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EXISTENCE AND LOCATION OF A UNIQUE SOLUTION OF CAPUTO-LIOUVILLE TYPE LANGEVIN EQUATION WITH FINITELY MANY NONLINEARITIES AND NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper, we discuss the existence of a unique solution of Caputo-Liouville type Langevin equation involving two fractional orders and finitely many nonlinearities, equipped with nonlocal boundary conditions via Banach contraction mapping principle. The location of the unique solution of the given problem is also presented. In addition, we discuss the existence of solutions for the problem at hand by means of Krasnosel'skiĭ's fixed point theorem. Examples are constructed for the illustration of the obtained results. The paper concludes with some interesting remarks.

1. Introduction

Fractional order differential equations received overwhelming attention of many researchers as these equations extensively appear in the mathematical modeling of several scientific and technical phenomena. Examples include physics, biology, chemistry, control theory, electrical circuits, wave propagation, blood flow phenomena, signal and image processing, etc. For further details, see [1]- [5].

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Langevin equation, formulated in terms of integer-order derivatives by Langevin [6] in 1908, is a well-known equation of mathematical physics, which is used to describe the evolution of physical phenomena, such as Brownian motion in fluctuating environments.

Langevin equation is also known as a stochastic differential equation as it is related to the fast motion of microscopic variables of the dynamical systems. However, the failure of classical Langevin equation to describe the complex systems led to its several generalizations, which successfully modeled the physical phenomena in disordered regions [7], anomalous diffusion processes in complex and viscoelastic environment [8, 9], etc. Among these generalizations includes the one obtained by replacing the ordinary derivative by fractional order derivative in it; the resulting form is known as fractional Langevin equation and can take care of the fractal and memory properties of the phenomena under investigation. Applications of fractional Langevin equation include motor control system [10], single-file diffusion [11], transformation of the Fokker-Planck equation into the Wiener process [12], association of Kramers-Fokker-Planck equation with Langevin equation [13], etc. In order to obtain a more flexible model for fractal processes, Lim et al. [14] introduced a new form of Langevin equation involving two different fractional orders. For some recent results on fractional Langevin equation, for instance, see [15]- [24] and references therein.

Modern tools of functional analysis have played a key role in developing the theory (existence and uniqueness of solutions) for fractional order initial and boundary value problems, for example, see [25]- [30].

In this paper, motivated by the recent development on Langevin equation, we study the following nonlocal boundary value problem involving Langevin equation with finitely many nonlinearities:

$$ {}^cD^{\alpha}({}^cD^{\beta}+\mu)y(t)=\sum_{i=1}^{m}a_i f_i(t,y(t)), \quad 0<\alpha\le 1, \quad 1<\beta\le 2, \tag{1.1}$$

$$ y(0)=0, \quad y(\xi_1)=0, \quad y(1)=\omega\ y(\xi_2), \quad 0<\xi_1<\xi_2<1, \tag{1.2}$$

where ${}^cD$ denotes the Caputo-Liouville fractional derivative operator, $f_i:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions, and $\omega\in\mathbb{R}$.

By using Banach contraction mapping principle we prove the existence of a unique solution of boundary value problem (1.1)-(1.2) and moreover we study the location of the unique solution. An existence result is also obtained via Krasnosel'skiĭ's fixed point theorem.

The paper is organized as follows. In Section 2 we recall some basic definitions and properties from fractional calculus and solve a linear variant of the boundary value problem (1.1)-(1.2). The main results are presented in Section 3. Examples illustrating the obtained results are also constructed.

2. Preliminaries

Let us recall some basic definitions on fractional calculus.

