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ON FIRMLY NON-EXPANSIVE MAPPINGS

JOSEPH FRANK GORDON[1,*], ESTHER OPOKU GYASI[2]

[1] *Department of Mathematics Education, Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, Kumasi, Ghana*

[2] *Department of Mathematical Sciences, University of Mines and Technology, Tarkwa, Ghana*

[*] *Corresponding author:* josephfrankgordon@gmail.com

Abstract. In this paper, it is shown that for a closed convex subset $\mathcal{C}$ and to every non-expansive mapping $T : \mathcal{C} \to \mathcal{C}$, one can associate a firmly non-expansive mapping with the same fixed point set as T in a given Banach space.

1. Introduction

The study of non-expansive mappings in the sixties have experimented a boost, basically motivated by Browder's work on the relationship between monotone operators, non-expansive mappings [1–3, 3–5] and the seminal paper by Kirk [6], where the significance of the geometric properties of the norm for the existence of fixed points for non-expansive mappings was highlighted.

Now the history of firmly non-expansive mappings goes back to the paper by Minty [7], where he implicitly used this class of mappings to study the resolvent of a monotone operator. Browder [3] first introduced firmly non-expansive mappings in the concept of Hilbert spaces $\mathcal{H}$. That is, given a $\mathcal{C}$ closed convex subset of a Hilbert space $\mathcal{H}$, a mapping $F : \mathcal{C} \to \mathcal{H}$ is firmly non-expansive if for all $x, y \in \mathcal{C}$

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle. \tag{1.1}$$

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In his study of non-expansive projections on subsets of Banach spaces, Bruck [8] defined a firmly non-expansive mapping $F : \mathcal{C} \to \mathcal{X}$, where $\mathcal{C}$ is a closed convex subset of a real Banach space $\mathcal{X}$, to be a mapping such that for all $x, y \in \mathcal{C}$ and $\alpha \geq 0$,

$$\|Fx - Fy\| \leq \|\alpha(x - y) + (1 - \alpha)(Fx - Fy)\|. \tag{1.2}$$

It is clear that equation (1.2) reduces to equation (1.1) when in Hilbert spaces and also when $\alpha = 1$, F becomes a non-expansive mappings, that is, for each x and y in $\mathcal{C}$, we have $\|Fx - Fy\| \leq \|x - y\|$. A trivial example of equation (1.2) is the identity mapping. A non-trivial example of equation (1.2) in a Hilbert space is given by the metric projection

$$P_{\mathcal{C}}x = \text{argmin}_{y\in\mathcal{C}}\{\|x - y\|\}. \tag{1.3}$$

To see this, recall that in a real Hilbert space $\mathcal{H}$, $\forall x, y \in \mathcal{H}$, then $\langle x, y\rangle \geq 0$ if and only if

$$\|x\| \leq \|x + ay\|, \tag{1.4}$$

for all $a \geq 0$. Now equation (1.2) can be written as

$$\|Fx - Fy\| \leq \|Fx - Fy + \alpha(x - y - Fx + Fy)\|. \tag{1.5}$$

Now applying equation (1.4) on equation (1.5), we obtain the following

$$\begin{aligned}
\langle Fx - Fy, x - y - Fx + Fy\rangle &\geq 0,\\
\langle x - y - Fx + Fy, Fx - Fy\rangle &\geq 0,\\
\langle x - y - (Fx - Fy), Fx - Fy\rangle &\geq 0,\\
\langle x - y, Fx - Fy\rangle - \langle Fx - Fy, Fx - Fy\rangle &\geq 0,\\
\langle x - y, Fx - Fy\rangle &\geq \langle Fx - Fy, Fx - Fy\rangle,\\
\langle x - y, Fx - Fy\rangle &\geq \|Fx - Fy\|^2.
\end{aligned}$$

Hence we have that in a real Hilbert space, a firmly non-expansive mapping F can be written as

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle. \tag{1.6}$$

But in a real Hilbert space, equation (1.3) satisfies the following inequality

$$\|P_{\mathcal{C}}x - P_{\mathcal{C}}y\|^2 \leq \langle x - y, P_{\mathcal{C}}x - P_{\mathcal{C}}y\rangle. \tag{1.7}$$

This means that from equations (1.6) and (1.7), we can simply conclude that $F = P_{\mathcal{C}}$ and so the metric projection $P_{\mathcal{C}}$ is a firmly non-expansive mapping in a real Hilbert space.

