

$$(4.1) \quad \sum_{i=1}^n |\xi_i| \leq \sqrt{n} \cdot \sqrt{\sum_{i=1}^n |\xi_i|^2}$$

Also we have,

$$\left(\sum_{i=1}^n |\xi_i|\right)^2 = \sum_{i=1}^n |\xi_i|^2 + 2 \cdot \sum_{1 \leq i < j \leq n} |\xi_i \cdot \xi_j| = 0$$

hence

$$(4.2) \quad \sum_{i=1}^n |\xi_i|^2 = -2 \cdot \sum_{1 \leq i < j \leq n} |\xi_i \cdot \xi_j|$$

On comparing the coefficients of $\prod_{i=1}^n (\xi - \xi_i) = |FZ - \xi.I|$, we get

$$(4.3) \quad \sum_{1 \leq i < j \leq n} |\xi_i \cdot \xi_j| = - \sum_{u_i, u_j \in \mu^*} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)]^2$$

Using (4.3), eq.(4.2) becomes

$$(4.4) \quad \sum_{i=1}^n |\xi_i|^2 = 2 \cdot \sum_{u_i, u_j \in \mu^*} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)]^2$$

hence eq.(4.1) becomes

$$\sum_{i=1}^n |\xi_i| \leq \sqrt{2n \cdot \sum_{u_i, u_j \in \mu^*} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)]^2}$$

$$(4.5) \quad FZE(G) \leq \sqrt{2n \cdot (FHM)}$$

Lower Bound:

$$\begin{aligned} (FZE(G))^2 &= \left(\sum_{i=1}^n |\xi_i| \right)^2 \\ &= \sum_{i=1}^n |\xi_i|^2 + 2 \cdot \sum_{1 \leq i < j \leq n} |\xi_i \cdot \xi_j| \end{aligned}$$

$$(FZE(G))^2 = 2 \cdot \sum_{u_i, u_j \in \mu^*} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)]^2 + 2 \cdot \frac{n(n-1)}{2} \cdot AM\{|\xi_i \cdot \xi_j|\}$$

hence

$$(4.6) \quad (FZE(G))^2 \geq 2 \cdot (FHM) + n(n-1) \cdot GM\{|\xi_i \cdot \xi_j|\}$$

where

$$\begin{aligned} GM\{|\xi_i \cdot \xi_j|\} &= \left(\prod_{1 \leq i < j \leq n} |\xi_i \cdot \xi_j| \right)^{\frac{2}{n(n-1)}} \\ &= \left(\prod_{i=1}^n |\xi_i|^{n-1} \right)^{\frac{2}{n(n-1)}} \\ &= \left(\prod_{i=1}^n |\xi_i| \right)^{2/n} \\ &= (\det FZ)^{2/n} \end{aligned}$$

hence eq.(4.6) becomes

$$(FZE(G))^2 \geq 2 \cdot (FHM) + n(n-1) \cdot (\det FZ)^{2/n}$$

$$(4.7) \quad FZE(G) \geq \sqrt{2 \cdot (FHM) + n(n-1) \cdot (\det FZ)^{2/n}}$$

□

Theorem 4.4. Let $G=(V, \sigma, \mu)$ be a fuzzy graph and $\sigma^* = \{u_1, u_2, \dots, u_n\}$. If FZ is the corresponding fuzzy Zagreb matrix with eigen values $\xi_1, \xi_2, \dots, \xi_n$ and FZE is its fuzzy Zagreb energy then

$$FZE \leq \frac{2(FHM)}{n} + \sqrt{(n-1) \left\{ 2(FHM) - \left(\frac{2}{n}(FHM) \right)^2 \right\}}$$

Proof. First we recall some elementary results. If $A = (a_{ij})_{n \times n}$ is a symmetric matrix with diagonal elements all zero then $\lambda_{\max} \geq \frac{2}{n} \sum_{1 \leq i < j \leq n} a_{ij}$, where λ_{\max} is the maximum eigenvalue of A. Hence in case of fuzzy Zagreb matrix FZ , if $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n$ are its eigen values, then

$$(4.8) \quad |\xi_1| \geq \frac{2}{n} \sum_{u_i u_j \in \mu^*} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)] = \frac{2}{n} FM_1$$