Definition 2.1. *([2, 3]). The Riemann–Liouville fractional integral $I_a^\alpha y$ of order $\alpha > 0$ for a function $y \in L_1[a,b], -\infty < a < b < +\infty$, existing almost everywhere on $[a,b]$, is defined by*

$$I_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2. *[2, 3]. Let $y, y^{(m)} \in L_1[a,b]$. Then the Riemann–Liouville fractional derivative $D_a^\alpha y$ of order $\alpha \in (m-1, m], m \in N$, existing almost everywhere on $[a,b]$, is defined as*

$$D_a^\alpha y(t) = \frac{d^m}{dt^m} I_a^{m-\alpha} y(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} y(s) ds.$$

The Caputo fractional derivative ${}^cD_a^\alpha y$ of order $\alpha \in (m-1, m], m \in N$ is defined as

$${}^cD_a^\alpha y(t) = D_a^\alpha \left[y(t) - y(a) - y'(a) \frac{(t-a)}{1!} - \ldots - y^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

Remark 2.1. *[2]. If $y \in AC^m[a,b]$, then the Caputo fractional derivative ${}^cD_a^\alpha y$ of order $\alpha \in (m-1, m], m \in N$, existing almost everywhere on $[a,b]$, is defined as*

$${}^cD_a^\alpha y(t) = I_a^{m-\alpha} y^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} y^{(m)}(s) ds.$$

Proposition 2.1. *([2]) For $\kappa > 0$ and $\alpha > 0$ with $n-1 < \alpha \le n$, and $y \in L_1[a,b]$, we have the following properties:*

(i) $I_a^\alpha I_a^\kappa y(t) = I_a^\kappa I_a^\alpha y(t) = I_a^{\alpha+\kappa} y(t)$;

(ii) $I_a^\alpha (t-a)^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)} (t-a)^{\alpha+\eta}, \ \eta > -1$;

(iii) ${}^cD_a^\alpha \left[I_a^\alpha y(t) \right] = y(t)$;

(iv) $I_a^\alpha \left[{}^cD_a^\alpha y(t) \right] = y(t) - \sum_{p=0}^{n-1} \frac{y^{(p)}(a)(t-a)^p}{p!}, \ y \in C^n[a,b]$.

In the sequel, we write I^σ and ${}^cD^\sigma$ instead of I_0^σ and ${}^cD_0^\sigma$ respectively.

To study the nonlinear problem (1.1)-(1.2), we first solve its linear variant in the following lemma.

Lemma 2.1. *For a given $\rho \in C([0,1], \mathbb{R})$, the unique solution of the boundary value problem*

$$ {}^cD^\alpha({}^cD^\beta + \mu) y(t) = \rho(t) \qquad 0 < \alpha \le 1, \qquad 1 < \beta \le 2, \tag{2.1}$$

$$ y(0) = 0, \qquad y(\xi_1) = 0, \qquad y(1) = \omega y(\xi_2), \tag{2.2}$$

is given by

$$
\begin{aligned}
y(t) = {} & \frac{1}{\Gamma(\beta+\alpha)} \int_0^t (t-s)^{\beta+\alpha-1} \rho(s) ds - \frac{\mu}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds \\
& + \rho_1(t) \Big[\int_0^{\xi_1} \frac{(\xi_1 - s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \rho(s) ds - \mu \int_0^{\xi_1} \frac{(\xi_1 - s)^{\beta-1}}{\Gamma(\beta)} y(s) ds \Big] \\
& + \rho_2(t) \Big[\int_0^1 \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \rho(s) ds - \mu \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds \\
& - \omega \int_0^{\xi_2} \frac{(\xi_2 - s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \rho(s) ds + \mu\omega \int_0^{\xi_2} \frac{(\xi_2 - s)^{\beta-1}}{\Gamma(\beta)} y(s) ds \Big],
\end{aligned} \tag{2.3}
$$

where

$$
\rho_1(t) = \frac{1}{\Delta\Gamma(\beta+1)} \Big(t(1-\omega\xi_2^\beta) - (1-\omega\xi_2) t^\beta \Big), \ \rho_2(t) = \frac{1}{\Delta\Gamma(\beta+1)} \Big(t^\beta \xi_1 - t\xi_1^\beta \Big), \tag{2.4}
$$

and it is assumed that

$$
\Delta = \frac{[\xi_1^\beta(1-\omega\xi_2) - \xi_1(1-\omega\xi_2^\beta)]}{\Gamma(\beta+1)} \neq 0. \tag{2.5}
$$

