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GENERALIZED PETROVIĆ'S INEQUALITIES FOR COORDINATED EXPONENTIALLY m-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce a new class of convex function, which is called coordinated exponentially m-convex functions. Some new Petrović's type inequalities for exponentially m-convex functions and coordinated exponentially m-convex functions are derived. Lagrange-type and Cauchy-type mean value theorems for exponentially $m-$convex and coordinated exponentially m-convex functions are also derived. Several special cases are discussed. We also prove the Lagrange type and Cauchy type mean value theorems for exponentially m-convex and coordinated exponentially $m-$convex functions. Results proved in this paper may stimulate further research in different areas of pure and applied sciences.

1. Introduction

Convex functions and their variant forms are being used to study a wide class of problems which arises in various branches of pure and applied sciences. For recent applications, generalizations and other aspects of convex functions, see [1–5, 11–23, 28, 33, 34] and the references therein. One of the most significant inequality is the Petrović's inequality [25]. Petrović's type inequality have been obtained by several authors, see [7, 10, 24–31] and reference therein.

In recent years, the convexity theory have been generalized in several directions using novel and innovative

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techniques. Toader [34] introduced the concepts of m-convex sets and m-convex functions, which appeared to be an interesting generalization of the convex sets and convex functions. Exponentially convex functions were introduced by Bernstein [6], which have applications in covariance analysis. Avriel [4] investigated this concept by imposing the condition of r-convex functions. It is well known that log- convex functions is closely related to exponentially convex functions, which have important and interesting applications in information theory and machine learning. Motivated and inspired by these applications, Noor et. al. [14] considered exponentially convex functions and explored their basic characterizations. They have shown that the optimality conditions of the differentiable exponentially convex functions can be characterized by variational inequalities, which have appeared an interesting field with applications in various fields of pure and applied sciences. For the applications and other aspects of the variational inequalities, see Noor et al.[12, 21] and references therein. Pal and Wong [23] provided the application of exponentially convex functions in information, optimization, statistical theory and related areas. For other aspects of exponentially convex functions and their generalizations, see [2, 5, 12, 14, 16, 18, 19, 21–23, 33].

It is worth mentioning that the exponentially convex functions and m-convex functions are clearly two different generalizations of the convex functions. It is natural to unify these classes. Motivated by these facts, Rashid et al [33] introduced the exponentially m-convex functions and derived some Hermite-Hadamrd type inequalities. These integral inequalities can be used to obtain the upper and lower bounds for the integrand, which have applications in material sciences and numerical analysis. Petrović's [25] derives some integral inequalities for convex functions, which are known as Petrović's type inequalities. For the applications and other aspects of Petrović's inequalities, see [10, 24–27, 29, 31]. In this paper, we introduce some new concepts of coordinated exponentially m-convex functions. We derive Petrović's type inequality for exponentially m-convex and coordinated exponentially m-convex functions. The Lagrange and Cauchy mean value results for the exponentially m-convex functions are derived. Several important cases are discussed as applications of the obtained results. We expect that the ideas and techniques of this paper may be staring point for further research in this areas.

2. PRELIMINARIES

In this section, we recall the basic definitions and concepts of the exponentially convex functions.

Definition 2.1. *A nonempty set $\Omega \subseteq \mathbb{R}$ is convex, if*

$$\sigma u + (1-\sigma)v \in \Omega, \quad \forall u, v \in \Omega, \quad \sigma \in [0,1].$$

Definition 2.2. *A function $\mathcal{F} : \Omega \to \mathbb{R}$ is convex, if*

$$\mathcal{F}(\sigma u + (1-\sigma)v) \leq \sigma \mathcal{F}(u) + (1-\sigma)\mathcal{F}(v), \quad \forall u, v \in \Omega, \quad \sigma \in [0,1].$$

Toader [34] introduced m-convex functions as follows:

Definition 2.3. *The function $f : [0, b] \to \mathbb{R}, b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have*

$$f(\sigma u + m(1-\sigma)v) \leqslant \sigma f(u) + m(1-\sigma)f(v), \quad \forall u, v \in [0, b], \sigma \in [0, 1].$$

Remark 2.1. *One can note that the notion of m-convexity reduces to convexity for $m = 1$. For $m = 0$, we obtain starshaped functions.*

Noor et al. [12, 14] introduced exponentially convex function as follows:

Definition 2.4. *A function $\mathcal{F}$ is called exponentially convex on Ω, if*

$$e^{\mathcal{F}(u+\sigma(v-u))} \leq (1-\sigma)e^{\mathcal{F}(u)} + \sigma e^{\mathcal{F}(v)}, \quad \forall u, v \in \Omega, \ \sigma \in [0, 1], \tag{2.1}$$

which can be written in the following equivalent form, which is due to Avriel [4].

Definition 2.5. *A function $\mathcal{F}$ is called exponentially convex function on Ω, if*

$$\mathcal{F}(u + \sigma(v-u)) \leq \log[(1-\sigma)e^{\mathcal{F}(u)} + \sigma e^{\mathcal{F}(v)}], \quad \forall u, v \in \Omega, \ \sigma \in [0, 1]. \tag{2.2}$$

For the applications of the exponentially convex functions in information theory and mathematical programming, see Antczak [3] and Alirezaei and Mathar [2].

Rashid el al.[33] introduced exponentially $m-$convex function as follows:

Definition 2.6. *A function $\mathcal{F} : \Omega \to \mathbb{R}$ on an interval of real line is exponentially m-convex, where $m \in (0, 1]$, if*

$$e^{\mathcal{F}(\sigma u+m(1-\sigma)v)} \leq \sigma e^{\mathcal{F}(u)} + m(1-\sigma)e^{\mathcal{F}(v)}, \forall u, v \in \Omega, \sigma \in [0, 1]. \tag{2.3}$$

From now onwards, we take $I_1 = [a_1, b_1]$ and $I_2 = [c_1, d_1]$ as intervals in $\mathbb{R}$.
Dragomir [8] introduced coordinated convex functions as follows:

Definition 2.7. *Let us consider the bidimensional interval $\Delta = I_1 \times I_2$ in $\mathbb{R}^2$ with $a_1 < b_1$ and $c_1 < d_1$. Also, let $\mathcal{F} : I_1 \times I_2 \to \mathbb{R}$ be a mapping. Define partial mappings as*

$$\mathcal{F}_v : I_1 \to \mathbb{R} \text{ defined by } \mathcal{F}_v(x) = \mathcal{F}(x, v) \tag{2.4}$$

and

$$\mathcal{F}_u : I_2 \to \mathbb{R} \text{ defined by } \mathcal{F}_u(y) = \mathcal{F}(u, y). \tag{2.5}$$

The function $\mathcal{F}$ is called coordinated convex, if the partial mappings defined in (2.7) and (2.8) are convex on I_1 and I_2 respectively, for all $v \in I_2$ and $u \in I_1$.

Definition 2.8. *The function $\mathcal{F} : \Delta \to \mathbb{R}$ is convex in Δ, if*

$$\mathcal{F}(\sigma u + (1-\sigma)z_1, \sigma v + (1-\sigma)w_1) \leq \sigma\mathcal{F}(u, v) + (1-\sigma)\mathcal{F}(z_1, w_1), \tag{2.6}$$

$$\forall (u, v), (z_1, w_1) \in \Delta, \sigma \in [0, 1].$$

Farid et al.[9] introduced coordinated m-convex functions as follows:

Definition 2.9. *Let $\Delta_1 = [0, b] \times [0, d] \subset [0, \infty)^2$, then a function $f : \Delta \to \mathbb{R}$ is $m-$ convex on coordinates if the partial mappings*

$$f_v : [0, b] \to \mathbb{R} \text{ defined by } f_v(x) = f(x, v) \tag{2.7}$$

and

$$f_u : [0, d] \to \mathbb{R} \text{ defined by } f_u(y) = f(u, y) \tag{2.8}$$

are $m-$ convex on $[0, b]$ and $[0, d]$ respectively for all $v \in [0, d]$ and $u \in [0, b]$.

We now introduce the concept of exponentially $m-$convex functions on coordinates, which is the main motivation of this paper.

Definition 2.10. *Let $\mathcal{F} : \Delta_1 \to \mathbb{R}$ be a positive mapping. The function $\mathcal{F}$ is coordinated exponentially $m-$convex, if the partial mappings defined in (2.7) and (2.8) are exponentially $m-$convex on $[0, b]$ and $[0, d]$ respectively, for all $v \in [0, d]$ and $u \in [0, b]$.*

Definition 2.11. *A positive mapping $\mathcal{F} : \Delta_1 \to \mathbb{R}$ is exponentially $m-$convex in Δ_1, if*

$$e^{\mathcal{F}(\sigma u + m(1-\sigma)z_1, \sigma v + m(1-\sigma)w_1)} \leq \sigma e^{\mathcal{F}(u,v)} + m(1-\sigma)e^{\mathcal{F}(z_1,w_1)}, \tag{2.9}$$

$$\forall (u, v), (z_1, w_1) \in \Delta_1, \sigma \in [0, 1], m \in (0, 1].$$

Lemma 2.1. *Every exponentially $m-$convex mapping $\mathcal{F} : \Delta_1 \to \mathbb{R}$ is coordinated exponentially $m-$convex, but converse is not true in general.*

Proof. Let a positive mapping $\mathcal{F} : \Delta_1 \to \mathbb{R}$ be an exponentially $m-$convex in Δ_1. Also, let $\mathcal{F}_u : [0, d] \to \mathbb{R}$ defined as $\mathcal{F}_u(v_1) := f(u, v_1)$. Then

$$\begin{aligned}
e^{\mathcal{F}_u(\sigma v_1 + m(1-\sigma)w_1)} &= e^{\mathcal{F}(u, \sigma v_1 + m(1-\sigma)w_1)} \\
&= e^{\mathcal{F}(\sigma u + m(1-\sigma)z_1, \sigma v_1 + m(1-\sigma)w_1)} \\
&\leq \sigma e^{\mathcal{F}(u,v_1)} + m(1-\sigma)e^{\mathcal{F}(z_1,w_1)} \\
&= \sigma e^{\mathcal{F}_u(v_1)} + m(1-\sigma)e^{\mathcal{F}_{z_1}(w_1)}, \ \ \forall \sigma \in [0, 1], v_1, w_1 \in [0, d],
\end{aligned}$$

which shows the exponentially $m-$convexity of $\mathcal{F}_u$.

