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APPLICATION AND GRAPHICAL INTERPRETATION OF A NEW TWO-DIMENSIONAL QUATERNION FRACTIONAL FOURIER TRANSFORM

KHINAL PARMAR*, **V. R. LAKSHMI GORTY**

NMIMS, MPSTME, Vile Parle - West, Mumbai - 400056, India

* *Corresponding author:* khinal.parmar@nmims.edu

Abstract. In this paper, a new two-dimensional quaternion fractional Fourier transform is developed. The properties such as linearity, shifting and derivatives of the quaternion-valued function are studied. The convolution theorem and inversion formula are also established. An example with graphical representation is solved. An application related to two-dimensional quaternion Fourier transform is also demonstrated.

1. Introduction

In 1853, quaternions were developed by W. R. Hamilton [10]. The necessity of enlarging the operations on three-dimensional vectors to include multiplication and division led Hamilton to introduce the four-dimensional algebra of quaternions. In 1993, Ell [6] introduced quaternion Fourier transform for application to two-dimensional linear time-invariant systems of partial differential equations. In 2001 [3], authors defined non-commutative hypercomplex Fourier transforms of multidimensional Signals. In 2007 [9], author introduced right side quaternion Fourier transform. In 2008 [8], the concept of fractional quaternion Fourier transform was presented. In [11], the author studied the uncertainty principle for the quaternion Fourier transform. Authors in [1] developed quaternion domain Fourier transforms and its application in mathematical statistics. In [4], Plancherel theorem and quaternion Fourier transform for square-integrable functions were studied.

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Quaternion Fourier transform transfers signals from the real-valued time domain to quaternion-valued frequency domain. But the proposed two-dimensional quaternion fractional Fourier transform will transfer the signal to unified time-frequency domains. Hence, it has a wide range of applications in the field of optics and signal processing.

The organization of the paper is as follows: In section 2, some basic facts of quaternions and quaternion-valued functions are illustrated. In section 3, the two-dimensional quaternion fractional Fourier transform is defined and its inversion formula and operational properties are developed. Graphical interpretation of two-dimensional quaternion fractional Fourier transform is also illustrated. In Section 4, the application of the two-dimensional quaternion fractional Fourier transform is shown.

2. Preliminary results

In quaternions, every element is a linear combination of a real scalar and three imaginary units $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ with real coefficients.

Let q be a quaternion defined in

$$\mathbb{H} = \{q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\} \tag{2.1}$$

be the division ring of quaternions, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy Hamilton's multiplication rules (see, e.g. [9])

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2.2}$$

The quaternion conjugate of q is defined by

$$\bar{q} = x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3;\ x_0, x_1, x_2, x_3 \in \mathbb{R}. \tag{2.3}$$

The norm of $q \in \mathbb{H}$ is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}. \tag{2.4}$$

Alternatively, in [13] the quaternions are defined as

$$\mathbb{H} = \{q = q_1 + jq_2 : q_1, q_2 \in \mathbb{C}\} \tag{2.5}$$

where j is the imaginary number satisfying following conditions:

$j^2 = -1,\ jr = rj,\ \forall\, r \in \mathbb{R},\ ji = -ij$, where i is the imaginary number.

From [13] $f \in L^2(\mathbb{R}^2; \mathbb{H})$, then the function is expressed as

$$f(u, v) = f_0(u, v) + \mathbf{i}f_1(u, v) + \mathbf{j}f_2(u, v) + \mathbf{k}f_3(u, v). \tag{2.6}$$

For some applications the quaternions can be rewritten by replacing $\mathbf{k}$ with $\mathbf{ij}$ as given in [9],

$$q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{i}x_3\mathbf{j}.$$

Another way of rewritting quaternion is

$$q = x_+ + x_-;\ x_\pm = \frac{1}{2}\left(q \pm \mathbf{i}q\mathbf{j}\right).$$

$x_\pm$ can also be expressed as

$$x_\pm = \{x_0 \pm x_3 + \mathbf{i}(x_1 \mp x_2)\}\,\frac{1 \pm \mathbf{k}}{2} = \frac{1 \pm \mathbf{k}}{2}\,\{x_0 \pm x_3 + \mathbf{j}(x_2 \mp x_1)\}.$$

The real scalar part of the quaternion can be written as [9],

$$x_0 = \langle q \rangle_0 . \tag{2.7}$$

We can also rewrite the function $f \in L^2(\mathbb{R}^2, \mathbb{H})$ as [9],

$$f = f_0 + \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{i}f_3\mathbf{j}.$$

We can also split the function as [9],

$$f = f_+ + f_-;\ f_+ = \frac{1}{2}\left(f + \mathbf{i}f\mathbf{j}\right),\ f_- = \frac{1}{2}\left(f - \mathbf{i}f\mathbf{j}\right).$$