Proof. Applying the Riemann-Liouville integral operator I^α to both sides of (2.1) and using Proposition 2.1 (iv), we obtain

$$
({}^cD^\beta + \mu) y(t) + c_0 = I^\alpha \rho(t), \tag{2.6}
$$

where $c_0 \in \mathbb{R}$ is an unknown constant. Next, operating I^β on both sides of (2.6), we obtain

$$
\begin{aligned}
y(t) = {} & \int_0^t \frac{(t-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \rho(s) ds - \frac{\mu}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds \\
& - \frac{c_0 t^\beta}{\Gamma(1+\beta)} - c_1 - c_2 t.
\end{aligned} \tag{2.7}
$$

Using the conditions $y(0) = 0$ and $y(\xi_1) = 0$ in (2.7) yields $c_1 = 0$ and

$$
\frac{c_0 \xi_1^\beta}{\Gamma(\beta+1)} + c_2 \xi_1 = A, \tag{2.8}
$$

where

$$
A = \int_0^{\xi_1} \frac{(\xi_1 - s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \rho(s) ds - \mu \int_0^{\xi_1} \frac{(\xi_1 - s)^{\beta-1}}{\Gamma(\beta)} y(s) ds.
$$

Now using $y(1) = \omega y(\xi_2)$ in (2.7), we obtain

$$
\frac{(1-\omega\xi_2^\beta)}{\Gamma(\beta+1)} c_0 + (1-\omega\xi_2) c_2 = B, \tag{2.9}
$$

where

$$
\begin{aligned}
B = {} & \int_0^1 \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \rho(s) ds - \mu \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds \\
& - \omega \int_0^{\xi_2} \frac{(\xi_2 - s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \rho(s) ds + \mu\omega \int_0^{\xi_2} \frac{(\xi_2 - s)^{\beta-1}}{\Gamma(\beta)} y(s) ds.
\end{aligned}
$$

Solving (2.8) and (2.9) for c_0 and c_2, we find that

$$c_0 = \frac{1}{\Delta}(\sigma_3 A - \xi_1 B), \ c_2 = \frac{1}{\Delta}(\sigma_1 B - \sigma_2 A), \tag{2.10}$$

where

$$\Delta = \sigma_3\sigma_1 - \xi_1\sigma_2, \ \sigma_1 = \frac{\xi_1^\beta}{\Gamma(\beta+1)}, \ \sigma_2 = \frac{(1-\omega\xi_2^\beta)}{\Gamma(\beta+1)}, \ \sigma_3 = (1-\omega\xi_2). \tag{2.11}$$

Inserting $c_1 = 0$ and the values of c_0 and c_2 from (2.10) into (2.7), together with (2.11), leads to the solution (2.3). The converse follows by direct computation. This completes the proof. □

Definition 2.3. *A function $y \in C([0,1], \mathbb{R})$ is a solution of the boundary value problem (1.1)-(1.2) if and only if it satisfies the integral equation:*

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds - \mu\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds \\ &+ \rho_1(t)\left[\int_0^{\xi_1} \frac{(\xi_1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds - \mu\int_0^{\xi_1} \frac{(\xi_1-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds\right] \\ &+ \rho_2(t)\left[\int_0^1 \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds - \mu\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds\right. \\ &\left. - \omega\int_0^{\xi_2} \frac{(\xi_2-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds + \mu\omega\int_0^{\xi_2} \frac{(\xi_2-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds\right]. \end{aligned} \tag{2.12}$$

3. Uniqueness and location results

In the following theorem, we prove the existence of a unique solution for the problem (1.1)-(1.2) by applying Banach contraction mapping principle.

Theorem 3.1. *Let $f_i(t,y) : [0,1]\times\mathbb{R} \to \mathbb{R}, i = 1,\dots,m$ be continuous functions satisfying the Lipschitz condition:*

(A_1) $|f_i(t,x) - f_i(t,y)| \le L_i |x-y|, \ \forall t \in [0,1], \ x,y \in \mathbb{R}, \ L_i > 0, \ i = 1,2,\dots,m.$