Similarly, one can show the exponentially $m-$convexity of $\mathcal{F}_v$.

Now, consider the positive mapping $\mathcal{F}_0 : [0,1]^2 \to [0,\infty)$ given by $e^{\mathcal{F}_0(u,v_1)} = uv_1$.

Clearly $\mathcal{F}$ is coordinated exponentially $m-$convex. But it is not exponentially $m-$convex on $[0,1]^2$.

Indeed, if $(u,0),(0,w_1) \in [0,1]^2$ and $\sigma \in [0,1]$. Then

$$e^{\mathcal{F}(\sigma(u,0)+m(1-\sigma)(0,w_1))} = e^{\mathcal{F}(\sigma u, m(1-\sigma)w_1)} = m\sigma(1-\sigma)uw_1$$

and

$$\sigma e^{\mathcal{F}(u,0)} + m(1-\sigma)e^{\mathcal{F}(0,w_1)} = 0.$$

Thus, $\forall\ \sigma \in (0,1)$, $u, w_1 \in (0,1)$, one has

$$e^{\mathcal{F}(\sigma(u,0)+m(1-\sigma)(0,w_1))} > \sigma e^{\mathcal{F}(u,0)} + m(1-\sigma)e^{\mathcal{F}(0,w_1)},$$

which shows that $\mathcal{F}$ is not exponentially $m-$convex. □

Petrović [25] derived some inequality for convex functions.

Theorem 2.1. *Let $(u_i, i = 1,2,...,n)$ be non-negative n-tuples and $(p_j, j = 1,2,...,n)$ be positive n-tuples such that $\sum_{j=1}^{n} p_j \geq 1$,*

$$\sum_{\kappa=1}^{n} p_\kappa u_\kappa \in [0,a_1] \text{ and } \sum_{\kappa=1}^{n} p_\kappa u_\kappa \geq u_l \text{ for each } l = 1,...,n.$$

Consider the function $\mathcal{F}$ is convex on $[0,a_1]$, then

$$\sum_{\kappa=1}^{n} p_\kappa \mathcal{F}(u_\kappa) \leq \mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa\right) + \left(\sum_{\kappa=1}^{n} p_\kappa - 1\right)\mathcal{F}(0). \tag{2.10}$$

Rehman et al. [31] gave the Petrović's inequality on coordinated convex functions.

Theorem 2.2. *Let $(u_i, i = 1,2,...,n)$ and $(v_j, j = 1,2,...,n)$ be non-negative n-tuples and $(p_k, k = 1,...,n)$ and $(q_l, l = 1,...,n)$ be positive n-tuples such that*

$$\sum_{\kappa=1}^{n} p_\kappa \geq 1, \quad 0 \neq \sum_{\kappa=1}^{n} p_\kappa u_\kappa \geq u_j \text{ for every } j = 1,2,...,n,$$

and

$$\sum_{l=1}^{n} q_l \geq 1, \quad 0 \neq \sum_{l=1}^{n} q_l v_l \geq v_j \text{ for every } i = 1,2,...,n.$$

Let $\mathcal{F}:[0,a_1)\times[0,b_1)\to\mathbb{R}$ be a convex on coordinates, then

$$\sum_{\kappa=1}^{n}\sum_{l=1}^{n}p_\kappa q_l\mathcal{F}(u_\kappa,v_l)\leqslant\mathcal{F}\left(\sum_{\kappa=1}^{n}p_\kappa u_\kappa,\sum_{l=1}^{n}q_lv_l\right)+\left(\sum_{l=1}^{n}q_l-1\right)\mathcal{F}\left(\sum_{\kappa=1}^{n}p_\kappa u_\kappa,0\right)$$
$$+\left(\sum_{\kappa=1}^{n}p_\kappa-1\right)\left(\mathcal{F}\left(0,\sum_{l=1}^{n}q_lv_l\right)+\left(\sum_{l=1}^{n}q_l-1\right)\mathcal{F}(0,0)\right). \tag{2.11}$$

3. Main Results

In this section, we prove an important lemma, which plays a key role for proving our next results.

Lemma 3.1. *Let $(u_i, i=1,2,...,n)$ be non-negative n-tuples and $(p_j, j=1,2,...,n)$ be positive n-tuples such that $\sum_{j=1}^{n}p_j\geq 1$, $\theta\in[0,a_1]$,*

$$\sum_{\kappa=1}^{n}p_\kappa u_\kappa\in[0,a_1] \text{ and } \sum_{\kappa=1}^{n}p_\kappa u_\kappa\geq u_l>m\theta \text{ for each } l=1,...,n.$$

Suppose a positive function $\mathcal{F}:[0,a_1]\to\mathbb{R}$ is exponentially $m-$convex. If $\frac{e^{\mathcal{F}(u)}}{u-m\theta}$ is increasing on $[0,a_1]$, then

$$e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n}p_\kappa u_\kappa\right)}\geq\frac{\left(\sum\limits_{\kappa=1}^{n}p_\kappa u_\kappa-m\theta\right)}{\sum\limits_{\kappa=1}^{n}p_\kappa(u_\kappa-m\theta)}\sum_{\kappa=1}^{n}p_\kappa e^{\mathcal{F}(u_\kappa)}. \tag{3.1}$$

Proof. Since $\sum_{\kappa=1}^{n}p_\kappa u_\kappa\geq u_l>m\theta$ for all $l=1,...,n$ and $\frac{e^{\mathcal{F}(u)}}{u-m\theta}$ is increasing on $[0,a_1]$, we have

$$\frac{e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n}p_\kappa u_\kappa\right)}}{\left(\sum\limits_{\kappa=1}^{n}p_\kappa u_\kappa-m\theta\right)}\geq\frac{e^{\mathcal{F}(u_\kappa)}}{(u_\kappa-m\theta)}.$$

This implies

$$(u_\kappa-m\theta)e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n}p_\kappa u_\kappa\right)}\geq\left(\sum_{\kappa=1}^{n}p_\kappa u_\kappa-m\theta\right)e^{\mathcal{F}(u_\kappa)}.$$

Multiplying above inequality by p_κ and taking sum for $\kappa=1,...,n$, one has

$$\sum_{\kappa=1}^{n}p_\kappa(u_\kappa-m\theta)e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n}p_\kappa u_\kappa\right)}\geq\left(\sum_{\kappa=1}^{n}p_\kappa u_\kappa-m\theta\right)\sum_{l=1}^{n}p_\kappa e^{\mathcal{F}(u_\kappa)},$$

from which, one has the required result. □

We now derive the Petrović's type inequality for exponentially $m-$convex functions.

Theorem 3.1. *Let $(u_i, i = 1, 2, ..., n)$ be non-negative n-tuples and $(p_j, j = 1, 2, ..., n)$ be positive n-tuples such that $\sum_{j=1}^{n} p_j \geq 1$, $\theta \in [0, a_1]$,*

$$\sum_{\kappa=1}^{n} p_\kappa u_\kappa \in [0, a_1] \text{ and } \sum_{\kappa=1}^{n} p_\kappa u_\kappa \geq u_l > \theta \text{ for each } l = 1, ..., n.$$

Let a positive function $\mathcal{F} : [0, \infty) \to \mathbb{R}$ be an exponentially $m-$convex. Then

$$\sum_{l=1}^{n} p_l e^{\mathcal{F}(u_\kappa)} \leq A e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa\right)} + m\left(\sum_{l=1}^{n} p_l - A\right) e^{\mathcal{F}(\theta)} \quad , \tag{3.2}$$

where

$$A = \left(\frac{\sum\limits_{l=1}^{n} p_l(u_l - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta}\right).$$

Proof. Let a function $\mathcal{F}$ be an exponentially $m-$convex and

$$\aleph_{(x)} = \frac{e^{f(x)} - me^{f(a)}}{x - ma}.$$

We take $y > x > ma$ and $x = ma + \sigma(y - ma)$, where $\sigma \in (0, 1)$. Then

$$\begin{aligned} \aleph_{(x)} &= \frac{e^{f(\sigma y + m(1-\sigma)a)} - me^{f(a)}}{\sigma y + m(1-\sigma)a - ma)} \\ &\leq \frac{\sigma e^{f(y)} + m(1-\sigma)e^{f(a)} - me^{f(a)}}{\sigma(y - ma)} \\ &= \frac{e^{f(y)} - me^{f(a)}}{y - ma}. \end{aligned}$$

This implies

$$\aleph_{(x)} \leq \aleph_{(y)}.$$

Hence $\aleph_{(x)}$ is increasing on $[0, a]$.

