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NUMERICAL STUDY OF RAYLEIGH-BENARD PROBLEM UNDER THE EFFECT OF MAGNETIC FIELD

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Abstract. In this paper, a linear stability analysis is studied for Rayliegh-Benard problem with the effect of magnetic field, a perturbation equations is solved numerically by using spectral Chebyshev tau method, the boundaries are considered both are free, both are rigid, the lower is free and the upper is rigid, the results were illustrated graphically and compared with previous studies.

1. Introduction

The aim of this work is to solve the perturbation equations that represent Rayleigh-Benard problem under the effect of magnetic field ([1], Page (160, 161)) as follows

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\left(\frac{\delta p}{\rho_0} + \mu\frac{\mathbf{H}\cdot\mathbf{h}}{4\pi\rho_0}\right) + g\alpha\theta\underline{k} + v\nabla^2\mathbf{u} + \frac{\mu}{4\pi\rho_0}\left(\mathbf{H}\cdot\nabla\right)\mathbf{h}, \tag{1.1}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \left(\mathbf{H}\cdot\nabla\right)\mathbf{u} + \eta\nabla^2\mathbf{h}, \tag{1.2}$$

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$$\nabla \boldsymbol{.} \mathbf{h} = 0, \tag{1.3}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa \nabla^2 \theta, \tag{1.4}$$

$$\nabla \boldsymbol{.} \mathbf{u} = 0, \tag{1.5}$$

where, $\mathbf{u}$, δp, θ and $\mathbf{h}$ are the perturbation in the velocity, pressure, temperature and magnetic field respectively, ρ_0 is the density at mean temperature T_0, v is the kinematic viscosity, κ is the thermal diffusivity, α the coefficient of volume expansion, μ is the magnetic permeability, η is the resistivity and β is the temperature gradient defined as

$$\beta = \frac{T_0 - T_d}{d}, \tag{1.6}$$

The problem is studied analytically by various authors ([1]- [4], [12]- [14]). Also many numerical methods were used, one of these method is Galerkin method which discussed in [14]. In this paper we used the numerical method namely Spectral Chebyshev tau method to determine the conditions of instability for various cases of the boundary conditions. The paper outlined as follows.

In section 1 we formulate the governing mathematical perturbation equations. In section 2 a linear stability analysis for the perturbation equations. In section 3 we describe the method of solution Spectral Chebyshev Tau method for the three cases of the boundary conditions. Finally, we present our numerical results in which are computed using MATLAB.

2. Linear Stability analysis

By taking the curl operator of equation (1.1), we get

$$\frac{\partial \omega}{\partial t} = g\alpha \left(\frac{\partial \theta}{\partial y} \underline{i} - \frac{\partial \theta}{\partial x} \underline{j} \right) + v \nabla^2 \omega + \frac{\mu}{4\pi\rho_0} \left(\mathbf{H} \boldsymbol{.} \nabla \right) \mathbf{v}, \tag{2.1}$$

where $\omega = \nabla \times \mathbf{u}$ is the vorticity vector and $\mathbf{v} = \nabla \times \mathbf{h}$ is the current density induced by the perturbation. Taking the curl operator of (2.1) again, we get

$$\frac{\partial \nabla^2 \mathbf{u}}{\partial t} = -g\alpha \left(\frac{\partial^2 \theta}{\partial z \partial x} \underline{i} + \frac{\partial^2 \theta}{\partial z \partial y} \underline{j} - \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) \underline{k} \right) + v \nabla^4 \mathbf{u} + \frac{\mu}{4\pi\rho_0} \left(\mathbf{H} \boldsymbol{.} \nabla \right) \nabla^2 \mathbf{h}. \tag{2.2}$$

Take the curl of equation (1.2), we get

$$\frac{\partial \mathbf{v}}{\partial t} = \left(\mathbf{H} \boldsymbol{.} \nabla \right) \omega + \eta \nabla^2 \mathbf{v}. \tag{2.3}$$