Then, the boundary value problem (1.1)-(1.2) has a unique solution on $[0,1]$ if

$$\Omega_1 < 1, \tag{3.1}$$

where

$$\begin{aligned} \Omega_1 = &\frac{(\sum_{i=1}^m |a_i| L_i)}{\Gamma(\beta+\alpha+1)}\left[1 + \bar{\rho_1}\xi_1^{\beta+\alpha} + \bar{\rho_2}\left(1 + |\omega|\xi_2^{\beta+\alpha}\right)\right] \\ &+ \frac{|\mu|}{\Gamma(\beta+1)}\left[1 + \bar{\rho_1}\xi_1^\beta + \bar{\rho_2}\left(1+|\omega|\xi_2^\beta\right)\right], \end{aligned} \tag{3.2}$$

and

$$\bar{\rho_1} = \max_{t\in[0,1]} |\rho_1(t)|, \ \bar{\rho_2} = \max_{t\in[0,1]} |\rho_2(t)|.$$

Proof. To transform the problem (1.1) and (1.2) into a fixed-point problem, we introduce an operator $\mathcal{N}\colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ as

$$
\begin{aligned}
&(\mathcal{N}y)(t)\\
&= \int_0^t \frac{(t-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds - \mu\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds\\
&+\rho_1(t)\left[\int_0^{\xi_1} \frac{(\xi_1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds - \mu\int_0^{\xi_1} \frac{(\xi_1-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds\right]\\
&+\rho_2(t)\left[\int_0^1 \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds - \mu\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds\right.\\
&\left.-\omega\int_0^{\xi_2} \frac{(\xi_2-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds + \mu\omega\int_0^{\xi_2} \frac{(\xi_2-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds\right],
\end{aligned}
\tag{3.3}
$$

where $C([0,1],\mathbb{R})$ is the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ equipped with the norm $\|y\| = \sup_{t\in[0,1]}|y(t)|$. Observe that the fixed points of the operator $\mathcal{N}$ are solutions of the problem (1.1) and (1.2) by Definition 2.3. Further, it is an immediate consequence of the dominated convergence theorem that $\mathcal{N}y \in C([0,1],\mathbb{R})$ for every $y \in C([0,1],\mathbb{R})$. The proof will be complete by means of Banach contraction mapping principle once it is shown that that the operator $\mathcal{N}$ defined by (3.3) is a contraction. For $x, y \in \mathbb{R}$ and $\forall t \in [0,1]$, we have

$$
\begin{aligned}
&\|(\mathcal{N}x)-(\mathcal{N}y)\|\\
= &\sup_{t\in[0,1]}\left|\int_0^t \frac{(t-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i[f_i(s,x(s))-f_i(s,y(s))]\Big)ds\right.\\
&-\mu\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}[x(s)-y(s)]ds\\
&+\rho_1(t)\left[\int_0^{\xi_1} \frac{(\xi_1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i[f_i(s,x(s))-f_i(s,y(s))]\Big)ds\right.\\
&\left.-\mu\int_0^{\xi_1} \frac{(\xi_1-s)^{\beta-1}}{\Gamma(\beta)}[x(s)-y(s)]ds\right]\\
&+\rho_2(t)\left[\int_0^1 \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i[f_i(s,x(s))-f_i(s,y(s))]\Big)ds\right.\\
&-\mu\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}[x(s)-y(s)]ds\\
&-\omega\int_0^{\xi_2} \frac{(\xi_2-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i[f_i(s,x(s))-f_i(s,y(s))]\Big)ds\\
&\left.\left.+\mu\omega\int_0^{\xi_2} \frac{(\xi_2-s)^{\beta-1}}{\Gamma(\beta)}[x(s)-y(s)]ds\right]\right|\\
\leq &\frac{\left(\sum_{i=1}^m |a_i|\, L_i\right)}{\Gamma(\beta+\alpha+1)}\left[1+\bar{\rho_1}\xi_1^{\beta+\alpha}+\bar{\rho_2}\left(1+|\omega|\,\xi_2^{\beta+\alpha}\right)\right]\|x-y\|
\end{aligned}
$$

$$+\frac{|\mu|}{\Gamma(\beta+1)}\left[1+\bar{\rho_1}\xi_1^{\beta}+\bar{\rho_2}\left(1+|\omega|\,\xi_2^{\beta}\right)\right]\|x-y\|,$$

which leads to

$$\|(\mathcal{N}x)-(\mathcal{N}y)\|\leq\Omega_1\|x-y\|.$$

Evidently, we deduce by (3.1) that $\mathcal{N}:C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is a contraction. Hence, by Banach contraction mapping principle, the operator $\mathcal{N}$ has a unique fixed point, which corresponds to a unique solution of the boundary value problem (1.1)-(1.2) on $[0,1]$. This completes the proof. □