For $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ and $\mathbf{u} = (u, v) = ue_1 + ve_2 \in \mathbb{R}^2$ with $\{e_1, e_2\}$ as the basis of $\mathbb{R}^2$, the quaternion-valued inner product is defined in [9] as

$$(f, g) = \int_{\mathbb{R}^2} f(\mathbf{u})\bar{g}(\mathbf{u})d^2\mathbf{u}, \tag{2.8}$$

with real symmetric part

$$\langle f, g \rangle = \frac{1}{2}\left[(f, g) + (g, f)\right] = \int_{\mathbb{R}^2} \langle f(\mathbf{u})\bar{g}(\mathbf{u})\rangle_0\, d^2\mathbf{u}. \tag{2.9}$$

The norm of $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined as

$$||f|| = \sqrt{(f, f)} = \sqrt{\langle f, f \rangle} = \int_{\mathbb{R}^2} |f(\mathbf{u})|^2 d^2\mathbf{u}. \tag{2.10}$$

3. Main Results

Definition 3.1. *Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then two-dimensional quaternion fractional Fourier transform (2D-QFrFT) of particular order α, β using [9, 12] is defined as*

$$\hat{f}_{\alpha,\beta}\left(w_1, w_2\right) = F_{\alpha,\beta}\left[f\left(u, v\right); w_1, w_2\right] = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f\left(u, v\right) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv \tag{3.1}$$

where $0 < \alpha, \beta \leq 1$.

Analogous to [5, page 112],the integral will converge for values of w_1 and w_2 in the strips $-s_1 < Im(w_1) < s_1$ and $-s_2 < Im(w_2) < s_2$ respectively; where $s_1 < Re(p_1),\ s_2 < Re(p_2)$, for $p_1 = iw_1, p_2 = jw_2$.

The sufficient condition for $f(u,v)$ to have 2D-QFrFT is that $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|f(u,v)|dudv$ exists.

Inversion formula: Consider the inverse formula of quaternion Fourier transform as defined in [9]

$$f(u,v)=\frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{\mathbf{i}xu}\hat{f}(x,y)e^{\mathbf{j}yv}dxdy.$$

Substituting $x=w_1^{\frac{1}{\alpha}}$ and $y=w_2^{\frac{1}{\beta}}$.

Then,

$$f(u,v)=\frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{\mathbf{i}w_1^{\frac{1}{\alpha}}u}w_1^{\frac{1}{\alpha}-1}\hat{f}_{\alpha,\beta}(w_1,w_2)\,e^{\mathbf{j}w_2^{\frac{1}{\beta}}v}w_2^{\frac{1}{\beta}-1}\frac{dw_1}{\alpha}\frac{dw_2}{\beta}$$

$$f(u,v)=\frac{1}{(2\pi)^2\alpha\beta}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{\mathbf{i}w_1^{\frac{1}{\alpha}}u}w_1^{\frac{1-\alpha}{\alpha}}\hat{f}_{\alpha,\beta}(w_1,w_2)\,e^{\mathbf{j}w_2^{\frac{1}{\beta}}v}w_2^{\frac{1-\beta}{\beta}}dw_1dw_2.$$

Hence, the inversion formula is defined as

$$\begin{aligned}F_{\alpha,\beta}^{-1}\left[\hat{f}_{\alpha,\beta}(w_1,w_2)\right]&=f(u,v)\\ &=\frac{1}{(2\pi)^2\alpha\beta}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{\mathbf{i}w_1^{\frac{1}{\alpha}}u}w_1^{\frac{1-\alpha}{\alpha}}\hat{f}_{\alpha,\beta}(w_1,w_2)\,e^{\mathbf{j}w_2^{\frac{1}{\beta}}v}w_2^{\frac{1-\beta}{\beta}}dw_1dw_2.\end{aligned}\tag{3.2}$$

Property 3.1 (Left linearity)**.** *For $f_1,f_2\in L^2(\mathbb{R}^2,\mathbb{H})$ and*

$k_1,k_2\in\{q|q=x_0+\boldsymbol{i}x_1,x_0,x_1\in\mathbb{R}\}$;

$$F_{\alpha,\beta}\left[k_1f_1(u,v)+k_2f_2(u,v)\right]=k_1F_{\alpha,\beta}\left[f_1(u,v)\right]+k_2F_{\alpha,\beta}\left[f_2(u,v)\right].\tag{3.3}$$

Proof. For $f_1,f_2\in L^2(\mathbb{R}^2,\mathbb{H})$; $k_1,k_2\in\mathbb{R}$ and using (3.1), we get

$$\begin{aligned}&F_{\alpha,\beta}\left[k_1f_1(u,v)+k_2f_2(u,v)\right]\\ &=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\left[k_1f_1(u,v)+k_2f_2(u,v)\right]e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\\ &=k_1\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\left[f_1(u,v)\right]e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\\ &+k_2\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\left[f_2(u,v)\right]e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\\ &=k_1F_{\alpha,\beta}\left[f_1(u,v)\right]+k_2F_{\alpha,\beta}\left[f_2(u,v)\right].\end{aligned}$$