In this paper, we give a simple proof showing that to any non-expansive self-mapping $T : \mathcal{C} \to \mathcal{C}$ that has fixed points, one can associate a large family of firmly non-expansive mappings having the same fixed point set as T. That is, from the point of view of the existence of fixed points on closed convex sets, non-expansive and firmly non-expansive mappings exhibit a similar behavior. However, this is no longer true in non-convex domains [9].

2. Main Results

Let T be a non-expansive mapping defined on a closed convex subset $\mathcal{C}$ of a normed space $\mathcal{X}$, thus, $T : \mathcal{C} \to \mathcal{C}$. For a fixed $r \in \mathbb{R}_{>1}$, we can define the following mapping

$$T_r : \mathcal{C} \to \mathcal{C} \quad \text{by} \quad x \mapsto \Big(1 - \frac{1}{r}\Big)x + \frac{1}{r}T(T_r x). \tag{2.1}$$

Now we observe that equation (2.1) (the new mapping T_r) always exist. To see this, one can create an internal contraction $F : \mathcal{C} \to \mathcal{C}$ such that

$$F(y) = \Big(1 - \frac{1}{r}\Big)x + \frac{1}{r}Ty, \quad \text{where} \quad x \quad \text{is fixed.}$$

Now $\|F(y) - F(z)\| = \frac{1}{r}\|Ty - Tz\| \le \frac{1}{r}\|y - z\|$. Hence F is a contraction mapping and by the Banach contraction mapping theorem [10], there exists $u \in \mathcal{C}$ such that $F(u) = u$, thus, $u = (1 - \frac{1}{r})x + \frac{1}{r}Tu$. Since for every $x \in \mathcal{C}$, we can find a unique u such that $u = T_r x$, then equation (2.1) always exists. Now we have the following claims.

Claim 1: T_r is a non-expansive mapping. To see this, we have the following:

$$\begin{aligned}
\|T_r x - T_r y\| &= \|(1 - \frac{1}{r})(x - y) + \frac{1}{r}(T(T_r x) - T(T_r y))\|, \\
&\le (1 - \frac{1}{r})\|x - y\| + \frac{1}{r}\|T(T_r x - T(T_r y))\|, \\
&\le (1 - \frac{1}{r})\|x - y\| + \frac{1}{r}\|T_r x - T_r y)\|, \\
\|T_r x - T_r y\| - \frac{1}{r}\|T_r x - T_r y\| &\le (1 - \frac{1}{r})\|x - y\|, \\
(1 - \frac{1}{r})\|T_r x - T_r y\| &\le \|(1 - \frac{1}{r})\|\|x - y\|.
\end{aligned}$$

So we have that T_r is a non-expansive mapping since $r > 1$.

Claim 2: Now we prove that $T_r x$ is a firmly non-expansive mapping.

Now for $r > 1, \alpha \in (0, 1)$ and $\beta > 0$, we have the following evaluation:

$$\|T_r x - T_r y\| = \|\beta[\alpha(x - y) + (1 - \alpha)(T_r x - T_r y)] - \beta\alpha(x - y)$$
$$+ (T_r x - T_r y) - \beta(1 - \alpha)(T_r x - T_r y)\|.$$

But

$$
\begin{aligned}
(T_rx - T_ry) - \beta(1-\alpha)(T_rx - T_ry) &= \Big(1 - \frac{1}{r}\Big)(x-y) + \frac{1}{r}\Big(T(T_rx) - T(T_ry)\Big) \\
&\quad - \beta(1-\alpha)\Big[\Big(1 - \frac{1}{r}\Big)(x-y) + \frac{1}{r}\Big(T(T_rx) - T(T_ry)\Big)\Big], \\
&= \Big(1 - \frac{1}{r}\Big)(x-y) - \beta(1-\alpha)\Big(1 - \frac{1}{r}\Big)(x-y) + \frac{1}{r}\Big(T(T_rx) - T(T_ry)\Big) \\
&\quad - \beta(1-\alpha)\frac{1}{r}\Big(T(T_rx) - T(T_ry)\Big), \\
&= \Big(1 - \frac{1}{r}\Big)(x-y)[1 - \beta(1-\alpha)] \\
&\quad + \frac{1}{r}\Big(1 - \beta(1-\alpha)\Big)(T(T_rx) - T(T_ry)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|T_rx - T_ry\| &= \|\beta[\alpha(x-y) + (1-\alpha)(T_rx - T_ry)] - \beta\alpha(x-y) + \Big(1 - \frac{1}{r}\Big)(x-y)[1 - \beta(1-\alpha)] \\
&\quad + \frac{1}{r}\Big(1 - \beta(1-\alpha)\Big)(T(T_rx) - T(T_ry))\|, \\
&= \|\beta[\alpha(x-y) + (1-\alpha)(T_rx - T_ry)] + [-\beta\alpha + \Big(1 - \frac{1}{r}\Big)(1 - \beta(1-\alpha))](x-y) \\
&\quad + \frac{1}{r}\Big(1 - \beta(1-\alpha\Big)(T(T_rx) - T(T_ry))\|.
\end{aligned}
$$