Example 3.1. *Consider the following boundary value problem:*

$$\begin{aligned}&{}^cD^{\frac{1}{2}}\Big({}^cD^{\frac{3}{2}}+\frac{1}{5}\Big)y(t)=\sum_{i=1}^{3}a_if_i(t,y(t)),\ t\in[0,1],\\&y(0)=0,\ y\Big(\frac{1}{3}\Big)=0,\ y(1)=y\Big(\frac{2}{3}\Big).\end{aligned}\tag{3.4}$$

Here $\alpha=1/2,\beta=3/2,\mu=1/5,\omega=1,\xi_1=1/3,\xi_2=2/3,m=3,a_1=1/2,a_2=1,a_3=3/4$ *and*

$$f_1(t,y)=\frac{1}{\sqrt{t^2+100}}\frac{|y|}{|y|+1}+e^{-t},\quad f_2(t,y)=\frac{1}{t^2+20}\tan^{-1}y+2,$$
$$f_3(t,y)=\frac{1}{15}\left(\frac{e^{-t}}{t^2+1}\right)\sin y+\frac{t^2}{1+t^2}.$$

It is easy to find that $|f_i(t,x)-f_i(t,y)|\leq L_i|x-y|, i=1,2,3,$ *with* $L_1=1/10,L_2=1/20,L_3=1/15$ *and* $\sum_{i=1}^{3}a_iL_i=3/20$. *Furthermore, we have*
$|\Delta|=\frac{|[\xi_1^{\beta}(1-\omega\xi_2)-\xi_1(1-\omega\xi_2^{\beta})]|}{\Gamma(\beta+1)}\approx0.000499$, $\bar{\rho_1}=\max_{t\in[0,1]}|\rho_1(t)|=\rho_1(t)|_{t=0.830537}\approx1.437777$,
$\bar{\rho_2}=\max_{t\in[0,1]}|\rho_2(t)|=\rho_2(t)|_{t=1}\approx1.605694$ *and* $\Omega_1=0.826088<1$. *Clearly all the assumptions of Theorem 3.1 are satisfied. Therefore the problem (3.4) has a unique solution on* $[0,1]$.

In the following result, we present location of the unique solution of the boundary value problem (1.1)-(1.2).

Theorem 3.2. *Let the hypotheses of Theorem 3.1 hold. Then the unique solution y of problem (1.1)-(1.2) satisfies*

$$\|y\|\leq\frac{\Omega_2}{1-\Omega_1},\tag{3.5}$$

where Ω_1 *is given by (3.2) and*

$$\Omega_2=\Big(\sum_{i=1}^{m}M_i\Big)\left\{\frac{1}{\Gamma(\beta+\alpha+1)}\left[1+\bar{\rho_1}\xi_1^{\beta+\alpha}+\bar{\rho_2}\left(1+|\omega|\,\xi_2^{\beta+\alpha}\right)\right]\right\},\tag{3.6}$$

with $M_i=\sup_{t\in[0,1]}|f_i(t,0)|$.

Proof. By Theorem 3.1, the solution (2.12) of the boundary value problem (1.1)-(1.2) is unique. In view of the inequality:

$$\Big|\sum_{i=1}^{m} a_i f_i(s, y(s))\Big| \leq \sum_{i=1}^{m} |a_i| \left(L_i \|y\| + M_i\right),$$