As we have proved that, if $\mathcal{F}$ is exponentially $m-$convex, then $\frac{e^{f(x)} - me^{f(a)}}{x - ma}$ is increasing for $x > m\theta$.

Substituting $e^{f(x)}$ by $e^{f(x)} - me^{f(\theta)}$ in Lemma 3.1, one has

$$e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa\right)} - me^{\mathcal{F}(\theta)} \geq \frac{\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta\right)}{\sum\limits_{l=1}^{n} p_l(u_l - m\theta)} \sum_{l=1}^{n} p_l \left(e^{\mathcal{F}(u_\kappa)} - me^{\mathcal{F}(\theta)}\right).$$

This gives us

$$\frac{\sum\limits_{l=1}^{n} p_l(u_l - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta}\left(e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa\right)} - me^{\mathcal{F}(\theta)}\right) \geq \sum_{l=1}^{n} p_l e^{\mathcal{F}(u_\kappa)} - m\sum_{l=1}^{n} p_l e^{\mathcal{F}(\theta)}.$$

This leads to

$$\frac{\sum_{l=1}^{n} p_l(u_l - m\theta)}{\sum_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa\right)} \geq \sum_{l=1}^{n} p_l e^{\mathcal{F}(u_\kappa)} - m\sum_{l=1}^{n} p_l e^{\mathcal{F}(\theta)}$$
$$+m\left(\frac{\sum_{l=1}^{n} p_l(u_l - m\theta)}{\sum_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta}\right) e^{\mathcal{F}(\theta)}.$$

Finally, we have

$$\frac{\sum_{l=1}^{n} p_l(u_l - m\theta)}{\sum_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa\right)} \geq \sum_{l=1}^{n} p_l e^{\mathcal{F}(u_\kappa)} -$$
$$m\left(\sum_{l=1}^{n} p_l - \frac{\sum_{l=1}^{n} p_l(u_l - m\theta)}{\sum_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta}\right) e^{\mathcal{F}(\theta)},$$

which is the required result. □

If $\theta = 0$, then Theorem 3.1 reduces to the following new result. It can be considered as Petrović's type inequality for exponentially $m-$convex function.

Theorem 3.2. *Let the conditions given in Theorem 3.1 be satisfied and let a positive function $\mathcal{F} : [0, \infty) \to \mathbb{R}$ be an exponentially $m-$convex. Then*

$$\sum_{l=1}^{n} p_l e^{\mathcal{F}(u_\kappa)} \leq e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa\right)} + m\left(\sum_{l=1}^{n} p_l - 1\right) e^{\mathcal{F}(0)}. \tag{3.3}$$

If $m = 1$, then Theorem 3.1 reduces to the following new result. It can be viewed as a new generalized Petrović's type inequality for exponentially convex function.

Theorem 3.3. *Let the conditions given in Theorem 3.1 be satisfied.*
Also, let a positive function $\mathcal{F} : [0, \infty) \to \mathbb{R}$ be an exponentially convex. Then

$$\begin{aligned}\sum_{l=1}^{n} p_l e^{\mathcal{F}(u_\kappa)} \leq &\left(\frac{\sum_{l=1}^{n} p_l(u_l - \theta)}{\sum_{\kappa=1}^{n} p_\kappa u_\kappa - \theta}\right) e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa\right)} \\ &+\left(\sum_{l=1}^{n} p_l - \left(\frac{\sum_{l=1}^{n} p_l(u_l - \theta)}{\sum_{\kappa=1}^{n} p_\kappa u_\kappa - \theta}\right)\right) e^{\mathcal{F}(\theta)}.\end{aligned} \tag{3.4}$$

If $m = 1$ and $\theta = 0$, then Theorem 3.1 reduces to the following new result. It can be considered as Petrović's type inequality for exponentially convex function.

Theorem 3.4. *Let the conditions given in Theorem 3.1 be satisfied and let a positive function $\mathcal{F} : [0, \infty) \to \mathbb{R}$ be an exponentially convex. Then*

$$\sum_{l=1}^{n} p_l e^{\mathcal{F}(u_\kappa)} \leq e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa\right)} + \left(\sum_{l=1}^{n} p_l - 1\right) e^{\mathcal{F}(0)}. \tag{3.5}$$

Now, we derive the generalized Petrović's type inequality for coordinated exponentially $m-$convex functions.

Theorem 3.5. *Let $(u_i, i = 1, 2, ..., n)$ and $(v_j, j = 1, 2, ..., n)$ be non-negative n-tuples and $(p_k, k = 1, 2, ..., n)$ and $(q_l, l = 1, ..., n)$ be positive n-tuples such that $\theta \in [0, a_1]$, $\sum_{\kappa=1}^{n} p_\kappa \geq 1$, $\sum_{l=1}^{n} q_l \geq 1$,*

$$\sum_{\kappa=1}^{n} p_\kappa u_\kappa \in [0, a_1), \quad 0 \neq \sum_{\kappa=1}^{n} p_\kappa u_\kappa \geq u_j > \theta \text{ for every } j = 1, 2, ..., n$$

and

$$\sum_{l=1}^{n} q_l v_l \in [0, b_1), \quad 0 \neq \sum_{l=1}^{n} q_l v_l \geq v_i > \theta \text{ for every } i = 1, 2, ..., n.$$

Let a positive function $\mathcal{F} : [0, \infty)^2 \to \mathbb{R}$ be coordinated exponentially $m-$convex function. Then

$$\begin{aligned} \sum_{\kappa=1}^{n} \sum_{l=1}^{n} p_\kappa q_l e^{\mathcal{F}(u_j, v_l)} \leq A &\left\{ B e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, \sum\limits_{l=1}^{n} q_l v_l\right)} + m \left(\sum_{l=1}^{n} q_l - B\right) e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, \theta\right)} \right\} \\ + m \left(\sum_{\kappa=1}^{n} p_\kappa - A\right) &\left\{ B e^{f\left(\theta, \sum\limits_{l=1}^{n} q_l v_l\right)} + m \left(\sum_{l=1}^{n} q_l - B\right) e^{f(\theta, \theta)} \right\}, \end{aligned} \tag{3.6}$$

where

$$A = \left(\frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) \tag{3.7}$$

and

$$B = \left(\frac{\sum\limits_{l=1}^{n} q_l (v_l - m\theta)}{\sum\limits_{l=1}^{n} q_l v_l - m\theta} \right). \tag{3.8}$$

Proof. Consider the partial mappings $\mathcal{F}_u : [0, a_1] \to \mathbb{R}$ and $\mathcal{F}_v : [0, b_1] \to \mathbb{R}$ defined by $\mathcal{F}_u(v_1) = \mathcal{F}(u, v)$ and $\mathcal{F}_v(u) = \mathcal{F}(u, v)$.

As $\mathcal{F}$ is coordinated exponentially $m-$convex on $[0, \infty)^2$. Therefore, the partial mapping $\mathcal{F}_v$ is exponentially $m-$convex on $[0, b_1]$. By Theorem 3.1, we have

$$\sum_{\kappa=1}^{n} p_\kappa e^{\mathcal{F}_v(u_j)} \leq \left(\frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) e^{\mathcal{F}_v\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa\right)}$$
$$+m \left(\sum_{\kappa=1}^{n} p_\kappa - \frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) e^{\mathcal{F}_v(\theta)}.$$

This is equivalent to

$$\sum_{\kappa=1}^{n} p_\kappa e^{\mathcal{F}(u_j,v)} \leq \left(\frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, v\right)}$$
$$+m \left(\sum_{\kappa=1}^{n} p_\kappa - \frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) e^{\mathcal{F}(\theta,v)}.$$

By setting $v = v_l$, we get

$$\sum_{\kappa=1}^{n} p_\kappa e^{\mathcal{F}(u_j,v_l)} \leq \left(\frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, v_l\right)}$$
$$+m \left(\sum_{\kappa=1}^{n} p_\kappa - \frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) e^{\mathcal{F}(\theta,v_l)}.$$

Multiplying above inequality by q_l and taking sum for $l = 1, ..., n$, one has

$$\begin{aligned} \sum_{\kappa=1}^{n} \sum_{l=1}^{n} p_\kappa q_l e^{\mathcal{F}(u_j,v_l)} \leq{}& \left(\frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) \sum_{l=1}^{n} q_l e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, v_l\right)} + \\ & m \left(\sum_{\kappa=1}^{n} p_\kappa - \frac{\sum\limits_{\kappa=1}^{n} p_\kappa (u_\kappa - m\theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - m\theta} \right) \sum_{l=1}^{n} q_l e^{\mathcal{F}(\theta,v_l)}. \end{aligned} \tag{3.9}$$

Now again by Theorem 3.1, we have

$$\sum_{l=1}^{n} q_l e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, v_l\right)} \leq \left(\frac{\sum\limits_{l=1}^{n} q_l (v_l - m\theta)}{\sum\limits_{l=1}^{n} q_l v_l - m\theta} \right) e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, \sum\limits_{l=1}^{n} q_l v_l\right)} +$$
$$m \left(\sum_{l=1}^{n} q_l - \frac{\sum\limits_{l=1}^{n} q_l (v_l - m\theta)}{\sum\limits_{l=1}^{n} q_l v_l - m\theta} \right) e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, \theta\right)}$$

and

$$\sum_{l=1}^{n} q_l e^{f(\theta,v_j)} \leq \left(\frac{\sum\limits_{l=1}^{n} q_l(v_l - m\theta)}{\sum\limits_{l=1}^{n} q_l v_l - m\theta} \right) e^{f\left(\theta, \sum\limits_{l=1}^{n} q_l v_l\right)}$$
$$+m \left(\sum_{l=1}^{n} q_l - \frac{\sum\limits_{l=1}^{n} q_l(v_l - m\theta)}{\sum\limits_{l=1}^{n} q_l v_l - m\theta} \right) e^{f(\theta,\theta)}.$$

Putting these values in inequality (3.9) and using the notations given in (3.7) and (3.8), we get the required result. □

If $m = 1$, then Theorem 3.5 reduces to the following new result.