Using eq.(4.4) of theorem 4.3,

$$(4.9) \quad \sum_{i=1}^n |\xi_i|^2 = 2 \cdot \sum_{u_i u_j \in \mu^*} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)]^2 = 2(FHM)$$

$$(4.10) \quad \sum_{i=2}^n |\xi_i|^2 = 2(FHM) - |\xi_1|^2$$

Using Cauchy-Schwartz inequality to the $(n-1)$ numbers $(1, 1, \dots, 1)$ and $(\xi_2, \xi_3, \dots, \xi_n)$ we get

$$(4.11) \quad \sum_{i=2}^n (1 \cdot |\xi_i|) \leq \left(\sum_{i=2}^n 1 \right)^{1/2} \cdot \left(\sum_{i=2}^n |\xi_i|^2 \right)^{1/2}$$

$$\sum_{i=2}^n |\xi_i| \leq \sqrt{(n-1) \cdot \left(\sum_{i=2}^n |\xi_i|^2 \right)}$$

Using eq.(4.10), eq.(4.11) becomes

$$\sum_{i=2}^n |\xi_i| \leq \sqrt{(n-1) \cdot \{ 2(FHM) - |\xi_1|^2 \}}$$

$$\sum_{i=2}^n |\xi_i| + |\xi_1| \leq |\xi_1| + \sqrt{(n-1) \cdot \{ 2(FHM) - |\xi_1|^2 \}}$$

$$(4.12) \quad FZE \leq |\xi_1| + \sqrt{(n-1) \cdot \{ 2(FHM) - |\xi_1|^2 \}}$$

As the function $g(x) = x + \sqrt{(n-1) \cdot \{ 2(FHM) - x^2 \}}$ is decreasing function of x in the interval $\frac{2}{n}(FHM) < x \leq \sqrt{2(FHM)}$ and also $|\xi_1| \geq \frac{2}{n} FM_1$, we get

$$\sqrt{\frac{2}{n}(FHM)} \leq \frac{2}{n}(FHM) \leq \frac{2}{n} FM_1 \leq |\xi_1| \leq \sqrt{2(FHM)}$$

Hence eq.(4.12) becomes

$$(4.13) \quad FZE \leq \frac{2}{n}(FHM) + \sqrt{(n-1) \cdot \left\{ 2(FHM) - \left(\frac{2}{n}(FHM) \right)^2 \right\}}$$

□

Theorem 4.5. *If $G=(V, \sigma, \mu)$ is a fuzzy graph and $\sigma^* = \{u_1, u_2, \dots, u_n\}$, FZ is the corresponding fuzzy Zagreb matrix with eigen values $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n > 0$ and FZE is its fuzzy Zagreb energy then*

$$FZE \geq \frac{2}{n}FM_1 + (n-1) + \ln \left[\frac{n|\det(FZ)|}{2FM_1} \right]$$

Proof. Consider $G=(V, \sigma, \mu)$ is a fuzzy graph and $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n > 0$ are eigen values of corresponding Fuzzy Zagreb matrix FZ . For simplicity, consider a function $f(x) = 1 - x - \ln(x)$, $x > 0$. Elementary calculations shows that $f(x)$ is decreasing function in $(0, 1]$ and it is increasing function for $x \geq 1$. Hence $f(x) \geq f(1) = 0$ for $x > 0$ gives, $x \geq 1 + \ln(x)$ for $x > 0$.

$$\begin{aligned} FZE &= \sum_{i=1}^n |\xi_i| = |\xi_1| + \sum_{i=2}^n |\xi_i| \\ &\geq |\xi_1| + \sum_{i=2}^n [1 + \ln |\xi_i|] \\ &= |\xi_1| + (n-1) + \ln \left[\prod_{i=2}^n |\xi_i| \right] \\ &= |\xi_1| + (n-1) + \ln \left[\prod_{i=1}^n |\xi_i| \right] - \ln |\xi_1| \\ &= |\xi_1| + (n-1) + \ln |\det(FZ)| - \ln |\xi_1| \end{aligned}$$

As the function $g(x) = x + (n-1) + \ln |\det(FZ)| - \ln x$ is increasing function in $1 \leq x \leq n$, hence for $|\xi_1| \geq \frac{2}{n}FM_1$,

$$FZE \geq \frac{2}{n}FM_1 + (n-1) + \ln |\det(FZ)| - \ln \left[\frac{2}{n}FM_1 \right]$$

$$(4.14) \quad FZE \geq \frac{2}{n}FM_1 + (n-1) + \ln \left[\frac{n|\det(FZ)|}{2FM_1} \right]$$

□

Theorem 4.6. *If $G=(V, \sigma, \mu)$ is a fuzzy graph and $\sigma^* = \{u_1, u_2, \dots, u_n\}$, FZ is the corresponding fuzzy Zagreb matrix with eigen values $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n > 0$ and FZE is its fuzzy Zagreb energy then*

$$FZE \leq 2(FHM) + \frac{2}{n}FM_1 - \left(\frac{2}{n}FM_1 \right)^2 - \ln \left[\frac{n|\det(FZ)|}{2FM_1} \right]$$

