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FIXED POINT THEOREM FOR MONOTONE NON-EXPANSIVE MAPPINGS

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Abstract. In this paper, we study the fixed point theorem for monotone nonexpansive mappings in the setting of a uniformly smooth and uniformly convex smooth Banach space.

1. Introduction

Given a complete metric space $(\mathcal{X}, d)$, the most well-studied types of self-maps are referred to as *Lipschitz mappings* (or *Lipschitz maps*, for short), which are given by the metric inequality

$$d(Tx, Ty) \leq kd(x, y), \tag{1.1}$$

for all $x, y \in \mathcal{X}$, where $k > 0$ is a real number, usually referred to as the *Lipschitz constant* of T. The metric inequality (1.1) can be classified into three categories, thus *contraction mappings* for the case where $k < 1$, *non-expansive mappings* for the case where $k = 1$ and *expansive mappings* for the case where $k > 1$. The most important property of (1.1) is that they are uniformly continuous. Thus, for any sequence $\{x_n\}_{n\geq 1}$ converging to x in $\mathcal{X}$, we have $d(Tx_n, Tx) = 0$ as $n \to \infty$. It is well known that when $\mathcal{X}$ is complete and T is a contraction mapping, then T has unique fixed point and the sequence of Picard iteration $T^n(x)$ converges to the fixed point of T as $n \to \infty$. Fixed points problems of contraction mappings always exist

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and it's unique due to Banach [1]. Edelstein [2] also showed when T is a contractive mapping (that is, $d(Tx, Ty) < d(x, y)$) on a compact metric space $\mathcal{X}$, then T has a unique fixed point and the fixed point can be iteratively approximated by the Picard iteration $x_{n+1} = Tx_n$.

In metric spaces, the only non-trivial thing one can say about nonexpansive mappings is that the Picard iteration is a bounded sequence, which is as a result of the inequality

$$d(T^n x, T^n y) \leq d(x, y), \forall n \geq 0, \quad x, y \in \mathcal{X},$$

where T is a nonexpansive mapping. Even in compact metric spaces, with the exception of the contractive mappings described above, generally one cannot find a fixed point (if it exists) by the Picard iteration. It is therefore imperative that one considers the more specialised complete metric spaces: that is, the Banach spaces, where linearity and homogeneity affords more structure to the nonexpansive mappings and their fixed points. We use $\texttt{Fix}(T)$ to denote the set of fixed points of the mapping T (that is, $\texttt{Fix}(T) = \{x \in \mathcal{C} : Tx = x\}$).

An approximate fixed point sequence of a nonexpansive self-map T on a closed convex subset $\mathcal{C}$ of Banach space $\mathcal{X}$ is any sequence $\{x_n\}_{n\geq 1} \subset \mathcal{C}$ such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

When $\mathcal{C}$ is bounded or $\texttt{Fix}(T) \neq \emptyset$, then such a sequence always exists. One of the ways to construct an approximate fixed point sequence for nonexpansive mappings is to use the Banach contraction mapping theorem [1] to obtain a sequence $\{x_n\}$ in $\mathcal{C}$ such that

$$x_n = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad n \geq 1$$

where the initial guess x_0 is taken arbitrarily in $\mathcal{C}$ and $\{\alpha_n\}$ is a sequence in the interval (0,1) such that $\alpha_n \to 0$ as $n \to \infty$. By assuming that $\texttt{Fix}(T) \neq \emptyset$, this sequence $\{x_n\}$ is bounded (indeed, $\|x_n - p\| \leq \|x_0 - p\|$ for all $p \in \texttt{Fix}(T)$). Hence

$$\|x_n - Tx_n\| = \alpha_n \|x_0 - Tx_n\| \to 0,$$

and $\{x_n\}$ is an approximate fixed point sequence for T. The immediate conclusion from the above deduction is the following result on compact star-convex sets.

Theorem 1.1. *Let T be a nonexpansive self-mapping on a compact star-convex subset of a Banach space. Then T has a fixed point.*

Theorem 1.1 is proved by means of the Banach contraction mapping theorem [1], and it is in this spirit that we employ the Monotone contraction mapping theorem [8] to prove the following weaker but generalized version of Theorem 1.1.