Theorem 3.6. *Let the conditions given in 3.5 be satisfied. Also, let a positive function $\mathcal{F} : [0,\infty)^2 \to \mathbb{R}$ be coordinated exponentially $m-$convex function, then*

$$\begin{aligned}\sum_{\kappa=1}^{n}\sum_{l=1}^{n} p_\kappa q_l e^{\mathcal{F}(u_j,v_l)} &\leq C \left\{ D e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, \sum\limits_{l=1}^{n} q_l v_l\right)} + \left(\sum_{l=1}^{n} q_l - D\right) e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, \theta\right)} \right\} \\ &+ \left(\sum_{\kappa=1}^{n} p_\kappa - C\right) \left\{ D e^{f\left(\theta, \sum\limits_{l=1}^{n} q_l v_l\right)} + \left(\sum_{l=1}^{n} q_l - D\right) e^{f(\theta,\theta)} \right\},\end{aligned} \tag{3.10}$$

where

$$C = \left(\frac{\sum\limits_{\kappa=1}^{n} p_\kappa(u_\kappa - \theta)}{\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa - \theta} \right)$$

and

$$D = \left(\frac{\sum\limits_{l=1}^{n} q_l(v_l - \theta)}{\sum\limits_{l=1}^{n} q_l v_l - \theta} \right).$$

If $\theta = 0$, then Theorem 3.5 reduces to the following new result.

Theorem 3.7. *Let the conditions given in Theorem 3.5 be satisfied. If $\mathcal{F} : [0,\infty)^2 \to \mathbb{R}$ be coordinated exponentially $m-$convex, then*

$$\begin{aligned}\sum_{\kappa=1}^{n}\sum_{l=1}^{n} p_\kappa q_l e^{\mathcal{F}(u_j,v_l)} &\leq \left\{ e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, \sum\limits_{l=1}^{n} q_l v_l\right)} + m \left(\sum_{l=1}^{n} q_l - 1\right) e^{\mathcal{F}\left(\sum\limits_{\kappa=1}^{n} p_\kappa u_\kappa, 0\right)} \right\} \\ &+ m \left(\sum_{\kappa=1}^{n} p_\kappa - 1\right) \left\{ e^{f\left(0, \sum\limits_{l=1}^{n} q_l v_l\right)} + m \left(\sum_{l=1}^{n} q_l - 1\right) e^{f(0,0)} \right\}.\end{aligned} \tag{3.11}$$

If $\theta = 0$ and $m = 1$, then Theorem 3.5 reduces to the following new result.

Theorem 3.8. *Let the conditions given in Theorem 3.5 be satisfied. If $\mathcal{F} : [0,\infty)^2 \to \mathbb{R}$ be coordinated exponentially convex, then*

$$\begin{aligned}\sum_{\kappa=1}^{n}\sum_{l=1}^{n} p_\kappa q_l e^{\mathcal{F}(u_j,v_l)} \le & \left\{ e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa, \sum_{l=1}^{n} q_l v_l\right)} + \left(\sum_{l=1}^{n} q_l - 1\right) e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa, 0\right)} \right\} \\ & + \left(\sum_{\kappa=1}^{n} p_\kappa - 1\right) \left\{ e^{f\left(0, \sum_{l=1}^{n} q_l v_l\right)} + \left(\sum_{l=1}^{n} q_l - 1\right) e^{f(0,0)} \right\}.\end{aligned} \tag{3.12}$$

By considering non-negative difference of (3.3), we define the following functional.

$$\mathcal{P}(e^{\mathcal{F}}) = e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa\right)} + m\left(\sum_{l=1}^{n} p_l - 1\right) e^{\mathcal{F}(0)} - \sum_{l=1}^{n} p_l e^{\mathcal{F}(u_\kappa)}. \tag{3.13}$$

Also by considering non-negative difference of (3.11), we define the following functional.

$$\begin{aligned}\Upsilon(e^{\mathcal{F}}) = & \left\{ e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa, \sum_{l=1}^{n} q_l v_l\right)} + m\left(\sum_{l=1}^{n} q_l - 1\right) e^{\mathcal{F}\left(\sum_{\kappa=1}^{n} p_\kappa u_\kappa, 0\right)} \right\} \\ & + m\left(\sum_{\kappa=1}^{n} p_\kappa - 1\right) \left\{ e^{f\left(0, \sum_{l=1}^{n} q_l v_l\right)} + m\left(\sum_{l=1}^{n} q_l - 1\right) e^{f(0,0)} \right\} - \sum_{\kappa=1}^{n}\sum_{l=1}^{n} p_\kappa q_l e^{\mathcal{F}(u_j,v_l)}.\end{aligned} \tag{3.14}$$

We need the following lemma.

Lemma 3.2. *Let a positive function $\mathcal{F} : [0, b_1] \to \mathbb{R}$ be an exponentially $m-$convex such that*

$$n_1 \leqslant \frac{(u - ma_1)e^{\mathcal{F}(u)}\mathcal{F}'(u) - e^{\mathcal{F}(u)} + me^{\mathcal{F}(a_1)}}{u^2 - 2ma_1 u + ma_1^2} \leqslant N_1,$$

$\forall\, u \in [0, b_1]\backslash\{a_1\}$ and $a_1 \in (0, b_1)$.

Let $\gamma_1, \gamma_2 : [0, b_1] \to \mathbb{R}$ be positive functions defined as

$$\gamma_1(u) = \log[N_1 u^2 - e^{\mathcal{F}(u)}]$$

and

$$\gamma_2(u) = \log[e^{\mathcal{F}(u)} - n_1 u^2],$$

then γ_1 and γ_2 are exponentially $m-$convex on $[0, b_1]$.

Proof. Suppose

$$P_{\gamma_1}(u) = \frac{e^{\gamma_1(u)} - me^{\gamma_1(a_1)}}{u - ma_1}$$

$$= \frac{N_1 u^2 - e^{\mathcal{F}(u)} - mN_1 {a_1}^2 + me^{\mathcal{F}(a_1)}}{u - ma_1}$$

$$= \frac{N_1(u^2 - m{a_1}^2)}{u - ma_1} - \frac{e^{\mathcal{F}(u)} - me^{\mathcal{F}(a_1)}}{u - ma_1}.$$

By differentiating with respect to u, one has

$$P'_{\gamma_1}(u) = N_1 \frac{(u - ma_1)2u - (u^2 - ma_1^2)}{(u - ma_1)^2} - \frac{(u - ma_1)e^{\mathcal{F}(u)}\mathcal{F}'(u) - e^{\mathcal{F}(u)} + me^{\mathcal{F}(a_1)}}{(u - ma_1)^2}.$$

Since

$$u^2 - 2ma_1 u + m^2 a_1^2 - m^2 a_1^2 + ma_1^2 = (u - ma_1)^2 - m(m-1)a_1^2 > 0,$$

by the given condition, one has

$$N_1(u^2 - ma_1 2u + ma_1^2) \geq (u - ma_1)e^{\mathcal{F}(u)}\mathcal{F}'(u) - e^{\mathcal{F}(u)} + me^{\mathcal{F}(a_1)}.$$

This implies

$$N_1 \frac{u^2 - 2ma_1 u + ma_1^2}{(u - ma_1)^2} \geq \frac{(u - ma_1)e^{\mathcal{F}(u)}\mathcal{F}(u) - e^{\mathcal{F}(u)} + me^{\mathcal{F}(a_1)}}{(u - ma_1)^2},$$

$$N_1 \frac{u^2 - 2ma_1 u + ma_1^2}{(u - ma_1)^2} - \frac{(u - ma_1)e^{\mathcal{F}(u)}\mathcal{F}(u) - e^{\mathcal{F}(u)} + me^{\mathcal{F}(a_1)}}{(u - ma_1)^2} \geq 0.$$

This implies

$$P'_{\gamma_1}(u) \geq 0, \ \ \forall u \in [0, a_1) \cup (a_1, b_1].$$

Similarly, one can show that

$$P'_{\gamma_2}(u) \geq 0, \ \ \forall u \in [0, a_1) \cup (a_1, b_1].$$

This implies that P_{γ_1} and P_{γ_2} are increasing on $u \in [0, a_1) \cup (a_1, b_1]$ for all $a \in (0, b_1)$.
Hence by (**??**), $\gamma_1(u)$ and $\gamma_2(u)$ are exponentially $m-$convex in $[0, b_1]$. □

Here we prove the mean value theorems related to functional defined for Petrović's inequality for exponentially $m-$convex functions.