Now let $-\beta\alpha + \Big(1 - \frac{1}{r}\Big)(1 - \beta(1-\alpha)) = 0$. This implies that

$$
\begin{aligned}
\beta &= \frac{r-1}{\alpha r + (\alpha - 1)(r-1)}, \\
\frac{1}{r}(1 - \beta(1-\alpha)) &= \frac{\alpha}{\alpha r + (1-\alpha)(r-1)}.
\end{aligned}
$$

Hence we have

$$
\|T_rx - T_ry\| \leq \beta\|\alpha(x-y) + (1-\alpha)(T_rx - T_ry)\| + \frac{\alpha}{\alpha r + (1-\alpha)(r-1)}\|T(T_rx) - T(T_ry)\|.
$$

So by the non-expansiveness of T,The above inequality becomes

$$
\begin{aligned}
\|T_rx - T_ry\| &\leq \beta\|\alpha(x-y) + (1-\alpha)(T_rx - T_ry)\| + \frac{\alpha}{\alpha r + (1-\alpha)(r-1)}\|T_rx - T_ry\|, \\
&= \frac{r-1}{\alpha r + (1-\alpha)(r-1)}\|\alpha(x-y) + (1-\alpha)(T_rx - T_ry)\| \\
&\quad + \frac{\alpha}{\alpha r + (1-\alpha)(r-1)}\|T_rx - T_ry\|.
\end{aligned}
$$

After simplifying we obtain the following results

$$
\begin{aligned}
\|T_r x - T_r y\| - \frac{\alpha}{\alpha r + (1-\alpha)(r-1)}\|T_r x - T_r y\| &\leq \frac{r-1}{\alpha r + (1-\alpha)(r-1)}\|\alpha(x-y) + (1-\alpha)(T_r x - T_r y)\|,\\
[1 - \frac{\alpha}{\alpha r + (1-\alpha)(r-1)}]\|T_r x - T_r y\| &\leq \frac{r-1}{\alpha r + (1-\alpha)(r-1)}\|\alpha(x-y) + (1-\alpha)(T_r x - T_r y)\|,\\
\frac{r-1}{\alpha r + (1-\alpha)(r-1)}\|T_r x - T_r y\| &\leq \frac{r-1}{\alpha r + (1-\alpha)(r-1)}\|\alpha(x-y) + (1-\alpha)(T_r x - T_r y)\|,\\
\|T_r x - T_r y\| &\leq \|\alpha(x-y) + (1-\alpha)(T_r x - T_r y)\|,
\end{aligned}
$$

since $\beta = \frac{r-1}{\alpha r + (1-\alpha)(r-1)} > 0$. Hence T_r is firmly non-expansive mapping.

Claim 3: We have that z is a fixed point of T if and only if it is also a fixed point of T_r.

Proof. Now suppose that $Tz = z$. Then we have the following evaluation:

$$
\begin{aligned}
\|T_r z - z\| &= \Big\|\Big(1 - \frac{1}{r}\Big)z + \frac{1}{r}T(T_r z) - z\Big\|,\\
&= \Big\|\frac{1}{r}T(T_r z) - \frac{1}{r}z\Big\|,\\
&= \frac{1}{r}\Big\|T(T_r z) - z\Big\|,\\
&= \frac{1}{r}\Big\|T(T_r z) - Tz\Big\|,\\
&\leq \frac{1}{r}\Big\|T_r z - z\Big\|.
\end{aligned}
$$

Hence $\|T_r z - z\| \leq \frac{1}{r}\|T_r z - z\|$ which is not possible since $r > 1$. it is possible when $\|T_r z - z\| = 0 \Rightarrow T_r z = z$. So z is a fixed point of $T_r z$. On the other hand, let us suppose that $T_r z = z$, that is z is a fixed point of T_r. Then

$$
\begin{aligned}
z &= T_r z,\\
&= \Big(1 - \frac{1}{r}\Big)z + \frac{1}{r}T(T_r z),\\
&= \Big(1 - \frac{1}{r}\Big)z + \frac{1}{r}T(z).
\end{aligned}
$$

This gives us

$$
\Big[1 - \Big(1 - \frac{1}{r}\Big)\Big]z = \frac{1}{r}T(z).
$$

Hence z is a fixed point of T and that concludes our main result. □

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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