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PRICING OPTIONS IN A DELAYED MARKET DRIVEN BY LE'VY NOISE

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Abstract. In this paper we studied stochastic delayed differential equations driven by Le'vy noise. The analogue of Itô formula is considered. The Black-Scholes formula analogue for Vanilla call option price formula is derived.

1. Introduction

In this paper we studied the Stochastic Delay Differential Equations driven by Le'vy noise which arise in many applications of stochastic analysis in finance specifically in pricing of options security markets. As known such systems are quite hard to study due to their lack of Markovianity which is a key property for the study of option prices. Basically, the difficulties arises from the fact that delay systems have, in general, an infinite dimensional nature.

The model for the stock price $\zeta(t)$ that we consider satisfies a stochastic delay differential equation driven by Le'vy noise with volatility σ depending on time t and the path $\zeta_t = \{\zeta(t+\theta), \theta \in [-\tau, 0]\}$ called a level and past-dependent volatility. An analogue of Itô's formula for such a stochastic systems is obtained.

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An option price value of the form

$$G(t,\zeta_t)=\int_{-\tau}^{0} e^{r\theta}F(\zeta(t+\theta),\zeta(t),t)d\theta$$

is studied when F is in $\mathbb{C}^{0,1,2}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^3)$. A special case of $G(t,\zeta_t)$ of the form

$$G(t,\zeta_t)=g_1(\zeta(t),t)+\int_{-\tau}^{0} e^{r\theta}g_2(\zeta(t+\theta),t)d\theta$$

when $g_1(\zeta(t),t)$ is a classical Black-Scholes call option is studied. The partial differential equation of Black-Scholes type is derived for such option.

2. Notations and Preliminaries

Let us consider a probability space $(\Omega,\mathcal{F},\rho)$ on which is defined $((B(t))_{t\geq 0},(\eta(t))_{t\geq 0})$ where $((B(t))_{t\geq 0}$ and $(\eta(t))_{t\geq 0})$ are independent stochastic processes

-$((B(t))_{t\geq 0}$ is a standard Brownian motion with respect to its natural filtration,

-$(\eta(t))_{t\geq 0})$ is a pure jump Le'vy process.

The Poisson random measure $N(t)$ of the process η is defined by

$$N(t,A)=\sum_{\zeta\in[0,t]} I_A(\eta(s)-\eta(s^-)),\quad A\subset\mathbb{R}.$$

The Le'vy measure ν of the process η is supposed to be $\int_{\mathbb{R}} z^2\nu(dz)<+\infty$, since $(\eta(t))_{t\geq 0})$ is a pure jump $\nu(\{0\})=0$.

Define the measure-valued process $(\tilde{N}(t))_{t\geq 0}$ by

$$\tilde{N}(t,dz):=N(t,dz)-\nu(dz)$$

so that the compensated poisson random measure is

$$\tilde{N}(dt,dz):=N(dt,dz)-\nu(dz)dt$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by the process $B(t)$ and $\eta(t)$ as defined above. Since B and η are assumed independent, then $B(.)$ and $\tilde{N}(.,A)$ are still Brownian motion and square integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

2.1. Stochastic Delay Differential Equations Driven By Le'vy Noise

Consider the following Sdde:

$$
\begin{aligned}
dx(t) =& \mu(t,x_t)dt + \sigma(t,x_t)dw(t) + \int_{\mathbb{R}} \gamma(x_t^-, z)\tilde{N}(dt,dz), t \in [0,T] \\
x_0 =& x(\theta), \quad \theta \in [-r,0]
\end{aligned}
\tag{2.1}
$$

where $\mu : [0,T] \times D \to \mathbb{R}^n, \sigma : [0,T] \times D \to \mathbb{R}^{nd}, \gamma : D \times D \to \mathbb{R}$ are predictable processes.

For the unknown process $(x(t))_{t\in[-r,T]}$ in $\mathbb{R}, T < \infty$ is a fixed finite time, $x(t)$ is the value of x at $t \in [0,T]$ and x_t its segment, i.e. its value in the past time interval $[t-r,t]$ i.e. $x_t(.) : [-r,0] \to \mathbb{R}$ defined by $x_t(\theta) : x(t+\theta)$ for all $\theta \in [-r,0]$.
The initial data $x(\theta)$ is assumed to be in the space $D := D([-r,0],\mathbb{R})$ of Càdlàg random variables from $[-r,0]$ to $\mathbb{R}$.