where $M_i = \sup_{t\in[0,1]} |f_i(t, 0)|$, it follows from (2.12) that

$$\begin{aligned}
\|y\| &\leq \sum_{i=1}^{m} |a_i| \left(L_i \|y\| + M_i\right) \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} ds \\
&+ |\rho_1(t)| \int_0^{\xi_1} \frac{(\xi_1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} ds \\
&\quad + |\rho_2(t)| \left[\int_0^1 \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} ds + |\omega| \int_0^{\xi_2} \frac{(\xi_2-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} ds \right] \Bigg\} \\
&\quad + |\mu| \|y\| \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds + |\rho_1(t)| \int_0^{\xi_1} \frac{(\xi_1-s)^{\beta-1}}{\Gamma(\beta)} ds \\
&\quad + |\rho_2(t)| \left[\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} ds + |\omega| \int_0^{\xi_2} \frac{(\xi_2-s)^{\beta-1}}{\Gamma(\beta)} ds \right] \Bigg\} \\
&\leq \|y\| \Bigg\{ \frac{\left(\sum_{i=1}^{m} |a_i| L_i\right)}{\Gamma(\beta+\alpha+1)} \left[1 + \bar{\rho}_1 \xi_1^{\beta+\alpha} + \bar{\rho}_2 \left(1 + |\omega| \xi_2^{\beta+\alpha}\right) \right] \\
&\quad + \frac{|\mu|}{\Gamma(\beta+1)} \left[1 + \bar{\rho}_1 \xi_1^{\beta} + \bar{\rho}_2 \left(1 + |\omega| \xi_2^{\beta}\right) \right] \Bigg\} \\
&\quad + \Big(\sum_{i=1}^{m} M_i \Big) \left\{ \frac{1}{\Gamma(\beta+\alpha+1)} \left[1 + \bar{\rho}_1 \xi_1^{\beta+\alpha} + \bar{\rho}_2 \left(1 + |\omega| \xi_2^{\beta+\alpha}\right) \right] \right\} \\
&= \|y\| \Omega_1 + \Omega_2.
\end{aligned}$$

Solving the above inequality for $\|y\|$ yields

$$\|y\| \leq \frac{\Omega_2}{1-\Omega_1}.$$

This completes the proof. □

Example 3.2. *In relation to Example 3.1, it is found that $\|y\| \leq \delta$, where $\delta \approx 35.008543$.*

4. An existence result

In this section, we establish an existence result for the boundary value problem (1.1)-(1.2) via Krasnosel'skiǐ's fixed point theorem [31].

Theorem 4.1. *Let $f_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the condition (A_2): $|f_i(t, y)| \leqslant h(t)$, $\forall\ i = 1, 2, \ldots, m$, $(t, y) \in [0,1] \times \mathbb{R}$, $h \in C([0,1], \mathbb{R}^+)$. Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0,1]$, provided that*

$$Q_1 := \frac{|\mu|}{\Gamma(\beta+1)} \left[1 + \bar{\rho}_1 \xi_1^{\beta} + \bar{\rho}_2 (1 + |\omega| \xi_2^{\beta}) \right] < 1. \tag{4.1}$$

Proof. Consider $B_r = \{y \in C([0,1],\mathbb{R}) : \|y\| \le r\}$, with $r > \dfrac{Q_2\|h\|}{1-Q_1}$, where

$$Q_2 := \sum_{i=1}^{m} |a_i| \left\{ \frac{1}{\Gamma(\beta+\alpha+1)} \left[1 + \bar{\rho_1}\xi_1^{\beta+\alpha} + \bar{\rho_2}\left(1 + |\omega|\, \xi_2^{\beta+\alpha}\right)\right]\right\} \tag{4.2}$$

and $\|h\| = \sup\limits_{t\in[0,1]} |h(t)|$. Then we define the operators $\mathcal{P}$ and $\mathcal{Q}$ on B_r to $C([0,1],\mathbb{R})$ as

$$\begin{aligned} \mathcal{P}(t) \;=\;& \int_0^t \frac{(t-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds \\ &+\rho_1(t)\int_0^{\xi_1} \frac{(\xi_1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds \\ &+\rho_2(t)\Bigg[\int_0^{1} \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds \\ &-\omega\int_0^{\xi_2} \frac{(\xi_2-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m a_i f_i(s,y(s))\Big)ds\Bigg], \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathcal{Q}(t) \;=\;& -\mu\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds - \mu\rho_1(t)\int_0^{\xi_1}\frac{(\xi_1-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds \\ &-\mu\rho_2(t)\Bigg[\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds - \omega\int_0^{\xi_2}\frac{(\xi_2-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds\Bigg]. \end{aligned} \tag{4.4}$$