Now by equating the z-component of equation (1.2), (2.1), (2.2) and (2.3) respectively, we get

$$\frac{\partial h_z}{\partial t} = \eta \nabla^2 h_z + (\mathbf{H} \centerdot \nabla)\, w, \tag{2.4}$$

$$\frac{\partial \zeta}{\partial t} = v \nabla^2 \zeta + \frac{\mu}{4\pi\rho_0} (\mathbf{H} \centerdot \nabla)\, \xi, \tag{2.5}$$

$$\frac{\partial \nabla^2 w}{\partial t} = g\alpha \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + v \nabla^4 w + \frac{\mu}{4\pi\rho_0} (\mathbf{H} \centerdot \nabla)\, h_z, \tag{2.6}$$

$$\frac{\partial \xi}{\partial t} = \eta \nabla^2 \xi + (\mathbf{H} \centerdot \nabla)\, \zeta, \tag{2.7}$$

where w, ζ and ξ and h_z are the z-components of the velocity, the vorticity, the current density and the magnetic field respectively. When the direction of the magnetic field coincides with the vertical direction $\mathbf{H} = (0, 0, H)$, the required perturbation equations become

$$\frac{\partial}{\partial t} \nabla^2 w = g\alpha \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + v \nabla^4 w + \frac{\mu H}{4\pi\rho_0} \frac{\partial}{\partial z} \nabla^2 h_z, \tag{2.8}$$

$$\frac{\partial h_z}{\partial t} = \eta \nabla^2 h_z + H \frac{\partial w}{\partial z}, \tag{2.9}$$

$$\frac{\partial \xi}{\partial t} = \eta \nabla^2 \xi + H \frac{\partial \zeta}{\partial z}, \tag{2.10}$$

$$\frac{\partial \zeta}{\partial t} = v \nabla^2 \zeta + \frac{\mu H}{4\pi\rho_0} \frac{\partial \xi}{\partial z}. \tag{2.11}$$

In addition to equation (1.4).

The boundary conditions of θ, w and ζ for the three cases free-free, rigid-rigid and rigid-free are

$$\begin{cases} \theta = 0, w = 0 \quad \text{at } z = 0 \text{ and } z = d, \\ \frac{d\zeta}{dz} = 0, \frac{d^2 w}{dz^2} = 0 \quad \text{at free boundary}, \\ \zeta = 0, \frac{dw}{dz} = 0 \quad \text{at rigid boundary}. \end{cases} \tag{2.12}$$

3. Normal Mode Analysis

Let w, θ, ζ, h_z and ξ be defined as

$$\begin{pmatrix} z\theta \\ \zeta \\ w \\ \xi \\ h_z \end{pmatrix} = \begin{pmatrix} \Theta(z) \\ Z(z) \\ W(z) \\ X(z) \\ K(z) \end{pmatrix} \exp\left(i(k_x x + k_y y) + \gamma t\right), \tag{3.1}$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wave number and γ is a constant. Substitute (3.1) into the system (1.4), (2.8)-(2.11), the system become

$$\gamma\Theta = \beta W + \kappa\left(\frac{d^2}{dz^2} - k^2\right)\Theta, \tag{3.2}$$

$$\gamma K = \eta\left(\frac{d^2}{dz^2} - k^2\right)K + H\frac{dW}{dz}, \tag{3.3}$$

$$\gamma\left(\frac{d^2}{dz^2} - k^2\right)W = -g\alpha k^2\Theta + v\left(\frac{d^2}{dz^2} - k^2\right)^2 W + \frac{\mu H}{4\pi\rho_0}\frac{d}{dz}\left(\frac{d^2}{dz^2} - k^2\right)K, \tag{3.4}$$

$$\gamma X = \eta\left(\frac{d^2}{dz^2} - k^2\right)X + H\frac{dZ}{dz}, \tag{3.5}$$

$$\gamma Z = v\left(\frac{d^2}{dz^2} - k^2\right)Z + \frac{\mu H}{4\pi\rho_0}\frac{dX}{dz}. \tag{3.6}$$

And the boundary conditions (2.12) become

$$\begin{cases} \Theta = 0, W = 0 \quad \text{at } z = 0 \text{ and } z = d, \\ \frac{dZ}{dz} = 0, \frac{d^2W}{dz^2} = 0 \quad \text{at free boundary}, \\ Z = 0, \frac{dW}{dz} = 0 \quad \text{at rigid boundary}. \end{cases} \tag{3.7}$$