Hypotheses 2.1 (i). There exists constant $L > 0$ such that for all $t \in [0,T]$ and for all $x_1, x_2 \in D$,

$$|\mu(t,x_1) - \mu(t,x_2)| + |\sigma(t,x_1) - \sigma(t,x_2) + \int_{\mathbb{R}} |\gamma(x_1,z) - \gamma(x_2,z)|\nu(dz) \leq L|x_1 - x_2|.$$

(ii). The functions μ, σ, γ satisfy the linear growth condition, i.e. there exists a constant $K > 0$ such that for all $x \in D$,

$$|\mu(t,x)| + |\sigma(t,x)| + \int_{\mathbb{R}} |\gamma(x,z)|\nu(dz) \leq K(1 + |x|).$$

Theorem 2.1. Suppose that hypotheses (i) and (ii) hold. Then there exists a unique $Ca'dla'g$ adapted solution to equation (2.1). For proof we refer to [7].

The infinitesimal operator of the solution of equation (2.1)

In reformulation of equation (2.1) in infinite dimension, the following linear stochastic evolution equation in the space H

$$dx(t) = Ax(t)dt + < \sigma, x(t) > -\hat{n}dw(t) + \int_{\mathbb{R}} < r(z), x(t^-) > -\hat{n}\tilde{N}(dt,dz) \quad (2.1)'$$

will represents equation (2.1) in the sense that

$$x(t) = (x_0(t), x_1(t)) = (x(t), x(t+\theta) : \theta \in [-T,0], \forall t \geq 0$$

where A is defined on

$$D(A = \{y = (y_0, y_1(.)) \in H; y_1(.) \in W^{1,2}([-1,0],\mathbb{R}), y_0 = y_1(0)\}$$

by $Ay = (\mu_0 y_0 + \int_{-T}^{0} \mu_1(\theta) y_1(\theta) d\theta + \mu_2 y_1(-T), y_1'(.))$ is the generator of a strongly continuous semigroup $(s(t))_{t\geq 0}$ on H. Here $\hat{n} = (1,0) \in H$ and $x(t^-) := \lim_{s\uparrow t} x(s) = \lim_{s\uparrow t}(x(s), s(s+.))$ with the limit taken in H.

Theorem 2.1. Let $y \in C$ and let $x(.;y), x(.;y)$ representing the solutions of (2.1) and $(2.1)'$ respectively. Then $x(.;y)$ represents $s(.;y)$ in the sense that

$$x(t;y) = (x(t;y), x(t+\theta;y) : \theta \in [-T,0]), \forall t \geq 0.$$

For proof see [11].

The infinitesimal operator of the process $x(.,0)$ is formally defined as

$$[L\phi](y) :=< Ay, \varphi_y(y) > + \frac{1}{2} < \sigma, y > \varphi_{y_0 y_0}(y) +$$
$$\int_{\mathbb{R}} [\varphi(y+ < r(z), y > \hat{n}) - \varphi(y) - \varphi_{y_0}(y) < r(z), y >]\nu(dz), \quad y \in D(A), \varphi \in C^2(H;\mathbb{R}).$$

3. Option Price Formula

Assume that the stock price satisfy the following stochastic delay differential equation of the form

$$\begin{aligned} ds(t) =& rs(t)dt + \sigma(t, s_t)dw(t) + \int_{\mathbb{R}} \gamma(s_t^-, z)\tilde{N}(dt,dz), t \in [0,T] \\ & s(0) = y_0, s(\theta) = y_1(\theta), \quad \theta \in [-r,0] \end{aligned} \tag{3.1}$$

where $y := (y_0, y_1(.)) \in C$ is positive. Here C is the subspace of the Hilbert space $H := \mathbb{R} \times L^2_{-r} := \mathbb{R} \times L^2([-r,0],\mathbb{R})$ whose inner product $< .,. >$ is

$$< .,. >=< .,. >_{\mathbb{R}} + < .,., >_{L^2_r}$$

$C := y \in H : y_1(.)$ admits a $C\grave{a}dla'g$ representative.
In (3.1) $r \in \mathbb{R}, \sigma := (\sigma_0, \sigma_1(.) \in H, \gamma(.) := (\gamma_0, \gamma_1(.)(.)) \in L^2(\mathbb{R}, \nu; H)$ are functional parameters.

Analogue of Black-Scholes formula for Vanilla call option price:

Suppose that the financial market under consideration as follows: (i). A risk free asset given by

$$ds_0(t) = r(s_0)(t)dt; t \in [0.T].$$

(ii). A risky asset given by equation (3.1)-(3.2).

A portfolio in such market is an F_t predictable process $\Pi(t)$ representing the number of units held at time t of the assets number $0, 1, \dots, n$ respectively, then the wealth process $x(t) = x^{\Pi}(t)$ associated to the portfolio Π is defined to be:

$$x^{\Pi}(t) = \Pi(t)s(t) = \sum_{i=1}^{n} \Pi_i(t)s_i(t).$$

Absence of arbitrage

In order to have no arbitrage opportunities in the considered market, the return from the portfolio must be risk-free with interest rate r.