Theorem 3.9. *Let $(u_1, ..., u_n) \in [0, b_1]$, and $(p_1, ..., p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k u_k \geq u_j$ for each $j = 1, 2, ..., n$.*
Also let $\phi(u) = \log u^2$.
If a positive exponentially $m-$convex function $\mathcal{F} \in C^1([0, b_1])$, then there exist $\gamma \in (0, b_1)$ such that

$$\mathcal{P}(e^{\mathcal{F}}) = \frac{(\gamma - ma_1)e^{\mathcal{F}(\gamma)}\mathcal{F}'(\gamma) - e^{\mathcal{F}(\gamma)} + me^{\mathcal{F}(a_1)}}{(\gamma^2 - 2ma_1\gamma + ma_1^2)} \mathcal{P}(e^{\phi}), \tag{3.15}$$

provided that $\mathcal{P}(e^{\phi})$ is non zero and $a \in (0, b_1)$.

Proof. As $\mathcal{F} \in C^1([0, b_1])$, so there exist real numbers n_1 and N_1 such that

$$n_1 \leqslant \frac{(u - ma_1)e^{\mathcal{F}(u)}\mathcal{F}'(u) - e^{\mathcal{F}(u)} + me^{\mathcal{F}(a_1)}}{(u^2 - 2ma_1u + ma_1^2)} \leqslant N_1, \ \ \forall u \in [0, b_1] \text{ and } a_1 \in (0, b_1).$$

Consider the functions γ_1 and γ_2 defined in Lemma 3.2.

As γ_1 is exponentially $m-$convex in $[0, b_1]$, so

$$\mathcal{P}(e^{\gamma_1}) \geq 0,$$

that is

$$\mathcal{P}\left(N_1u^2 - e^{\mathcal{F}(u)}\right) \geq 0,$$

this gives

$$N_1\mathcal{P}(e^{\phi}) \geq \mathcal{P}(e^{\mathcal{F}}). \tag{3.16}$$

Similarly γ_2 is exponentially $m-$convex $[0, b_1]$, therefore one has

$$n_1\mathcal{P}(\phi) \leqslant \mathcal{P}(e^{\mathcal{F}}). \tag{3.17}$$

By assumption $\mathcal{P}(e^{\phi})$ is non zero, combining inequalities (3.16) and (3.17), one has

$$n_1 \leqslant \frac{\mathcal{P}(e^{\mathcal{F}})}{\mathcal{P}(e^{\phi})} \leqslant N_1.$$

Hence there exist $v \in (0, b_1)$ such that

$$\frac{\mathcal{P}(e^{\mathcal{F}})}{\mathcal{P}(e^{\phi})} = \frac{(\gamma - ma_1)e^{\mathcal{F}(\gamma)}\mathcal{F}'(\gamma) - e^{\mathcal{F}(\gamma)} + me^{\mathcal{F}(a_1)}}{(\gamma^2 - 2ma_1\gamma + ma_1^2)},$$

which is the required result. □

If we take $m = 1$, then Theorem 3.9 reduces to the following result.

Theorem 3.10. *Let the conditions given in Theorem 3.9 be satisfied. If $\mathcal{F} \in C^1([0, b_1])$ is a positive exponentially convex function, then there exist $\gamma \in (0, b_1)$ such that*

$$\mathcal{P}(e^{\mathcal{F}}) = \frac{(\gamma - a_1)e^{\mathcal{F}(\gamma)}\mathcal{F}'(\gamma) - e^{\mathcal{F}(\gamma)} + e^{\mathcal{F}(a_1)}}{(\gamma - a_1)^2}\mathcal{P}(e^{\phi}), \tag{3.18}$$

provided that $\mathcal{P}(e^{\phi})$ is non zero and $a \in (0, b_1)$.

Theorem 3.11. *Let the conditions given in Theorem 3.9 be satisfied. Suppose the positive exponentially $m-$convex functions $\mathcal{F}_1, \mathcal{F}_2 \in C^1([0, b_1])$, then there exist $\gamma \in (0, b_1)$ such that*

$$\frac{\mathcal{P}(e^{\mathcal{F}_1})}{\mathcal{P}(e^{\mathcal{F}_2})} = \frac{(\gamma - ma_1)e^{\mathcal{F}_1(\gamma)}\mathcal{F}_1'(\gamma) - e^{\mathcal{F}_1(\gamma)} + me^{\mathcal{F}_1(a)}}{(\gamma - ma_1)e^{\mathcal{F}_2(\gamma)}\mathcal{F}_2{}'(\gamma) - e^{\mathcal{F}_2(\gamma)} + me^{\mathcal{F}_2(a)}},$$

provided that the denominators are non-zero and $a_1 \in (0, b_1)$.

Proof. Suppose $k \in C^1([0, b_1])$ be a function defined as

$$k = \log\left(c_1 e^{\mathcal{F}_1} - c_2 e^{\mathcal{F}_2}\right),$$

where c_1 and c_2 are defined as

$$c_1 = \mathcal{P}(e^{\mathcal{F}_2}),$$

$$c_2 = \mathcal{P}(e^{\mathcal{F}_1}).$$

Then using Theorem 3.9 with $\mathcal{F} = k$, one has

$$(\gamma - ma_1)e^{\log(c_1 e^{\mathcal{F}_1(\gamma)} - c_2 e^{\mathcal{F}_2(\gamma)})}(\log(c_1 e^{\mathcal{F}_1(\gamma)} - c_2 e^{\mathcal{F}_2(\gamma)}))' - (c_1 e^{\mathcal{F}_1(\gamma)} - c_2 e^{\mathcal{F}_2(\gamma)})$$
$$+ m(c_1 e^{\mathcal{F}_1(a)} - c_2 e^{\mathcal{F}_2(a)}) = 0,$$

this gives

$$(\gamma - ma_1)(c_1 e^{\mathcal{F}_1(\gamma)} - c_2 e^{\mathcal{F}_2(\gamma)})' - c_1 e^{\mathcal{F}_1(\gamma)} + c_2 e^{\mathcal{F}_2(\gamma)} + mc_1 e^{\mathcal{F}_1(a)}$$
$$- mc_2 e^{\mathcal{F}_2(a)} = 0,$$

that is

$$(\gamma - ma_1)(c_1 e^{\mathcal{F}_1(\gamma)}\mathcal{F}_1'(\gamma) - c_2 e^{\mathcal{F}_2(\gamma)}\mathcal{F}_2'(\gamma)) - c_1 e^{\mathcal{F}_1(\gamma)} + c_2 e^{\mathcal{F}_2(\gamma)}$$
$$+ mc_1 e^{\mathcal{F}_1(a)} - mc_2 e^{\mathcal{F}_2(a)} = 0,$$

this gives

$$(\gamma - ma_1)c_1 e^{\mathcal{F}_1(\gamma)}\mathcal{F}_1'(\gamma) - (\gamma - ma_1)c_2 e^{\mathcal{F}_2(\gamma)}\mathcal{F}_2'(\gamma) - c_1 e^{\mathcal{F}_1(\gamma)} + c_2 e^{\mathcal{F}_2(\gamma)}$$
$$+ mc_1 e^{\mathcal{F}_1(a)} - mc_2 e^{\mathcal{F}_2(a)} = 0,$$

which implies

$$c_1 \left\{(\gamma - ma_1)f_1'(\gamma) - f_1(\gamma) + mf_1(a)\right\} - c_2 \left\{(\gamma - ma_1)f_2'(\gamma) + \mathcal{F}_2(\gamma) - m\mathcal{F}_2(a)\right\} = 0$$
$$c_1 \left\{(\gamma - ma_1)e^{\mathcal{F}_1(\gamma)}\mathcal{F}_1'(\gamma) - e^{\mathcal{F}_1(\gamma)} + me^{\mathcal{F}_1(a)}\right\} =$$
$$c_2 \left\{(\gamma - ma_1)e^{\mathcal{F}_2(\gamma)}\mathcal{F}_2'(\gamma) - e^{\mathcal{F}_2(\gamma)} + me^{\mathcal{F}_2(a)}\right\}.$$

This gives

$$\frac{c_2}{c_1} = \frac{(\gamma - ma_1)e^{\mathcal{F}_1(\gamma)}\mathcal{F}_1'(\gamma) - e^{\mathcal{F}_1(\gamma)} + me^{\mathcal{F}_1(a)}}{(\gamma - ma_1)e^{\mathcal{F}_2(\gamma)}\mathcal{F}_2'(\gamma) - e^{\mathcal{F}_2(\gamma)} + me^{\mathcal{F}_2(a)}}.$$

Putting the values of c_1 and c_2, one has the required result. □

If we take $m = 1$, then Theorem 3.11 reduces to the following result.