Define the following non-dimensional variables,

$$a = kd, \quad \sigma = \frac{\gamma d^2}{v}, \quad z^* = \frac{z}{d}, \quad P_1 = \frac{v}{\kappa}, \quad \text{and} \quad P_2 = \frac{v}{\eta}, \tag{3.8}$$

the operators $\frac{d}{dz} = \frac{1}{d}\frac{d}{dz^*}$ and $\frac{d^2}{dz^2} = \frac{1}{d^2}\frac{d^2}{dz^{*2}}$, assume $D = \frac{d}{dz^*}$, then by substituting (3.8) into the system (3.2)-(3.6), we get

$$\left(D^2 - a^2 - P_1\sigma\right)\Theta = -\left(\frac{\beta d^2}{\kappa}\right)W, \tag{3.9}$$

$$\left(D^2 - a^2 - P_2\sigma\right) K = -\left(\frac{Hd}{\eta}\right) DW, \tag{3.10}$$

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \sigma\right) W + \left(\frac{\mu Hd}{4\pi\rho_0 v}\right) D\left(D^2 - a^2\right) K = \left(\frac{g\alpha d^2}{v}\right) a^2\Theta, \tag{3.11}$$

$$\left(D^2 - a^2 - P_2\sigma\right) X = -\left(\frac{Hd}{\eta}\right) DZ, \tag{3.12}$$

$$\left(D^2 - a^2 - \sigma\right) Z = -\left(\frac{\mu Hd}{4\pi\rho_0 v}\right) DX, \tag{3.13}$$

At the marginal state ($\sigma = 0$), then equations (3.9)-(3.13) become

$$\left(D^2 - a^2\right)\Theta = -\left(\frac{\beta d^2}{\kappa}\right) W, \tag{3.14}$$

$$\left(D^2 - a^2\right) K = -\left(\frac{Hd}{\eta}\right) DW, \tag{3.15}$$

$$\left(D^2 - a^2\right)^2 W + \left(\frac{\mu Hd}{4\pi\rho_0 v}\right) D\left(D^2 - a^2\right) K = \left(\frac{g\alpha d^2}{v}\right) a^2\Theta \tag{3.16}$$

$$\left(D^2 - a^2\right) X = -\left(\frac{Hd}{\eta}\right) DZ, \tag{3.17}$$

$$\left(D^2 - a^2\right) Z = -\left(\frac{\mu Hd}{4\pi\rho_0 v}\right) DX, \tag{3.18}$$

and the boundary conditions (3.7) become

$$\begin{cases} \Theta = 0, \quad W = 0 \quad \text{at } z = 0 \text{ and } z = 1, \\ DZ = 0, \quad D^2W = 0 \quad \text{at free boundary}, \\ Z = 0, \quad DW = 0 \quad \text{at rigid boundary}. \end{cases} \tag{3.19}$$

By Taking the operator D for equation (3.15) and substituting in (3.16), we get

$$\left(D^2 - a^2\right)^2 W - QD^2W = \left(\frac{g\alpha d^2}{v}\right) a^2\Theta, \tag{3.20}$$

where,

$$Q = \frac{\mu H^2 d^2}{4\pi \rho_0 \upsilon \eta}, \tag{3.21}$$

is Chandrasekhar number [1]. Taking the operator $(D^2 - a^2)$ for equation (3.20), and using equation (3.14), we get

$$(D^2 - a^2)\left[\left(D^2 - a^2\right)^2 - QD^2\right] W = -Ra^2 W, \tag{3.22}$$

where,

$$R = \frac{g\alpha\beta d^4}{\kappa \upsilon}, \tag{3.23}$$

is Rayleigh number.

By Taking the operator $(D^2 - a^2)$ for equation (3.18) and using (3.17), we get

$$\left[\left(D^2 - a^2\right)^2 - QD^2\right] Z = 0, \tag{3.24}$$

we must seek the solution of (3.22) and (3.24) subject to the boundary conditions (3.19).

4. Spectral Chebyshev tau Method

Spectral Chebyshev tau method is a numerical method to solve the differential equations and the eigen values problems, (see [6], [7], [8] and [10]).