Theorem 1.2 (Main theorem)**.** *Let $\mathcal{X}$ be a uniformly smooth, uniformly convex smooth Banach space with a sequentially weakly continuous normalized duality mapping, $\mathcal{C} \subset \mathcal{X}$ be a weakly-compact star-domain such that $0 \in \ker \mathcal{C}$. Then every monotone nonexpansive mapping, $T : \mathcal{C} \to \mathcal{C}$ has a fixed point.*

It is not clear to the author if the 'sequentially weakly continuity' condition can be removed, which would be desirable; however, all attempts to do so presently has not been successful and we hope that we may be able to remove it in subsequent work. Throughout this paper, $\Re$ denotes the real part of a complex number. We also use $\ker(\mathcal{C})$ to denote the kernel of a star convex subset $\mathcal{C}$ (equivalently, star-domain) of a normed linear space, that is $\{x \in \mathcal{C} : ax + (1-a)y \in \mathcal{C}, \forall \in [0,1], y \in \mathcal{C}\}$.

Definition 1.1 (Normalised Duality Mapping, see Lunner [7],1961)**.** *Let $\mathcal{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$. Denote $\langle \cdot, \cdot \rangle$ as the duality product. The normalised duality mapping J from $\mathcal{X}$ to $\mathcal{X}^*$ is defined by*

$$Jx := \{f \in \mathcal{X}^* : \|f\|_*^2 = \|x\|^2 = \langle x, f \rangle = fx\},$$

for all $x \in \mathcal{X}$. Hahn Banach theorem guarantees that $Jx \neq \emptyset$ for every $x \in \mathcal{X}$. For our purposes in this work, our interest mostly lies on the case when Jx is single-valued for all $x \in \mathcal{X}$, which is equivalent to the statement that $\mathcal{X}$ is a smooth Banach space. We say that the normalized duality map J of a Banach space $\mathcal{X}$ is *sequentially weakly continuous* if a sequence $\{x_n\}_{n\geq 1}$ in $\mathcal{X}$ is weakly convergent to x, then the sequence $\{Jx_n\}_{n\geq 1}$ in $\mathcal{X}^*$ is weak-star convergent to Jx. That is, given that $x_n \rightharpoonup x \in \mathcal{X}$, then $\{Jx_n\}_{n\geq 1} \overset{*}{\rightharpoonup} Jx \in \mathcal{X}^*$.

Remark 1.1. By virtue of the Riesz-Representation theorem, it follows that $Jx = x$ (J is the identity map) when we are in a Hilbert space.

Definition 1.2 (Monotone Contraction Mapping, see Gordon [8], 2020)**.** *Let $\mathcal{X}$ be a smooth Banach space and let $\mathcal{C}$ be a closed subset of $\mathcal{X}$. Then the mapping $T : \mathcal{C} \to \mathcal{C}$ is said to be a monotone contraction mapping if there exists $0 \leq c < 1$ such that for all $x, y \in \mathcal{C}$, the following two conditions are satisfied:*

1. $\Re\langle Tx - Ty, JTx - JTy\rangle \leq c\Re\langle x - y, Jx - Jy\rangle$,

2. $\Re\langle T^{m+1}x - T^m y, JT^{n+1}x - JT^n y\rangle \leq 0$,

where J is the normalised duality mapping and for all $m, n \geq 0$ with $m \neq n$.

In this paper, we consider the case where $c = 1$ in the above definition and introduce the following set of new mappings.

Definition 1.3 (Monotone Nonexpansive Mapping)**.** *Let $\mathcal{X}$ be a smooth Banach space and let $\mathcal{C}$ be a closed subset of $\mathcal{X}$. Then the mapping $T : \mathcal{C} \to \mathcal{C}$ is said to be a monotone nonexpansive mapping if the following two conditions are satisfied:*

1. $\Re\langle Tx - Ty, JTx - JTy\rangle \leq \Re\langle x - y, Jx - Jy\rangle$,

2. $\Re\langle T^{m+1}x - T^m y, JT^{n+1}x - JT^n y\rangle \leq 0$,

where J is the normalised duality mapping and for all $m, n \geq 0$ with $m \neq n$.