Theorem 3.12. *Let the conditions given in Theorem 3.11 be satisfied. Suppose the positive exponentially convex functions $\mathcal{F}_1, \mathcal{F}_2 \in C^1([0, b_1])$, then there exist $\gamma \in (0, b_1)$ such that*

$$\frac{\mathcal{P}(e^{\mathcal{F}_1})}{\mathcal{P}(e^{\mathcal{F}_2})} = \frac{(\gamma - a_1)e^{\mathcal{F}_1(\gamma)}\mathcal{F}_1'(\gamma) - e^{\mathcal{F}_1(\gamma)} + e^{\mathcal{F}_1(a)}}{(\gamma - a_1)e^{\mathcal{F}_2(\gamma)}{\mathcal{F}_2}'(\gamma) - e^{\mathcal{F}_2(\gamma)} + e^{\mathcal{F}_2(a)}},$$

provided that the denominators are non-zero and $a_1 \in (0, b_1)$.

Here we state an important lemma that is helpful in proving mean value theorems related to the non-negative functional of Petroviĉ's inequality for coordinated exponentially $m-$convex functions.

Lemma 3.3. *Let $\Delta = [0, b_1] \times [0, d_1]$. Also, let $\mathcal{F} : \Delta \to \mathbb{R}$ be a positive coordinated exponentially $m-$convex function such that*

$$n_1 \leqslant \frac{(u - ma_1)e^{\mathcal{F}(u,v)}\frac{\partial}{\partial u}\mathcal{F}(u, v) - e^{\mathcal{F}(u,v)} + me^{\mathcal{F}(a_1,v)}}{(u^2 - 2ma_1u + ma_1^2)v^2} \leqslant N_1$$

and

$$n_2 \leqslant \frac{(v - mc_1)e^{\mathcal{F}(u,v)}\frac{\partial}{\partial v}\mathcal{F}(u, v) - e^{\mathcal{F}(u,v)} + me^{\mathcal{F}(u,c_1)}}{(v^2 - 2ma_1v + mc_1^2)u^2} \leqslant N_2$$

$\forall u \in [0, b_1]\backslash\{a_1\}$, $a_1 \in (0, b_1)$ *and* $v \in [0, d_1]\backslash\{c_1\}$, $c \in (0, d_1)$.

Consider the functions $\alpha_v : [0, b_1] \to \mathbb{R}$, and $\alpha_u : [0, d_1] \to \mathbb{R}$, defined as

$$\alpha(u, v) = \log[\max\{N_1, N_2\}u^2v^2 - e^{\mathcal{F}(u,v)}]$$

and

$$\beta(u, v) = \log[e^{\mathcal{F}(u,v)} - \min\{n_1, n_2\}u^2v^2].$$

Then α and β are coordinated exponentially $m-$convex.

Proof. Suppose the partial mappings $\alpha_v : [0, b_1] \to \mathbb{R}$ and $\alpha_u : [0, d_1] \to \mathbb{R}$ defined as $\alpha_v(u) := \alpha(u, v)$ for all $u \in (0, b_1]$ and $\alpha_u(v) := \alpha(u, v)$ for all $v \in (0, d]$.

$$\begin{aligned}
P_{\alpha_v}(u) &= \frac{e^{\alpha_v(u)} - me^{\alpha_v(a_1)}}{u - ma_1} \\
&= \frac{e^{\alpha(u,v)} - me^{\alpha(a_1,v)}}{u - ma_1} \\
&= \frac{e^{\log[\max\{N_1,N_2\}u^2v^2 - me^{\mathcal{F}(u,v)}]} - me^{\log[\max\{N_1,N_2\}a_1^2v^2 - e^{\mathcal{F}(a_1,v)}]}}{u - ma_1} \\
&= \frac{N_1u^2v^2 - e^{\mathcal{F}(u,v)} - mN_1a_1^2v^2 + me^{\mathcal{F}(a_1,v)}}{u - ma_1} \\
&= N_1\frac{(u^2 - ma_1^2)v^2}{u - ma_1} - \frac{e^{\mathcal{F}(u,v)} - me^{\mathcal{F}(a_1,v)}}{u - ma_1}.
\end{aligned}$$

Differentiating partially with respect to u, one has

$$\begin{aligned} P'_{\alpha_v}(u) &= N_1 v^2 \frac{(u-ma_1)2u-(u^2-ma_1^2)}{(u-ma_1)^2} \\ &- \frac{(u-ma_1)e^{\mathcal{F}(u,v)}\frac{\partial}{\partial u}\mathcal{F}(u,v)-e^{\mathcal{F}(u,v)}+me^{\mathcal{F}(a_1,v)}}{(u-ma_1)^2} \\ &= N_1 v^2 \frac{(u^2-2ma_1u+ma_1^2)}{(u-ma_1)^2} \\ &- \frac{(u-ma_1)e^{\mathcal{F}(u,v)}\frac{\partial}{\partial u}\mathcal{F}(u,v)-e^{\mathcal{F}(u,v)}+me^{\mathcal{F}(a_1,v)}}{(u-ma_1)^2} \end{aligned}$$

By the given condition, one has

$$N_1 \geq \frac{(u-ma_1)e^{\mathcal{F}(u,v)}\frac{\partial}{\partial u}\mathcal{F}(u,v)-e^{\mathcal{F}(u,v)}+me^{\mathcal{F}(a_1,v)}}{(u^2-2ma_1u+ma_1^2)v^2}.$$

Since

$$(u^2-2ma_1u+ma_1^2)v^2 > 0.$$

This implies

$$N_1\frac{(u^2-2ma_1u+ma_1^2)v^2}{(u-ma_1)^2} \geq \frac{(u-ma_1)e^{\mathcal{F}(u,v)}\frac{\partial}{\partial u}\mathcal{F}(u,v)-e^{\mathcal{F}(u,v)}+me^{\mathcal{F}(a_1,v)}}{(u-ma_1)^2},$$

$$N_1\frac{(u^2-2ma_1u+ma_1^2)v^2}{(u-ma_1)^2} - \frac{(u-ma_1)e^{\mathcal{F}(u,v)}\frac{\partial}{\partial u}\mathcal{F}(u,v)-e^{\mathcal{F}(u,v)}+me^{\mathcal{F}(a_1,v)}}{(u-ma_1)^2} \geq 0.$$

This implies

$$P'_{\alpha_v}(u) \geq 0, \ \ \forall u \in [0, ma_1) \cup (ma_1, b_1].$$

Similarly, one can show that

$$P'_{\alpha_u}(v) \geq 0, \ \ \forall u \in [0, mc_1) \cup (mc_1, d_1].$$

This ensure that P_{α_v} is increasing on $[0, ma_1) \cup (ma_1, b_1]$ for all $a_1 \in [0, b_1]$ and P_{α_u} is increasing on $[0, mc_1) \cup (mc_1, d_1]$ for all $c_1 \in [0, d_1]$.

By (**??**), α is exponentially $m-$convex. Hence by Lemma 2.1, α is coordinated exponentially $m-$convex.

Similarly, one can show that β is coordinated exponentially $m-$convex. □

Here we give mean value theorems related to the functional defined for Petroviȁ's type inequality for coordinated exponentially $m-$convex functions.

Theorem 3.13. *Let $(u_1, ..., u_n) \in [0, b_1]$, $(v_1, ..., v_n) \in [0, d_1]$ be non-negative n-tuples and $(q_1, ..., q_n)$, $(p_1, ..., p_n)$ be positive n-tuples such that*
$\sum_{k=1}^{n} p_k u_k \geq u_j$ for each $j = 1, 2, ..., n$. Also let $\varphi(u, v) = \log(u^2 v^2)$.
Let a positive coordinated exponentially $m-$convex function $\mathcal{F} \in C^1(\Delta)$, then there exist (γ_1, ζ_1) and (γ_2, ζ_2) in the interior of Δ, such that

$$\Upsilon(e^{\mathcal{F}}) = \frac{(\gamma_1 - ma)e^{\mathcal{F}(\gamma_1,\zeta_1)} \frac{\partial}{\partial u}\mathcal{F}(\gamma_1, \zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + me^{\mathcal{F}(a,\zeta_1)}}{(\gamma_1^2 - 2ma\gamma_1 + ma^2)\zeta_1^2} \Upsilon(e^{\varphi}) \tag{3.19}$$

and

$$\Upsilon(e^{\mathcal{F}}) = \frac{(\gamma_2 - ma)e^{\mathcal{F}(\gamma_2,\zeta_2)} \frac{\partial}{\partial v}\mathcal{F}(\gamma_2, \zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + me^{\mathcal{F}(a,\zeta_2)}}{(\gamma_2^2 - 2ma\gamma_2 + ma^2)\zeta_2^2} \Upsilon(e^{\varphi}), \tag{3.20}$$

provided that $\Upsilon(e^{\varphi})$ is non-zero and $a \in (0, b_1)$.

Proof. As $\mathcal{F}$ has continuous first order partial derivative in Δ, so there exist real numbers n_1, n_2, N_1 and N_2 such that

$$n_1 \leqslant \frac{(u - ma_1)e^{\mathcal{F}(u,v)} \frac{\partial}{\partial u}\mathcal{F}(u, v) - e^{\mathcal{F}(u,v)} + e^{\mathcal{F}(a,v)}}{(u^2 - 2ma_1 u + ma_1^2)v^2} \leqslant N_1$$

and

$$n_2 \leqslant \frac{(v - ma_1)e^{\mathcal{F}(u,v)} \frac{\partial}{\partial v}\mathcal{F}(u, v) - e^{\mathcal{F}(u,v)} + e^{\mathcal{F}(u,a)}}{(v^2 - 2ma_1 v + ma_1^2)u^2} \leqslant N_2,$$

$\forall u \in (0, b_1]$, $v \in (0, d]$ and $a \in (0, b_1)$.