To solve equations (3.22) and (3.24) subject to the boundary conditions (3.19), first convert the domain to Chebyshev polynomials domain $[-1, 1]$, use the relation $x = 2z - 1$, if $z \in [0, 1]$ implies $x \in [-1, 1]$ and the derivative $\frac{d}{dz} = 2\frac{d}{dx}$, $\frac{d^2}{dz^2} = 4\frac{d^2}{dx^2}$, then (3.22), (3.24) and (3.19) become

$$(4D^2 - a^2)\left[\left(4D^2 - a^2\right)^2 - 4QD^2\right] W = -Ra^2 W, \tag{4.1}$$

$$\left[\left(4D^2 - a^2\right)^2 - 4QD^2\right] Z = 0, \tag{4.2}$$

$$\begin{cases} W = 0 \quad \text{at } x = -1 \text{ and } x = 1, \\ DZ = 0, \quad D^2W = 0 \quad \text{at free boundary}, \\ Z = 0, \quad DW = 0 \quad \text{at rigid boundary}, \end{cases} \tag{4.3}$$

where $D = \frac{d}{dx}$.

Let $S = \left[\left(4D^2 - a^2\right)^2 - 4QD^2\right] W$ then we can write (4.1) - (4.3) as

$$\left[\left(4D^2 - a^2\right)^2 - 4QD^2\right] W - S = 0, \tag{4.4}$$

$$\left(4D^2 - a^2\right) S = -Ra^2 W, \tag{4.5}$$

$$\left[\left(4D^2 - a^2\right)^2 - 4QD^2\right] Z = 0, \tag{4.6}$$

$$\begin{cases} W = 0, S = 0 \quad \text{at } x = -1 \text{ and } x = 1, \\ DZ = 0, \quad D^2W = 0 \quad \text{at free boundary}, \\ Z = 0, \quad DW = 0 \quad \text{at rigid boundary}. \end{cases} \tag{4.7}$$

Now, expand W, S and Z as Chebyshev polynomials

$$W = \sum_{n=0}^{N} w_n T_n(x) = \begin{bmatrix} w_0 & \cdots & w_N \end{bmatrix} \begin{bmatrix} T_0 \\ \vdots \\ T_N \end{bmatrix} = \mathbf{W}\phi, \tag{4.8}$$

$$S = \sum_{n=0}^{N} s_n T_n(x) = \begin{bmatrix} s_0 & \cdots & s_N \end{bmatrix} \begin{bmatrix} T_0 \\ \vdots \\ T_N \end{bmatrix} = \mathbf{S}\phi, \tag{4.9}$$

$$Z = \sum_{n=0}^{N} z_n T_n(x) = \begin{bmatrix} z_0 & \cdots & z_N \end{bmatrix} \begin{bmatrix} T_0 \\ \vdots \\ T_N \end{bmatrix} = \mathbf{Z}\phi, \tag{4.10}$$

where W, S and Z are row vectors represent the coefficients of W, S and Z respectively, and ϕ is the vector of chebyshev polynomials T_0 up to T_N. Furthermore we can expand the derivatives DW, D^2W and D^4W as chebyshev polynomials as

$$DW = \sum_{n=0}^{N} w_n^{(1)} T_n(x) = \mathbf{WD}\phi, \tag{4.11}$$

$$D^2W = \sum_{n=0}^{N} w_n^{(2)} T_n(x) = \mathbf{WD}^2\phi, \tag{4.12}$$

$$D^4W = \sum_{n=0}^{N} w_n^{(4)} T_n(x) = \mathbf{WD}^4\phi, \tag{4.13}$$

similarly for the derivatives of S and Z

$$D^2 S = \sum_{n=0}^{N} s_n^{(2)} T_n(x) = \mathbf{S}\mathbf{D}^2\phi, \tag{4.14}$$

$$D^2 Z = \sum_{n=0}^{N} z_n^{(2)} T_n(x) = \mathbf{Z}\mathbf{D}^2\phi, \tag{4.15}$$

Where $\mathbf{D}$ and $\mathbf{D^2}$ are $(N+1)\times(N+1)$ matrices represent the coefficients of the first and second chebyshev derivatives, (see [6] and [10] for more details of formulations of $\mathbf{D}$ and $\mathbf{D}^2$).