We should note here that, monotone nonexpansive mappings reduce to the nonexpansive type of mappings in (1.1) when in Hilbert spaces because in Hilbert spaces J is the identity mapping.

These references Browder [3], Göhde [4], Alpach [5] and Kirk [6] can be consulted for fixed point problems on nonexpansive mappings.

2. Preliminaries

We introduce the following theorem, proposition and lemmas that will be used in the proof of our main result. As before, all notations employed remain as defined.

Theorem 2.1 (Monotone contraction mapping theorem, see Gordon [8], 2020)**.** *Let $\mathcal{C}$ be a closed subset of a uniformly convex smooth Banach space $\mathcal{X}$ and let $T : \mathcal{C} \to \mathcal{C}$ be a monotone contraction mapping. Then T has a unique fixed point, that is, $\texttt{Fix}(T) = \{p\}$ and that the Picard iteration associated to T, that is, the sequence defined by $x_n = T(x_{n-1}) = T^n(x_0)$ for all $n \geq 1$ converges to p for any initial guess $x_0 \in \mathcal{X}$.*

Proposition 2.1 (see for instance Ezearn, [9])**.** *Let $\mathcal{X}$ be a normed linear space. Then for any $jx \in Jx, jy \in Jy$*

$$(\|x\| - \|y\|)^2 \leq \Re\langle x - y, jx - jy\rangle \leq \|x - y\|(\|x\| + \|y\|).$$

Thus, $\Re\langle x - y, jx - jy\rangle \geq 0$. Moreover if

$$\Re\langle x - y, jx - jy\rangle = 0,$$

then $jx \in Jy$ and $jy \in Jx$; in particular, when $\mathcal{X}$ is smooth (resp. strictly convex) then equality occurs if and only if $jx = jy$ (resp. x=y).

Proposition 2.2 (see for instance Ezearn, [9])**.** *Let $\mathcal{X}$ be a Banach space and let $\mathcal{X}^*$ be the dual space of X. Denote $\langle \cdot, \cdot \rangle$ the duality product. Now for $\{x_n\}_{n\geq 1} \subset \mathcal{X}$ and $\{f_n\}_{n\geq 1} \subset \mathcal{X}^*$, suppose either of the following conditions hold*

- $\{x_n\} \rightharpoonup x$ *and* $\{f_n\} \to f$
- $\{x_n\} \to x$ *and* $\{f_n\} \overset{*}{\rightharpoonup} f$

Then $\lim_{n\to\infty}\langle x_n, f_n\rangle = \langle x, f\rangle$.

Lemma 2.1 (Uniform Continuity in Uniformly Smooth Spaces)**.** *Let $\mathcal{X}$ be a uniformly smooth Banach space. Then the normalised duality map $J : \mathcal{X} \to \mathcal{X}^*$ is norm-to-norm uniformly continuous.*

3. Main Results

In this section, we first give the proof of Theorem 1.1 following the proof of our main result, Theorem 1.2.

Proof of Theorem 1.1. Let $\mathcal{C}$ be a compact star convex subset of a Banach space $\mathcal{X}$ with a distinguished point 'p'. Let $T : \mathcal{C} \to \mathcal{C}$ be a non-expansive mapping on $\mathcal{C}$. For $n \geq 1$, define $T_n : \mathcal{C} \to \mathcal{C}$ by,

$$T_n x = \Big(\frac{n}{n+1}\Big)Tx + \Big(\frac{1}{n+1}\Big)p, \forall x \in \mathcal{C}.$$

Obviously, T_n is a contraction mapping on $\mathcal{C}$. Therefore, by the Banach contraction mapping theorem [1], T_n has a unique fixed point x_n in $\mathcal{C}$. Now consider,

$$\begin{aligned}\|Tx_n - x_n\| &= \|Tx_n - T_n x_n\|,\\ &= \Big\|Tx_n - \Big(\frac{n}{n+1}\Big)Tx_n - \Big(\frac{1}{n+1}\Big)p\Big\|,\\ &= \Big(\frac{1}{n+1}\Big)\Big\|Tx_n - p\Big\|, \forall n \geq 1.\end{aligned}$$