Consider the functions α and β defined in Lemma 3.3.

As α is coordinated exponentially $m-$convex, then

$$\Upsilon(e^{\alpha}) \geq 0,$$

that is

$$\Upsilon\Big(N_1 u^2 v^2 - e^{\mathcal{F}(u,v)}\Big) \geq 0,$$

this gives

$$N_1 \Upsilon(e^{\varphi}) \geq \Upsilon(e^{\mathcal{F}}). \tag{3.21}$$

Similarly β is coordinated exponentially $m-$convex, therefore one has

$$n_1 \Upsilon(e^{\varphi}) \leqslant \Upsilon(e^{\mathcal{F}}). \tag{3.22}$$

By assumption $\Upsilon(e^{\varphi})$ is non-zero, so combining inequalities (3.21) and (3.22), one has

$$n_1 \leqslant \frac{\Upsilon(e^{\mathcal{F}})}{\Upsilon(e^{\varphi})} \leqslant N_1.$$

Hence there exists (γ_1, ζ_1) in the interior of Δ, such that

$$\Upsilon(e^{\mathcal{F}}) = \frac{(\gamma_1 - ma)e^{\mathcal{F}(\gamma_1,\zeta_1)}\frac{\partial}{\partial u}\mathcal{F}(\gamma_1,\zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + me^{\mathcal{F}(a,\zeta_1)}}{(\gamma_1^2 - 2ma\gamma_1 + ma^2)\zeta_1^2}\Upsilon(e^{\varphi}).$$

Similarly, one can show that

$$\Upsilon(e^{\mathcal{F}}) = \frac{(\gamma_2 - ma)e^{\mathcal{F}(\gamma_2,\zeta_2)}\frac{\partial}{\partial v}\mathcal{F}(\gamma_2,\zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + me^{\mathcal{F}(a,\zeta_2)}}{(\gamma_2^2 - 2ma\gamma_2 + ma^2)\zeta_2^2}\Upsilon(e^{\varphi}),$$

which is the required result. □

If we take $m = 1$, then Theorem 3.13 reduces to the following result.

Theorem 3.14. *Let the conditions given in Theorem 3.13 be satisfied. Also, let a positive coordinated exponentially convex function $\mathcal{F} \in C^1(\Delta)$, then there exist (γ_1, ζ_1) and (γ_2, ζ_2) in the interior of Δ, such that*

$$\Upsilon(e^{\mathcal{F}}) = \frac{(\gamma_1 - a)e^{\mathcal{F}(\gamma_1,\zeta_1)}\frac{\partial}{\partial u}\mathcal{F}(\gamma_1,\zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + e^{\mathcal{F}(a,\zeta_1)}}{(\gamma_1^2 - 2a\gamma_1 + a^2)\zeta_1^2}\Upsilon(e^{\varphi}) \tag{3.23}$$

and

$$\Upsilon(e^{\mathcal{F}}) = \frac{(\gamma_2 - a)e^{\mathcal{F}(\gamma_2,\zeta_2)}\frac{\partial}{\partial v}\mathcal{F}(\gamma_2,\zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + e^{\mathcal{F}(a,\zeta_2)}}{(\gamma_2^2 - 2a\gamma_2 + a^2)\zeta_2^2}\Upsilon(e^{\varphi}), \tag{3.24}$$

provided that $\Upsilon(e^{\varphi})$ is non-zero and $a \in (0, b_1)$.

Theorem 3.15. *Let the conditions given in Theorem 3.13 be satisfied. Also let the positive coordinated exponentially $m-$convex functions $\mathcal{F}_1, \mathcal{F}_2 \in C^1(\Delta)$, then there exist (γ_1, ζ_1) and (γ_2, ζ_2) in the interior of Δ, such that*

$$\frac{\Upsilon(e^{\mathcal{F}_1})}{\Upsilon(e^{\mathcal{F}_2})} = \frac{(\gamma_1 - ma)e^{\mathcal{F}(\gamma_1,\zeta_1)}\frac{\partial}{\partial u}\mathcal{F}(\gamma_1,\zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + me^{\mathcal{F}(a,\zeta_1)}}{(\gamma_2 - ma)e^{\mathcal{F}(\gamma_2,\zeta_2)}\frac{\partial}{\partial u}\mathcal{F}(\gamma_2,\zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + me^{\mathcal{F}(a,\zeta_2)}}$$

and

$$\frac{\Upsilon(e^{\mathcal{F}_1})}{\Upsilon(e^{\mathcal{F}_2})} = \frac{(\gamma_1 - ma)e^{\mathcal{F}(\gamma_1,\zeta_1)}\frac{\partial}{\partial v}\mathcal{F}(\gamma_1,\zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + me^{\mathcal{F}(a,\zeta_1)}}{(\gamma_2 - ma)e^{\mathcal{F}(\gamma_2,\zeta_2)}\frac{\partial}{\partial v}\mathcal{F}(\gamma_2,\zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + me^{\mathcal{F}(a,\zeta_2)}},$$

provided that the denominators are non-zero and $a \in (0, b_1)$.

Proof. Suppose

$$k = \log\left(c_1 e^{\mathcal{F}_1} - c_2 e^{\mathcal{F}_2}\right),$$

where c_1 and c_2 are defined as

$$c_1 = \Upsilon(e^{\mathcal{F}_2}),$$

$$c_2 = \Upsilon(e^{\mathcal{F}_1}).$$

Using Theorem 3.13 with $\mathcal{F} = k$, one has

$$(\gamma - ma)e^{\log(c_1e^{\mathcal{F}_1}-c_2e^{\mathcal{F}_2})(\gamma,\zeta)}\frac{\partial}{\partial u}\log(c_1e^{\mathcal{F}_1} - c_2e^{\mathcal{F}_2})(\gamma,\zeta) - e^{\log(c_1e^{\mathcal{F}_1}-c_2e^{\mathcal{F}_2})(\gamma,\zeta)}$$
$$+ me^{\log(c_1e^{\mathcal{F}_1}-c_2e^{\mathcal{F}_2})(a,\zeta)} = 0,$$
$$(\gamma - ma)\frac{\partial}{\partial u}(c_1e^{\mathcal{F}_1} - c_2e^{\mathcal{F}_2})(\gamma,\zeta) - (c_1e^{\mathcal{F}_1} - c_2e^{\mathcal{F}_2})(\gamma,\zeta)$$
$$+ m(c_1e^{\mathcal{F}_1} - c_2e^{\mathcal{F}_2})(a,\zeta) = 0,$$
$$(\gamma_1 - ma)c_1e^{\mathcal{F}_1(\gamma_1,\zeta_1)}\frac{\partial}{\partial u}\mathcal{F}_1(\gamma_1,\zeta_1) - (\gamma_2 - ma)c_2e^{\mathcal{F}_2(\gamma_2,\zeta_2)}\frac{\partial}{\partial u}\mathcal{F}_2(\gamma_2,\zeta_2)$$
$$- c_1e^{\mathcal{F}_1(\gamma_1,\zeta_1)} + c_2e^{\mathcal{F}_2(\gamma_2,\zeta_2)} + mc_1e^{\mathcal{F}_1(a,\zeta_1)} - mc_2e^{\mathcal{F}_2(a,\zeta_2)} = 0,$$
$$c_1\left\{(\gamma_1 - ma)e^{\mathcal{F}_1(\gamma_1,\zeta_1)}\frac{\partial}{\partial u}\mathcal{F}_1(\gamma_1,\zeta_1) - e^{\mathcal{F}_1(\gamma_1,\zeta_1)} + me^{\mathcal{F}_1(a,\zeta_1)}\right\}$$
$$- c_2\left\{(\gamma_2 - ma)e^{\mathcal{F}_2(\gamma_2,\zeta_2)}\frac{\partial}{\partial u}\mathcal{F}_2(\gamma_2,\zeta_2) - e^{\mathcal{F}_2(\gamma_2,\zeta_2)} + me^{\mathcal{F}_2(a,\zeta_2)}\right\} = 0,$$
$$c_1\left\{(\gamma_1 - ma)e^{\mathcal{F}_1(\gamma_1,\zeta_1)}\frac{\partial}{\partial u}\mathcal{F}_1(\gamma_1,\zeta_1) - e^{\mathcal{F}_1(\gamma_1,\zeta_1)} + me^{\mathcal{F}_1(a,\zeta_1)}\right\}$$
$$= c_2\left\{(\gamma_2 - ma)e^{\mathcal{F}_2(\gamma_2,\zeta_2)}\frac{\partial}{\partial u}\mathcal{F}_2(\gamma_2,\zeta_2) - e^{\mathcal{F}_2(\gamma_2,\zeta_2)} + me^{\mathcal{F}_2(a,\zeta_2)}\right\},$$
$$c_1\left\{(\gamma_1 - ma)\frac{\partial}{\partial u}e^{\mathcal{F}_1(v,u)} - e^{\mathcal{F}_1(v,u)} + e^{\mathcal{F}_1(a,u)}\right\} = c_2\left\{(\gamma_1 - ma)\frac{\partial}{\partial u}e^{\mathcal{F}_2(v,u)}\right.$$
$$\left.-e^{\mathcal{F}_2(v,u)} + me^{\mathcal{F}_2(a,u)}\right\},$$
$$\frac{c_2}{c_1} = \frac{(\gamma_1 - ma)e^{\mathcal{F}(\gamma_1,\zeta_1)}\frac{\partial}{\partial u}\mathcal{F}(\gamma_1,\zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + me^{\mathcal{F}(a,\zeta_1)}}{(\gamma_2 - ma)e^{\mathcal{F}(\gamma_2,\zeta_2)}\frac{\partial}{\partial u}\mathcal{F}(\gamma_2,\zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + me^{\mathcal{F}(a,\zeta_2)}}.$$