In what follows we adopt that $\Pi(t)$ is riskless during $[t, t + dt]$ and instantaneously earn the same rate of return as other short-term risk-free assets.
These assumptions on $\Pi(t)$ gives $d\Pi(t) = r\Pi(t)dt$.

Let the option price value has the form

$$G(t, s_t) = \int_{-T}^{0} e^{-r\theta} F(s(t+\theta), s(t), t)d\theta \tag{3.2}$$

where $F \in C^{0,2,1}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^+)$.

Lemma 3.1. (It$\hat{o}$ Formula)

Suppose $s(t)$ is given by (3.1) and a functional $G : \mathbb{R}^+ \times C \to \mathbb{R}$ has the form

$$G(t, s_t) = \int_{-\tau}^{0} g(\theta)F(s_t(\theta), s_t(0), t)d\theta, \tag{3.3}$$

where $F \in C^{0,2,1}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ and $g \in C^1([-\tau, 0], \mathbb{R})$.
Hence in view of the classical It$\hat{o}$ formula, we have

$$\begin{aligned} G(t, s_t) =& G(0, \zeta) + \int_0^t AG(s, S_s)ds + \int_0^t \sigma(s, S_s)S(s)BG(s, S_s)dw(s) \\ &+ \int_0^t \int_{\mathbb{R}} \{F(t, s(t^-) + \gamma(s, z)) - F(t, s(t^-)) - F(t, S(t)^-)\}N(dt, dz) \end{aligned} \tag{3.4}$$

where for $(t, x) \in \mathbb{R}^+ \times C$.

$$\begin{aligned} AG(t, y) =& g(0)F(x_0, x_0, t) - g(-\tau)F(x(-\tau), x_0, t) - \\ & \int_{-\tau}^{0} g'(\theta)F(x(\theta), x0, t)d\theta + \int_{-\tau}^{0} g(\theta)LF(x(\theta), x0, t)d\theta \\ & + \int_{\mathbb{R}} \{F(t, x(t^-) + \gamma(x, z)) - F(t, x(t^-))\}N(dt, dz), \end{aligned}$$

with

$$\begin{aligned}LF(x(\theta,x(0),t) =& rx(0)F_2'(x(\theta,x(0),t)+\\ &\frac{\sigma^2(t,x)x^2(0)}{2}F_{22}''(x(\theta,x(0),t)+F_3'(x(\theta,x(0),t)\\ &+\int_{\mathbb{R}}\{F(x+\gamma(x,z))-F(x)-F_2'.\gamma(x,z)\}\nu(dz),\end{aligned}$$

where $F_i', i=1,2,3$ represents the derivative of F with respect to the i^{th} argument.

Portfolio concepts for financial markets driven by Le'vy process:

Theorem 3.1. The option price value given by (3.2) satisfies the equation

$$0=F|_{\theta=0}-e^{r\theta}F|_{\theta=-\tau}+\int_{-\tau}^{0}e^{-r\theta}\{(F_3'+rs(t)F_2'+\frac{1}{2}\sigma^2(t,s(t))s^2(t)F_{22}'')d\theta+ \int_{\mathbb{R}}\{F(s+\gamma(s,z))-F(s))-F_2'\gamma(s,z)\}\nu(dz)\}d\theta.$$

Proof. We sketch the proof as in [12] as follows:

By considering a portfolio consists of -1 derivative and $BG(t,s_t)$ shares. Then if $\Pi(t)$ represents the portfolio value we have

$$\Pi(t)=-G(t,s_t)+BG(t,s_t)s(t)=-dG+d(BGS)=d(BG)S+BGdS-dF.$$

Hence

$$d\Pi(t)=-dG+BGdS.$$

Since we assume BG is held constant during the time-step dt yields $d(BG)$ equal zero.

Substituting for dG and dS from equation (3.4) and (3.1) we get

$$d\Pi=-\overline{A}Gdt-\sigma sBGdw+BG(rsdt+\sigma sdw) \tag{3.5}$$

where

$$\overline{A}=A-\int_{\mathbb{R}}^{t}\{F(t,s(t^-)+\gamma(s,z))-F(t,s(t^-))-F(t,S(t)^-)\}N(dt,dz).$$

By considering risk-free during the time dt gives

$$d\Pi=r\Pi dt. \tag{3.6}$$

By equating the last equations we get

$$\overline{A}G(t,s_t) = rG(t,s_t).$$

which gives an equation for $F(S(t+\theta),S(t),t)$ in the form

$$0 = F|_{\theta=0} - e^{r\theta}F|_{\theta=-\tau} + \int_{-\tau}^{0} e^{-r\theta}\{(F_3' + rs(t)F_2' + \frac{1}{2}\sigma^2(t,s(t))s^2(t)F_{22}'')d\theta + \int_{\mathbb{R}}\{F(s+\gamma(s,z)) - F(s)) - F_2'\gamma(s,z)\}\nu(dz)\}d\theta.$$