Hence $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ since $\mathcal{C}$ is bounded. Since $\mathcal{C}$ is compact, the sequence $\{x_n\}_{n\geq 1}$ has a convergence subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges to some $x^* \in \mathcal{C}$ and by continuity of T, $Tx_{n_k} \to Tx^*$. Then consider,

$$Tx_{n_k}x_{n_k} = x_{n_k} = \Big(\frac{n_k}{n_k+1}\Big)Tx_{n_k} + \Big(\frac{1}{n_k+1}\Big)p.$$

By passing $k \to \infty$, we have $Tx^* = x^*$ and hence x^* is a fixed point of T in $\mathcal{C}$ and that completes the proof.

The proof of our theorem uses the ideas of the proof of Theorem 1.1 by creating an internal contraction in order to obtain an approximate fixed point sequence for these new mappings. The proof of our main result is as follows.

Proof of Theorem 1.2. Now for every natural number $n \geq 1$, define a new mapping $T_n : \mathcal{C} \to \mathcal{C}$ as

$$T_n(x) = \Big(1 - \frac{1}{n}\Big)Tx.$$

Clearly, T_n is a self-mapping since $0 \in \ker \mathcal{C}$. Now we have the following:

$$Tx = \frac{1}{\Big(1 - \frac{1}{n}\Big)}T_n x \quad \text{and} \quad Ty = \frac{1}{\Big(1 - \frac{1}{n}\Big)}T_n y.$$

By substituting Tx and Ty into Definition 1.3, we have the following:

$$\Re\langle\Big(1-\frac{1}{n}\Big)^{-1}T_n x - \Big(1-\frac{1}{n}\Big)^{-1}T_n y, \Big(1-\frac{1}{n}\Big)^{-1}J(T_n x) - \Big(1-\frac{1}{n}\Big)^{-1}J(T_n y)\rangle \leq \Re\langle x - y, Jx - Jy\rangle,$$
$$\Big(1-\frac{1}{n}\Big)^{-2}\Re\langle T_n x - T_n y, JT_n x - JT_n y\rangle \leq \Re\langle x - y, Jx - Jy\rangle.$$

Multiply the last inequality by $\Big(1 - \frac{1}{n}\Big)^2$ to obtain

$$\Re\langle T_n x - T_n y, JT_n x - JT_n y\rangle \leq \Big(1 - \frac{1}{n}\Big)^2 \Re\langle x - y, Jx - Jy\rangle.$$

Since $0 \leq \Big(1 - \frac{1}{n}\Big)^2 < 1$, then by Theorem 2.1, T_n has a unique fixed point say x_n, that is, $x_n = Tx_n = \Big(1 - \frac{1}{n}\Big)Tx_n$ and therefore $\|x_n - Tx_n\| = \frac{1}{n}\|Tx_n\|$.
Since $\mathcal{C}$ is bounded, then $\sup_{n\geq 1} \|Tx_n\| = D < \infty$ where D is constant.
Hence

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0, \tag{3.1}$$

where $\{x_n\}_{n\geq 1}$ is an approximate fixed point sequence for the monotone nonexpansive mapping T. Clearly, equation (3.1) implies $x_n - Tx_n \to 0$ as $n \to \infty$. Since $\mathcal{C}$ is weakly-compact, then the sequence $\{x_n\}_{n\geq 1}$ has a weakly converging subsequence. Without loss of generality, let $\{x_n\}_{n\geq 1}$ be the weakly converging subsequence and $x \in \mathcal{C}$ be the weak limit of this subsequence, that is, $x_n \rightharpoonup x$ as $n \to \infty$. Given that $x_n - Tx_n \rightharpoonup 0$ (strong convergence implies weak convergence) and $x_n \rightharpoonup x$, then $Tx_n \rightharpoonup x$ as $n \to \infty$.