Similarly, one can show that

$$\frac{c_2}{c_1} = \frac{(\gamma_1 - ma)e^{\mathcal{F}(\gamma_1,\zeta_1)}\frac{\partial}{\partial v}\mathcal{F}(\gamma_1,\zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + me^{\mathcal{F}(a,\zeta_1)}}{(\gamma_2 - ma)e^{\mathcal{F}(\gamma_2,\zeta_2)}\frac{\partial}{\partial v}\mathcal{F}(\gamma_2,\zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + me^{\mathcal{F}(a,\zeta_2)}}.$$

Putting the values of c_1 and c_2, one has the required result. □

If we take $m = 1$, then Theorem 3.15 reduces to the following result.

Theorem 3.16. *Let the conditions given in Theorem 3.13 be satisfied. Also let the positive coordinated exponentially convex functions $\mathcal{F}_1, \mathcal{F}_2 \in C^1(\Delta)$, then there exist (γ_1,ζ_1) and (γ_2,ζ_2) in the interior of Δ, such that*

$$\frac{\Upsilon(e^{\mathcal{F}_1})}{\Upsilon(e^{\mathcal{F}_2})} = \frac{(\gamma_1 - a)e^{\mathcal{F}(\gamma_1,\zeta_1)}\frac{\partial}{\partial u}\mathcal{F}(\gamma_1,\zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + e^{\mathcal{F}(a,\zeta_1)}}{(\gamma_2 - a)e^{\mathcal{F}(\gamma_2,\zeta_2)}\frac{\partial}{\partial u}\mathcal{F}(\gamma_2,\zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + e^{\mathcal{F}(a,\zeta_2)}}$$

and

$$\frac{\Upsilon(e^{\mathcal{F}_1})}{\Upsilon(e^{\mathcal{F}_2})} = \frac{(\gamma_1 - a)e^{\mathcal{F}(\gamma_1,\zeta_1)}\frac{\partial}{\partial v}\mathcal{F}(\gamma_1,\zeta_1) - e^{\mathcal{F}(\gamma_1,\zeta_1)} + e^{\mathcal{F}(a,\zeta_1)}}{(\gamma_2 - a)e^{\mathcal{F}(\gamma_2,\zeta_2)}\frac{\partial}{\partial v}\mathcal{F}(\gamma_2,\zeta_2) - e^{\mathcal{F}(\gamma_2,\zeta_2)} + e^{\mathcal{F}(a,\zeta_2)}},$$

provided that the denominators are non-zero and $a \in (0, b_1)$.

4. Conclusion

We have defined the coordinated exponentially m-convex functions. Petrović's type inequality for exponentially m-convex and coordinated exponentially m-convex functions have been derived. We obtained Lagrange-type and Cauchy-type mean value theorems for exponentially $m-$convex and coordinated exponentially m-convex functions. Some new special cases are discovered. It is expected the ideas and techniques of this paper may motivate the researchers working in functional analysis, information theory and statistical theory to find some applications. This is a new path for future research.

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References

[1] M. A. Latif, M. Alomar, On Hadmard-type inequalities for h-convex functions on the co-ordinates, Int. J. Math. Anal. 3(33)(2009), 1645–1656.

[2] G. Alirezaei, R. Mathar, On Exponentially Concave Functions and Their Impact in Information Theory, in: 2018 Information Theory and Applications Workshop (ITA), IEEE, San Diego, CA, 2018: pp. 1–10.

[3] T. Antczak, (p, r)-Invex Sets and Functions, J. Math. Anal. Appl. 263 (2001), 355–379.

[4] M. Avriel, r-convex functions, Math. Program. 2 (1972), 309–323.

[5] M.U. Awan, M.A. Noor, K.I. Noor, Hermite-Hadamard inequalities for exponentially convex functions, Appl. Math. Inform. Sci. 12(2) (2018), 405–409.

[6] S.N. Bernstein, Sur les fonctions absolument monotones, Acta Math. 52 (1929), 1–66.

[7] S. Butt, J. Pečarić, A.U. Rehman, Exponential convexity of Petrović and related functional, J. Inequal. Appl. 2011 (2011), 89.

[8] S.S. Dragomir, On Hadamards inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwan. J. Math. 5(4) (2001), 775–788.

[9] G. Farid, M. Marwan, A.U. Rehman, New mean value theorems and generalization of Hadamard inequality via coordinated m-convex functions, J. Inequal. Appl. 2015 (2015), 283.

[10] W. Iqbal, K.M. Awan, A.U. Rehman, G. Farid, An extension of Petrović's inequality for $h-$convex ($h-$concave) functions in plane, Open J. Math. Sci. 3(2019), 398–403.

[11] V.G. Mihesan, A generalization of the convexity, Seminar on functional equations, approximation and convexity, Cluj-Napoca, Romania, 1993.

[12] M.A. Noor, Advanced Convex Analysis and Variational Inequalities, Lecture Notes, COMSATS University Islamabad, Islmabad, Pakistan, (2006-2020).

[13] M.A. Noor, Some new classes of nonconvex functions, Nonlinear Funct. Anal. Appl. 11(1)(2006), 165–171.

[14] M.A. Noor, K.I. Noor, Exponentially general convex functions, Transylvanian J. Math. Mech. 11(1-2) (2019), 141-153.

[15] M.A. Noor, K.I. Noor, On exponentially convex functions, J. Orissa Math. Soc. 38(1-2)(2019), 33-51.

[16] M.A. Noor, K.I. Noor, New classes of strongly exponentially preinvex functions, AIMS Math. 4(6) (2019), 1554–1568.

[17] M.A. Noor, K.I. Noor, Strongly exponentially convex functions and their properties, J. Adv. Math. Stud. 12(2) (2019), 177–185.

[18] M.A. Noor, K.I. Noor, Some properties of exponentially preinvex functions, FACTA Univ. (NIS), 34(5) (2019), 941–955.

[19] M.A. Noor, K.I. Noor, Strongly exponentially convex functions, U.P.B. Sci. Bull. Ser. A. 81(4) (2019), 75–84.

[20] M.A. Noor, K.I. Noor, M.U. Awan, Fractional Ostrowski Inequalities for s-Godunova-Levin Functions, Int. J. Anal. Appl. 5(2)(2014), 167–173.

[21] M.A. Noor, K.I. Noor, M.Th. Rassias, New trends in general variational inequalities, Acta Appl. Math. 170(1) (2020), 986-1046.

[22] C. Niclulescu, L.E. Persson, Convex functions and Their Applications, Springer, New York, 2018.

[23] S. Pal, T.K. Wong, On exponentially concave functions and a new information geometry, Ann. Probab. 46(2) (2018), 1070-1113.

[24] J.E. Pečarić, On the Petrović's inequality for convex functions, Glasnik Mat. 18(38) (1983), 77–85.

[25] M. Petrović's, Sur une fontionnelle, Publ. Math. Univ. Belgrade, 1 (1932), 146–149.

[26] J. Pečarić, V. Čuljak, Inequality of Petrović and Giaccardi for convex function of higher order, Southeast Asian Bull. Math. 26(1) (2003), 57–61.

[27] J. Pečarić, J. Peric, Improvements of the Giaccardi and the Petrovic inequality and related Stolarsky type means, Ann. Univ. Craiova, Math. Computer Sci. Ser. 39(1)(2012), 65–75.

[28] J.E. Pečarić, F. Proschan, Y.L. Tong, Convex functions, partial orderings, and statistical applications, Academic Press, Boston, 1992.

[29] A.U. Rehman, G. Farid, V.N. Mishra, Generalized convex function and associated Petrović's inequality, Int. J. Anal. Appl. 17(1)(2019), 122–131.

[30] A.U. Rehman, G. Farid, W. Iqbal, More About Petrovic's Inequality on Coordinates via m-Convex Functions and Related Results, Kragujevac J. Math. 44(3)(2020), 335–351.

[31] A.U. Rehman, M. Mudessir, H.T. Fazal, G. Farid, Petrović's inequality on coordinates and related results, Cogent Math. 3 (2016), 1227298.

[32] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, Integral inequalities for generalized preinvex function, Punjab Univ. J. Math. 51(10) (2019), 77–91.

[33] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, Fractional exponentially m-convex functions and inequalities, Int. J. Anal. Appl. 17(3)(2019), 464–478.

[34] G. Toader, Some generalizations of the convexity, In: Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca, Cluj-Napoca, 1985.