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & 0 & 0 & \cdots & 0 \\ 3 & 0 & 6 & 0 & 0 & \cdots & 0 \\ 0 & 8 & 0 & 8 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ N & 0 & 2N & 0 & 2N & \cdots & 0 \end{pmatrix}, \tag{4.16}$$

$$\mathbf{D}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 24 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 32 & 0 & 48 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 120 & 0 & 80 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & N(N^2-1) & 0 & N(N^2-9) & 0 & N(N^2-25) & \cdots & 0 \end{pmatrix}, \tag{4.17}$$

and $\mathbf{D}^4 = (\mathbf{D})^4 = \mathbf{D}^2\mathbf{D}^2$. By substitute (4.8)–(4.15) into equations (4.4)-(4.6) we get,

$$\mathbf{W}\left[16\mathbf{D}^4 - (8a^2+4Q)\mathbf{D}^2 + a^4\mathbf{I}\right]\phi - \mathbf{S}\phi = 0, \tag{4.18}$$

$$\mathbf{S}\left[4\mathbf{D}^2 - a^2\mathbf{I}\right]\phi = -Ra^2\mathbf{W}\phi, \tag{4.19}$$

$$\mathbf{Z}\left[16\mathbf{D}^4 - (8a^2+4Q)\mathbf{D}^2 + a^4\mathbf{I}\right]\phi = 0, \tag{4.20}$$

where $\mathbf{I}$ is the identity matrix of order $(N+1)$. By taking the inner product with $T_n, n = 0, 1, .., N$, for each equation in the system (4.18) – (4.20) and using the property of orthogonality of Chebyshev polynomials, we obtain $3(N+1)$ equations as follows

$$\left[16w_n^{(4)} - \left(8a^2 + 4Q\right) w_n^{(2)} + a^4 w_n\right] - s_n = 0, \quad n = 0, 1, \ldots, N \tag{4.21}$$

$$4s_n^{(2)} - a^2 s_n = -Ra^2 w_n, \quad n = 0, 1, \ldots, N \tag{4.22}$$

$$16z_n^{(4)} - (8a^2 + 4Q)z_n^{(2)} + a^4 z_n = 0. \quad n = 0, 1, \ldots, N \tag{4.23}$$

Rewriting these equations in matrices form as

$$\mathbf{W}\left[16\mathbf{D}^4 - (8a^2 + 4Q)\mathbf{D}^2 + a^4\mathbf{I}\right] - \mathbf{S} = 0, \tag{4.24}$$

$$\mathbf{S}\left[4\mathbf{D}^2 - a^2\mathbf{I}\right] = -Ra^2\mathbf{W}, \tag{4.25}$$

$$\mathbf{Z}\left[16\mathbf{D}^4 - (8a^2 + 4Q)\mathbf{D}^2 + a^4\mathbf{I}\right] = 0, \tag{4.26}$$

or,

$$\mathbf{AX} = R\mathbf{BX}, \tag{4.27}$$

where $\mathbf{A}$ and $\mathbf{B}$ are square matrices of order $3(N+1)$ given as

$$\mathbf{A} = \begin{pmatrix} \mathbf{L}_1 & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -a^2\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \text{ and } \mathbf{X} = \begin{pmatrix} \mathbf{W}^T \\ \mathbf{S}^T \\ \mathbf{Z}^T \end{pmatrix} \tag{4.28}$$

where $\mathbf{L}_1 = \left[16\mathbf{D}^4 - (8a^2 + 4Q)\mathbf{D}^2 + a^4\mathbf{I}\right]^T$ and $\mathbf{L}_2 = \left[4\mathbf{D}^2 - a^2\mathbf{I}\right]^T$. Now returning to the boundary conditions (4.7), the p-derivative of Chebyshev polynomials at $x = \pm 1$ is given by the formula

$$\left.\frac{d^p T_n}{dx^p}\right|_{x=\pm 1} = (\pm 1)^{n+p} \prod_{k=0}^{p-1} \frac{n^2 - k^2}{2k+1}, \tag{4.29}$$

For Free-Free boundaries, We have eight boundary conditions as

$$W = 0, \quad D^2W = 0, \quad S = 0 \quad \text{and} \quad DZ = 0 \quad \text{for } x = \pm 1.$$

$$\text{BC1}: W(-1)=0 \Rightarrow \sum_{n=0}^{N} w_n T_n(-1)=0 \Rightarrow \sum_{n=0}^{N} (-1)^n\, w_n=0,$$

$$\text{BC2}: W(1)=0 \Rightarrow \sum_{n=0}^{N} w_n T_n(1)=0 \Rightarrow \sum_{n=0}^{N} w_n=0,$$

$$\text{BC3}: D^2W(-1)=0 \Rightarrow \sum_{n=0}^{N} w_n T_n^{''}(-1)=0 \Rightarrow \sum_{n=0}^{N} (-1)^{n+2}\, \frac{n^2(n^2-1)}{3} w_n=0,$$