Proof. Consider $G=(V, \sigma, \mu)$ is a fuzzy graph and $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n > 0$ are the eigen values of FZ . Recall that, $2FM_1 = 2 \sum_{u_i, u_j \in \mu^*} [\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)] \geq n$, hence $\frac{2}{n}FM_1 \geq 1$. For simplicity, consider a function $f(x) = x^2 - x - \ln(x)$, $x > 0$. $f(x)$ is decreasing function in $0 < x \leq 1$ and it is increasing function in $x \geq 1$. Hence $f(x) \geq f(1) = 0$ for $x > 0$, gives $x \leq x^2 - \ln(x)$ for $x > 0$.

$$\begin{aligned} FZE &= \sum_{i=1}^n |\xi_i| = |\xi_1| + \sum_{i=2}^n |\xi_i| \\ &\leq |\xi_1| + \sum_{i=2}^n [|\xi_i|^2 - \ln |\xi_i|] \\ &= |\xi_1| + \sum_{i=1}^n |\xi_i|^2 - |\xi_1|^2 - \ln \prod_{i=1}^n |\xi_i| + \ln |\xi_1| \\ &= 2(FHM) + |\xi_1| - |\xi_1|^2 - \ln \left[\frac{|\det(FZ)|}{|\xi_1|} \right] \end{aligned}$$

As the function $g(x) = 2(FHM) + x - x^2 - \ln [|\det(FZ)|] + \ln(x)$ is increasing function in $0 < x \leq 1$ and it is decreasing function in $x \geq 1$, also $x \geq \frac{2}{n}FM_1 \geq 1$ we get,

$$(4.15) \quad FZE \leq 2(FHM) + \frac{2}{n}FM_1 - \left(\frac{2}{n}FM_1\right)^2 - \ln \left[\frac{n|\det(FZ)|}{2FM_1} \right]$$

□

Illustration: Consider the fuzzy graph $G=(\sigma, \mu)$ as shown in *fig.1*. Here $FZE = 1.915$, $FHM = 0.468$ and $FM_1 = 1.68$

$$\begin{aligned} FZE &\geq \sqrt{2(FHM) + n(n-1)|\det(FZ)|^{\frac{2}{n}}} \\ &= \sqrt{2(0.468) + 6(5)|0.00000861|^{\frac{2}{6}}} \\ &= 1.2453 \end{aligned}$$

and

$$FZE \leq \sqrt{2n.(FHM)} = \sqrt{2(6).(0.468)} = 2.3698$$

hence it verifies the *theorem 4.3*.

$$\begin{aligned} FZE &\leq \frac{2(FHM)}{n} + \sqrt{(n-1) \left\{ 2(FHM) - \left(\frac{2}{n}(FHM)\right)^2 \right\}} \\ &= \frac{2(0.468)}{6} + \sqrt{(5) \left\{ 2(0.468) - \left(\frac{2}{6}(0.468)\right)^2 \right\}} \\ &= 2.291 \end{aligned}$$

hence it verifies the *theorem 4.4*.

$$\begin{aligned} FZE &\geq \frac{2}{n}FM_1 + (n-1) + \ln \left[\frac{n|\det(FZ)|}{2FM_1} \right] \\ &= \frac{2}{6}(1.68) + 5 + \ln \left[\frac{6(0.00000861)}{2(1.68)} \right] \\ &= -5.5228 \end{aligned}$$

hence the *theorem 4.5* is verified.

$$\begin{aligned} FZE &\leq 2(FHM) + \frac{2}{n}FM_1 - \left(\frac{2}{n}FM_1 \right)^2 - \ln \left[\frac{n|\det(FZ)|}{2FM_1} \right] \\ &= 2(0.468) + \frac{2}{6}(1.68) - \left(\frac{2}{6}(1.68) \right)^2 - \ln \left[\frac{6(0.00000861)}{2(1.68)} \right] \\ &= 12.2652 \end{aligned}$$

hence the *theorem 4.6* is verified.

5. Applications

In case of human trafficking, objects can be considered as vertices which are reasons for human trafficking while each link between these reasons can be considered as an edge. So each edge has strength of the routes between vertices. Concepts of indices can be applied to measure of susceptibility of certain routs which need to be eliminated with respect to human trafficking. Similarly, in case of internet routing, fuzzy Zagreb indices can be used to identify the nature of particular vertex or strength of the whole system so that it reduces the time consumption in the particular area.

6. Conclusion

Fuzzy Zagreb first, second and hyper indices as well as first, second fuzzy Zagreb matrices and their energies are studied. Some bounds for first fuzzy Zagreb energy are studied along with illustration. Further study on these fuzzy Zagreb indices may reveal more analogous results of these kind and will be discussed in the forthcoming papers.

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