4. Price Formula for European Call Option:

Theorem 4.1.(Black-Scholes PDE Type)

In view of (3.3) the option price value $G(T,S_T)$ will has the form

$$G(T,S_T) = \max(S(T) - K, 0). \tag{4.1}$$

For simplification we assume that $G(T,S_t)$ takes the form

$$G(t,s_t) = g_1(s(t),t) + \int_{-\tau}^{0} e^{-r\theta} g_2(s(t+\theta),t)d\theta, \tag{4.2}$$

where $g_1(s(t),t)$ is a classical Black-Scholes call option price with variance assumed equal to a long-run variance rate V, then

$$g_1(s(t),t) = s(t)N(d_1) - ke^{-r(T-t)}N(d_2), \tag{4.3}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}dx$$
$$d_1 = \ln(\frac{s(t)}{k}) + (r + \frac{V}{2})(T-t)$$
$$d_2 = d_1 - \sqrt{V(T-t)}.$$

Then $G(t,S_t)$ satisfies the following equation:

$$\frac{\partial G}{\partial t} + rS(t)\frac{\partial g_1}{\partial S} + \frac{1}{2}[\sigma^2(t,s_t)]s^2(t)\frac{\partial^2 g_1}{\partial s^2} + \int_{\mathbb{R}}\{F(s+\gamma(s,z)) - F(s)) - F_2'\gamma(s,z)\}\nu(dz) = rG.$$

Proof. By substituting (4.1) into (3.5) we obtain the equation for g_2 as follows

$$g_2(S(t),t) - e^{r\tau} g_2(S(t-\tau),t) + \int_{tau}^{0} e^{-r\theta} \frac{\partial g_2}{\partial t} d\theta$$
$$= \frac{1}{2}(V - \sigma^2(t, S(t)))S^2(t)\frac{\partial^2 g_1}{\partial S^2} +$$
$$\int_{\mathbb{R}} \{F(s+\gamma(s,z)) - F(s)) - F_2'\gamma(s,z)\}\nu(dz).$$

Since g_1 satisfies the classical Black-Scholes PDE we have:

$$\frac{\partial g_1}{\partial t} + rS(t)\frac{\partial g_1}{\partial S} + \frac{1}{2}[V]S^2(t)\frac{\partial^2 g_1}{\partial s^2} = rg_1.$$

By combining the last equation we get the following equation $G(t, S_t)$ of the form :

$$\frac{\partial G}{\partial t} + rS(t)\frac{\partial g_1}{\partial S} + \frac{1}{2}[\sigma^2(t, s_t)]s^2(t)\frac{\partial^2 g_1}{\partial s^2} +$$
$$\int_{\mathbb{R}} \{F(s+\gamma(s,z)) - F(s)) - F_2'\gamma(s,z)\}\nu(dz) = rG,$$

which is an integro-differential equation.

Remark. For pricing **American Put Option** we give the following discussion :

Suppose the functional (3.3) has the form $S(t) - \int_{-\tau}^{0} S(t+\xi)d\xi$ then the pricing formula for the European Put Option is to minimize the following functional overall F_t stoping time, $\tau < \infty$ a.s. for all $t \geq 0$:

$$F(\tau, S_\tau) = E[(K - (S(\tau) - \int_{-T}^{0} S(t+\xi)d\xi))^+]$$

where $K > 0$.

The dynamics of the stock prices are driven by the following stochastic differential equation:

$$dS(t) = \mu(S(t) - S(t-T))dt + \alpha(S(t) - \mu\int_{-T}^{0} S(t+\xi))dw(t) +$$
$$\beta\int_{\mathbb{R}} (S(t^-) - \mu\int_{-T}^{0} S(t^- + \xi)d\xi)z\tilde{N}dtdz$$

with the initial conditions $S(0) = y_0; S(\theta) = y_1(\theta), \theta \in [-T, 0)$, where μ, α, β are constants and $y := (y_0, y_1(.) \in C$.

The assumptions $y_0 - \mu\int_{-T}^{0} y_1(\xi)d\xi > 0$ and $\nu \equiv 0$ on $(-\infty, 0]$ are imposed to ensure that positivity of the solution and which it is exist and the allowance only for positive jumps.

The problem of finding the optimal exercise time of

$$F(\tau, S_\tau) = E[(K - (S(\tau) - \int_{-T}^{0} S(t+\xi)d\xi))^+]$$

is strictly connected to the corresponding optimal stoping problem of such stochastic system. This problem was studied by [13] where the solution is obtained by rewriting the problem in infinite dimension from which they found the condition under which the problem is reduced to one dimensional case which yields explicit solution.

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