$$\text{BC4}: D^2W(1)=0 \Rightarrow \sum_{n=0}^{N} w_n T_n^{''}(1)=0 \Rightarrow \sum_{n=0}^{N} \frac{n^2(n^2-1)}{3} w_n=0,$$

$$\text{BC5}: S(-1)=0 \Rightarrow \sum_{n=0}^{N} s_n T_n(-1)=0 \Rightarrow \sum_{n=0}^{N} (-1)^n\, s_n=0,$$

$$\text{BC6}: S(1)=0 \Rightarrow \sum_{n=0}^{N} s_n T_n(1)=0 \Rightarrow \sum_{n=0}^{N} s_n=0,$$

$$\text{BC7}: DZ(-1)=0 \Rightarrow \sum_{n=0}^{N} z_n T_n^{'}(-1)=0 \Rightarrow \sum_{n=0}^{N} (-1)^{n+1}\, n^2 z_n=0,$$

$$\text{BC8}: DZ(1)=0 \Rightarrow \sum_{n=0}^{N} z_n T_n^{'}(1)=0 \Rightarrow \sum_{n=0}^{N} n^2 z_n=0,$$

similarly, for rigid-rigid boundaries,

$$W=0, \quad DW=0, \quad S=0 \quad \text{and} \quad Z=0 \quad \text{for } x=\pm 1.$$

and for rigid-free boundaries,

$$W=0, \quad DW=0, \quad S=0 \quad \text{and} \quad Z=0 \quad \text{for } x=-1 \text{ rigid}.$$

$$W=0, \quad D^2W=0, \quad S=0 \quad \text{and} \quad DZ=0 \quad \text{for } x=1 \text{ free}.$$

For each case of the boundary conditions, insert $BC1$ up to $BC4$ into the rows $(N-2)^{th}$ up to $(N+1)^{th}$ of the first column in the matrix $\mathbf{A}$ in (4.27), $BC5$ and $BC6$ into the rows $(2N+1)^{th}$ and $(2N+2)^{th}$ of the second column in $\mathbf{A}$, $BC7$ and $BC8$ into the rows $(3N+2)^{th}$ and $(3N+3)^{th}$ of the third column in $\mathbf{A}$. The corresponding rows in the matrix $\mathbf{B}$ are zeros, then we can write the system (4.27) as follows

$$\begin{pmatrix} \overline{\mathbf{L}}_1 & -\overline{\mathbf{I}} & \overline{\mathbf{0}} \\ \text{BC1} & 0\dots0 & 0\dots0 \\ \text{BC2} & 0\dots0 & 0\dots0 \\ \text{BC3} & 0\dots0 & 0\dots0 \\ \text{BC4} & 0\dots0 & 0\dots0 \\ \overline{\mathbf{0}} & \overline{\mathbf{L}}_2 & \overline{\mathbf{0}} \\ 0\dots0 & \text{BC5} & 0\dots0 \\ 0\dots0 & \text{BC6} & 0\dots0 \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{L}}_1 \\ 0\dots0 & 0\dots0 & \text{BC7} \\ 0\dots0 & 0\dots0 & \text{BC8} \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_n \\ s_0 \\ \vdots \\ s_n \\ z_0 \\ \vdots \\ z_n \end{pmatrix} = R \begin{pmatrix} \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} \\ 0\dots0 & 0\dots0 & 0\dots0 \\ 0\dots0 & 0\dots0 & 0\dots0 \\ 0\dots0 & 0\dots0 & 0\dots0 \\ 0\dots0 & 0\dots0 & 0\dots0 \\ -a^2\overline{\mathbf{I}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} \\ 0\dots0 & 0\dots0 & 0\dots0 \\ 0\dots0 & 0\dots0 & 0\dots0 \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} \\ 0\dots0 & 0\dots0 & 0\dots0 \\ 0\dots0 & 0\dots0 & 0\dots0 \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_n \\ s_0 \\ \vdots \\ s_n \\ z_0 \\ \vdots \\ z_n \end{pmatrix} \tag{4.30}$$

where, $\overline{\mathbf{L_1}}, \overline{\mathbf{L_2}}, \overline{\mathbf{0}}$ and $\overline{\mathbf{I}}$ are the sub matrices of $\mathbf{L_1}, \mathbf{L_2}, \mathbf{0}$ and $\mathbf{I}$ respectively. Now we have generalized eigen value (4.30), we can find the minimum eigen values $R(a)$, then the critical Rayleigh number values R_c and the corresponding wave number a_c for various values of Q using MATLAB software, the results are illustrated in the next section.