We can clearly see that Definition 1.3 is equivalent to the following evaluation:

$$\Re\langle x - y + Tx - Ty, Jx - Jy - JTx + JTy\rangle \geq \Re\langle Tx - Ty, Jx - Jy\rangle - \Re\langle x - y, JTx - JTy\rangle \tag{3.2}$$

for all $x, y \in \mathcal{C}$. Since $\{x_n\}_{n\geq 1}$ and it weak limit are both contained in $\mathcal{C}$, then by replacing y with $= x_n$ in equation (3.2), we obtain the following

$$\begin{aligned} \Re\langle x - x_n + Tx - Tx_n, Jx - Jx_n - JTx + JTx_n\rangle \geq \Re\langle Tx - Tx_n, Jx - Jx_n\rangle \\ - \Re\langle x - x_n, JTx - JTx_n\rangle \end{aligned} \tag{3.3}$$

Taking limit as $n \to \infty$, the left hand side of equation (3.3) becomes

$$\lim_{n\to\infty} \Re\langle x - x_n + Tx - Tx_n, Jx - Jx_n - JTx + JTx_n\rangle. \tag{3.4}$$

Since $x_n - Tx_n \to 0$, then by Lemma 2.1, we have that $Jx_n - JTx_n \to 0$. Now with $Tx_n \rightharpoonup x$ and $Jx_n - JTx_n \to 0$, then by Proposition 2.2, as $n \to \infty$, equation (3.4) becomes

$$\Re\langle Tx - x, Jx - JTx\rangle = -\Re\langle x - Tx, Jx - JTx\rangle. \tag{3.5}$$

Again, taking limit of the right hand side of equation (3.3) as $n \to \infty$, we have

$$\lim_{n\to\infty} [\Re\langle Tx - Tx_n, Jx - Jx_n\rangle - \Re\langle x - x_n, JTx - JTx_n\rangle]. \tag{3.6}$$

Given that $x_n = \left(1 - \frac{1}{n}\right)Tx_n$ for all $n \geq 1$, then substituting this sequence into equation (3.6), we obtain

$$\lim_{n\to\infty} [\Re\langle Tx - Tx_n, Jx - (1 - \frac{1}{n})JTx_n\rangle - \Re\langle x - (1 - \frac{1}{n})Tx_n, JTx - JTx_n\rangle],$$

which by expansion gives the following:

$$\begin{aligned} \lim_{n\to\infty} [\Re\langle Tx, Jx\rangle - (1 - \frac{1}{n})\Re\langle Tx, JTx_n\rangle - \Re\langle Tx_n, Jx\rangle \\ - \Re\langle x, JTx\rangle + \Re\langle x, JTx_n\rangle + (1 - \frac{1}{n})\Re\langle Tx_n, JTx\rangle]. \end{aligned} \tag{3.7}$$

By the sequentially weakly continuity of $\mathcal{X}$, if $Tx_n \rightharpoonup x$ then $JTx_n \overset{*}{\rightharpoonup} Jx$ so that by Proposition 2.2, we have $\Re\langle Tx, JTx_n\rangle \to \Re\langle Tx, Jx\rangle$ and $\Re\langle Tx_n, Jx\rangle \to \Re\langle x, Jx\rangle$ as $n \to \infty$. Hence as $n \to \infty$, equation (3.7) reduces to:

$$\Re\langle Tx, Jx\rangle - \Re\langle Tx, Jx\rangle - \Re\langle x, Jx\rangle - \Re\langle x, JTx\rangle + \Re\langle x, Jx\rangle + \Re\langle x, JTx\rangle = 0. \tag{3.8}$$

By equation (3.5) and equation (3.8), equation (3.3) reduces to

$$\Re\langle x - Tx, Jx - JTx\rangle \leq 0,$$

which by Proposition 2.1 leads to

$$\Re\langle x - Tx, Jx - JTx\rangle = 0.$$

Since $\mathcal{X}$ is strictly convex, then by Proposition 2.1, we have $x - Tx = 0$ which implies $x \in \mathtt{Fix}(T)$ and that completes the proof.

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