We show that $\mathcal{P}x + \mathcal{Q}y \in B_r$. For $x, y \in B_r$, we find that

$$\begin{aligned} &\|(\mathcal{P}x) + (\mathcal{Q}y)\| \\ \le\;& \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m |a_i||f_i(s,y(s))|\Big)ds + |\mu|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|y(s)|ds \\ &+|\rho_1(t)|\Bigg[\int_0^{\xi_1}\frac{(\xi_1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m |a_i||f_i(s,y(s))|\Big)ds \\ &+|\mu|\int_0^{\xi_1}\frac{(\xi_1-s)^{\beta-1}}{\Gamma(\beta)}|y(s)|ds\Bigg] + |\rho_2(t)|\Bigg[\int_0^1 \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m |a_i||f_i(s,y(s))|\Big)ds \\ &+|\mu|\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}|y(s)|ds + |\omega|\int_0^{\xi_2}\frac{(\xi_2-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}\Big(\sum_{i=1}^m |a_i||f_i(s,y(s))|\Big)ds \\ &+|\mu||\omega|\int_0^{\xi_2}\frac{(\xi_2-s)^{\beta-1}}{\Gamma(\beta)}|y(s)|ds\Bigg]\Bigg\} \\ \le\;& \sum_{i=1}^m |a_i|\,\|h\|\left\{\frac{1}{\Gamma(\beta+\alpha+1)}\left[1+\bar{\rho_1}\xi_1^{\beta+\alpha}+\bar{\rho_2}\left(1+|\omega|\,\xi_2^{\beta+\alpha}\right)\right]\right\} \\ &+\frac{|\mu|\,r}{\Gamma(\beta+1)}\left[1+\bar{\rho_1}\xi_1^{\beta}+\bar{\rho_2}\left(1+|\omega|\,\xi_2^{\beta}\right)\right] \\ \le\;& Q_2\,\|h\| + Q_1 r \le r \end{aligned}$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_r$.

Next we show that $\mathcal{Q}$ is a contraction mapping. For $x, y \in C([0,1], \mathbb{R})$ and each $t \in [0,1]$, we obtain

$$\begin{aligned}
&\|(\mathcal{Q}x) - (\mathcal{Q}y)\| \\
\leq\ & |\mu| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |x(s) - y(s)| ds + |\mu||\rho_1(t)| \int_0^{\xi_1} \frac{(\xi_1 - s)^{\beta-1}}{\Gamma(\beta)} (x(s) - y(s)| ds \\
& + |\mu||\rho_2(t)| \Bigg[\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} |x(s) - y(s)| ds \\
& + |\omega| \int_0^{\xi_2} \frac{(\xi_2 - s)^{\beta-1}}{\Gamma(\beta)} |x(s) - y(s)| ds \Bigg] \\
\leq\ & \frac{|\mu|}{\Gamma(\beta+1)} \Big[1 + \bar{\rho}_1 \xi_1^\beta + \bar{\rho}_2 (1 + |\omega| \xi_2^\beta) \Big] \|x - y\|,
\end{aligned}$$

which is a contraction in view of the condition (4.1).

We show that $\mathcal{P}$ is compact and continuous. The continuity of f_i implies that the operator $\mathcal{P}$ is continuous. $\mathcal{P}$ is uniformly bounded on B_r as $\|\mathcal{P}x\| \leq Q_2 \|h\|$, where Q_2 is given by (5.1).

We shall prove now that $\mathcal{P}$ is equicontinuous. For $t_1, t_2 \in [0,1]$ with $t_1 > t_2$, we have