5. Results and Conclusion

- Spectral Chebyshev Tau method give results in full agreement with the results that obtained by the analytical solution given by Chandrasekhar [1].
- For the three cases of the boundaries free-free, rigid-rigid and rigid-free the critical Rayleigh number determine the stability. if $R < R_c$ the motion is stable and no convection, when $R = R_c$ it is stationary convection, and unstable motion when $R > R_c$.
- It is also observed the effect of the magnetic field, as Q increases, the value of the critical Rayleigh number and the critical wave number also increases.

Q	Chandrasekhar [1]		Present study	
	a_c	R_c	a_c	R_c
0	2.233	657.511	2.22	657.51
10	2.590	923.070	2.59	923.07
50	3.270	1762.04	3.27	1762.04
100	3.702	2653.71	3.7	2653.71
200	4.210	4258.49	4.21	4258.49
500	4.998	8578.88	5.00	8578.89
1000	5.684	15207.0	5.68	15207
2000	6.453	27699.9	6.46	27699.79
5000	7.585	63135.9	7.61	63135.48
10000	8.588	119832	8.61	119831.71
20000	9.706	230038	9.72	230038.48
40000	10.95	445507	10.96	445506.86

TABLE 1. The relation between between R_c and a_c when the boundaries are free-free and $Q = 0, 10, 50, \ldots, 40000$.

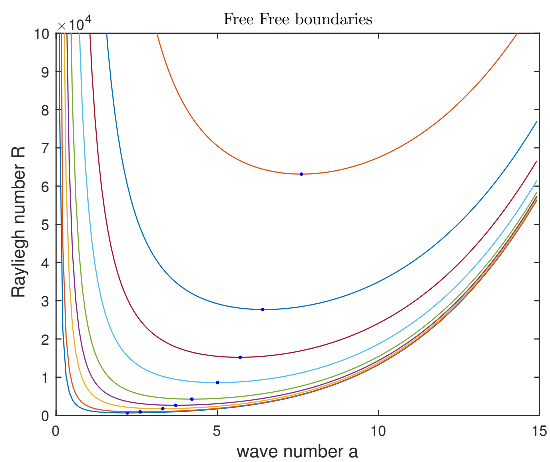

FIGURE 1. The relation between between R_c and a_c when the boundaries are free-free and $Q = 0, 10, 50, \ldots, 40000$.

Q	Chandrasekhar [1]		Present study	
	a_c	R_c	a_c	R_c
0	3.13	1707.8	3.12	1707.77
10	3.25	1945.9	3.27	1945.75
50	3.68	2803.1	3.68	2802.01
100	4.00	3757.4	4.01	3757.23
200	4.45	5488.6	4.45	5488.54
500	5.16	10110	5.16	10109.78
1000	5.80	17103	5.81	17102.85
2000	6.55	30125	6.56	30124.81
4000	7.40	54697	7.39	54700.75
6000	7.94	78391	7.93	78441.25
8000	8.34	101606	8.30	101930.17
10000	8.66	124509	8.52	125856.50

TABLE 2. The relation between between R_c and a_c when the boundaries are rigid-rigid and $Q = 0, 10, 50, \ldots, 10000$.

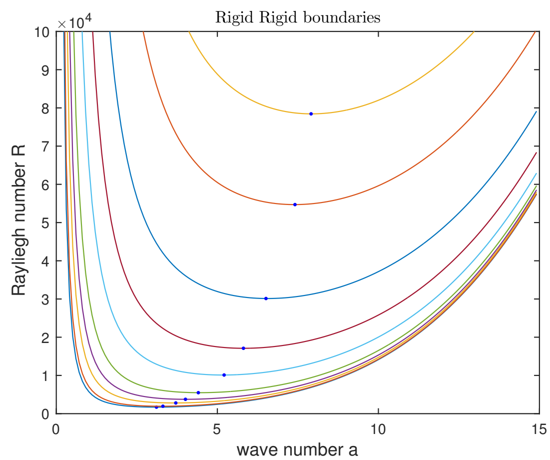

FIGURE 2. The relation between between R_c and a_c when the boundaries are rigid-rigid and $Q = 0, 10, \ldots, 10000$.

Q	Chandrasekhar [1]		Present study	
	a_c	R_c	a_c	R_c
0	2.68	1100.75	2.7	1100.81
2.5	2.75	1167.2	2.71	1167.45
12.5	2.97	1415.5	3.01	1415.81
25	3.17	1699.4	3.21	1699.79
50	3.45	2217.6	3.51	2217.98
125	4.00	3586.1	4.01	3585.85
250	4.50	5613.3	4.51	5612.93
500	5.10	9304.5	5.11	9303.95
1000	5.75	16119	5.71	16118.68
1500	6.20	22592	6.21	22591.30
2000	6.50	28879	6.51	28877.86
2500	6.75	35044	6.81	35043.57
5000	7.65	64847	7.61	64836.70
10000	8.65	122140	8.71	121512.72

TABLE 3. The relation between between R_c and a_c when the boundaries are rigid-free and $Q = 0, 2.5, 12.5, \ldots, 10000$.

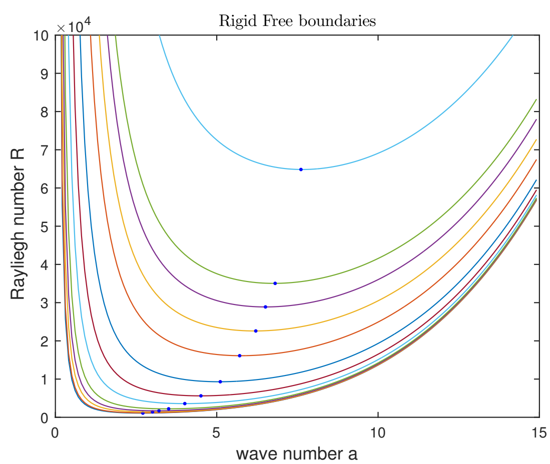

FIGURE 3. The relation between between R_c and a_c when the boundaries are rigid-free and $Q = 0, 2.5, 12.5 \ldots, 10000$.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Chandrasekhar, Hydrodynamic and hydromagnetic stability, Oxford University Press, Oxford, 1961.

[2] S. Chandrasekhar, The instability of a layer of fluid heated below and subject to Coriolis forces, Proc. R. Soc. Lond. A. 217 (1953), 306–327.

[3] S. Chandrasekhar, The instability of a layer of fluid heated below and subject to the simultaneous action of a magnetic field and rotation, Proc. R. Soc. Lond. A. 225 (1954), 173–184.

[4] S. Chandrasekhar, D.D. Elbert, The instability of a layer of fluid heated below and subject to Coriolis forces. II, Proc. R. Soc. Lond. A. 231 (1955), 198–210.

[5] A.J. Chorin, A numerical method for solving incompressible viscous flow problems, J. Comput. Phys. 2 (1967), 12–26.

[6] J.J. Dongarra, B. Straughan, D.W. Walker, Chebyshev tau-QZ algorithm methods for calculating spectra of hydrodynamic stability problems, Appl. Numer. Math. 22 (1996), 399–434.

[7] L. Fox, Chebyshev Methods for Ordinary Differential Equations, Computer J. 4 (1962), 318–331.

[8] M.A. Hernandez, Chebyshev's approximation algorithms and applications, Comput. Math. Appl. 41 (2001), 433–445.

[9] A.A. Hill, Convection in porous media and Legendre, Chebyshev Galerkin methods, Ph.D. Thesis, Durham University, 2005.

[10] D. Johnson, Chebyshev Polynomials in the Spectral Tau Method and Applications to Eigenvalue Problems, NASA Contractor Report 198451, 1996.

[11] S. Talukdar, A study of some steady and unsteady MHD flow problems with heat and mass transfer, Ph.D. Thesis, Gauhati University, 2013.

[12] D.Y. Tzou, Instability of Nanofluids in Natural Convection, J. Heat Transfer, 130 (2008), 072401.

[13] B.-F. Wang, D.-J. Ma, Z.-W. Guo, D.-J. Sun, Linear instability analysis of Rayleigh–Bénard convection in a cylinder with traveling magnetic field, J. Crystal Growth, 400 (2014), 49–53.

[14] D. Yadav, J. Wang, R. Bhargava, J. Lee, H.H. Cho, Numerical investigation of the effect of magnetic field on the onset of nanofluid convection, Appl. Therm. Eng. 103 (2016), 1441–1449.