$$\begin{aligned}
&|\mathcal{P}y(t_1) - \mathcal{P}y(t_2)| \\
\leq\ & \Big| \int_0^{t_1} \frac{(t_1 - s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \Big(\sum_{i=1}^m a_i f_i(s, y(s)) \Big) - \int_0^{t_2} \frac{(t_2 - s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \Big(\sum_{i=1}^m a_i f_i(s, y(s)) \Big) \Big| \\
& + |\rho_1(t_1) - \rho_1(t_2)| \int_0^{\xi_1} \frac{(\xi_1 - s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \Big(\sum_{i=1}^m a_i f_i(s, y(s)) \Big) ds \\
& + |\rho_2(t_1) - \rho_2(t_2)| \Bigg[\int_0^1 \frac{(1-s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \Big(\sum_{i=1}^m |a_i||f_i(s, y(s))| \Big) ds \\
& + |\omega| \int_0^{\xi_2} \frac{(\xi_2 - s)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} \Big(\sum_{i=1}^m |a_i||f_i(s, y(s))| \Big) ds \Bigg] \\
\leq\ & \frac{\sum_{i=1}^m |a_i| \|h\|}{\Gamma(\beta+\alpha+1)} \Big\{ 2 (t_1 - t_2)^{\beta+\alpha} + |t_1^{\beta+\alpha} - t_2^{\beta+\alpha}| + |\rho_1(t_1) - \rho_1(t_2)| \xi_1^{\beta+\alpha} \\
& + |\rho_2(t_1) - \rho_2(t_2)| (1 + |\omega| \xi_2^{\beta+\alpha}) \Big\},
\end{aligned}$$

which is independent of y and tends to zero as $t_1 - t_2 \to 0$. So $\mathcal{P}$ is equicontinuous. Hence, by Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on B_r. Thus all the assumptions of Krasnosel'skiĭ's fixed point theorem [31] are verified and hence its conclusion implies that the boundary value problem (1.1)-(1.2) has at least one solution on $[0,1]$. The proof is completed. □

5. Concluding remarks

We have discussed the solvability of Caputo-Liouville type Langevin equation involving two fractional orders and finitely many nonlinearities subject to nonlocal boundary conditions by means of standard fixed point theorems. It is imperative to notice that the right-hand side of (1.1) provides a leverage to consider a variety of nonlinearities, for instance, some of the terms in the given sum may be of the type $f_i(t,y) = \int_0^t k_i(t,s)y(s)ds$ or $\int_0^t g_i(s,y(s))ds$ or $I^p g_i(t,y(t))$ or some of the functions may be non-Lipschitz type. Here are two examples.

(a): Choosing the right-hand side of (1.1) as Riemann-Liouville type integral nonlinearities of the form: $\sum_{i=1}^m a_i I^{q_i} f_i(t,y(t))$, $q_i > 0$ instead of $\sum_{i=1}^m a_i f_i(t,y(t))$, the results for the problem (1.1) and (1.2) obtained in the previous sections become the ones for the problem with Riemann-Liouville type integral nonlinearities by replacing Ω_1 and Q_2 with $\widehat{\Omega}_1$ and $\widehat{Q}_2$ respectively, where

$$\begin{aligned}\widehat{\Omega}_1 = & \frac{1}{\Gamma(\beta+\alpha+1)} \sum_{i=1}^m \frac{(|a_i| L_i)}{\Gamma(q_i+1)} \left[1 + \bar{\rho}_1 \xi_1^{\beta+\alpha} + \bar{\rho}_2 \left(1 + |\omega| \xi_2^{\beta+\alpha}\right)\right] \\ & + \frac{|\mu|}{\Gamma(\beta+1)} \left[1 + \bar{\rho}_1 \xi_1^{\beta} + \bar{\rho}_2 \left(1 + |\omega| \xi_2^{\beta}\right)\right],\end{aligned}$$

$$\widehat{Q}_2 := \sum_{i=1}^m \left\{ \frac{|a_i|}{\Gamma(q_i+1)\Gamma(\beta+\alpha+1)} \left[1 + \bar{\rho}_1 \xi_1^{\beta+\alpha} + \bar{\rho}_2 \left(1 + |\omega| \xi_2^{\beta+\alpha}\right)\right] \right\}.$$

(b): By a simple manipulation, we can obtain the results for the problem (1.1) and (1.2) with the right-hand side of (1.1) of the form:

$$\sum_{i=1}^m a_i f_i(t,y(t)) + \sum_{j=1}^{\kappa} a_j I^{q_j} g_j(t,y(t)),\ q_j > 0,$$

where f_i and g_j are given continuous functions.

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