□

Property 3.2 (Right linearity)**.** *For $f_1,f_2\in L^2(\mathbb{R}^2,\mathbb{H})$ and*

$k_1',k_2'\in\{q|q=x_0+\boldsymbol{j}x_2,x_0,x_2\in\mathbb{R}\}$;

$$F_{\alpha,\beta}\left[f_1(u,v)k_1'+f_2(u,v)k_2'\right]=F_{\alpha,\beta}\left[f_1(u,v)\right]k_1'+F_{\alpha,\beta}\left[f_2(u,v)\right]k_2'.\tag{3.4}$$

The proof is similar to property 3.1.

Property 3.3 (Shifting)**.** *For $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $a, b \in \mathbb{R}$;*

$$F_{\alpha,\beta}\left[f(u-a, v-b)\right] = e^{-\boldsymbol{i}w_1^{\frac{1}{\alpha}}a} F_{\alpha,\beta}\left[f(u,v)\right] e^{-\boldsymbol{j}w_2^{\frac{1}{\beta}}b} \tag{3.5}$$

Proof. For $f \in L^2(\mathbb{R}^2, \mathbb{H})$; $a, b \in \mathbb{R}$ and using (3.1), we get

$$F_{\alpha,\beta}\left[f(u-a, v-b)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u-a, v-b) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv.$$

Substituting $u - a = s$ and $v - b = t$ gives

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}(s+a)} f(s,t) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}(t+b)} dsdt$$

$$= e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}a} F_{\alpha,\beta}\left[f(s,t)\right] e^{-\mathbf{j}w_2^{\frac{1}{\beta}}b}.$$

□

Property 3.4 (2D-QFrFT of derivatives)**.** *For $f \in L^2(\mathbb{R}^2, \mathbb{H})$, the two-dimensional quaternion fractional Fourier transform with derivatives of $f(u,v)$ are as follows:*

$$i)\ F_{\alpha,\beta}\left[\frac{\partial}{\partial u} f(u,v)\right] = \left(\boldsymbol{i}w_1^{\frac{1}{\alpha}}\right) F_{\alpha,\beta}\left[f(u,v)\right]. \tag{3.6}$$

$$ii)\ F_{\alpha,\beta}\left[\frac{\partial}{\partial v} f(u,v)\right] = F_{\alpha,\beta}\left[f(u,v)\right]\left(\boldsymbol{j}w_2^{\frac{1}{\beta}}\right). \tag{3.7}$$

$$iii)\ F_{\alpha,\beta}\left[\frac{\partial^2}{\partial u \partial v} f(u,v)\right] = \left(\boldsymbol{i}w_1^{\frac{1}{\alpha}}\right) F_{\alpha,\beta}\left[f(u,v)\right]\left(\boldsymbol{j}w_2^{\frac{1}{\beta}}\right). \tag{3.8}$$

In general

$$iv)\ F_{\alpha,\beta}\left[\frac{\partial^n}{\partial u^n} f(u,v)\right] = \left(\boldsymbol{i}w_1^{\frac{1}{\alpha}}\right)^n F_{\alpha,\beta}\left[f(u,v)\right]. \tag{3.9}$$

$$v)\ F_{\alpha,\beta}\left[\frac{\partial^n}{\partial v^n} f(u,v)\right] = F_{\alpha,\beta}\left[f(u,v)\right]\left(\boldsymbol{j}w_2^{\frac{1}{\beta}}\right)^n. \tag{3.10}$$

$$vi)\ F_{\alpha,\beta}\left[\frac{\partial^n}{\partial u^n}\frac{\partial^m}{\partial v^m} f(u,v)\right] = \left(\boldsymbol{i}w_1^{\frac{1}{\alpha}}\right)^n F_{\alpha,\beta}\left[f(u,v)\right]\left(\boldsymbol{j}w_2^{\frac{1}{\beta}}\right)^m. \tag{3.11}$$

Proof. $i)$ For $f \in L^2(\mathbb{R}^2, \mathbb{H})$, the first order derivative over $f(u,v)$ w.r.t. u is given by

$$\begin{aligned}
&F_{\alpha,\beta}\left[\frac{\partial}{\partial u} f(u,v)\right]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\frac{\partial}{\partial u} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv\\
&= \int_{-\infty}^{\infty}\left\{\left[e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v)\right] - \int_{-\infty}^{\infty} -\mathbf{i}w_1^{\frac{1}{\alpha}} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v) du\right\} e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dv\\
&= \mathbf{i}w_1^{\frac{1}{\alpha}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv\\
&= \left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right) F_{\alpha,\beta}\left[f(u,v)\right].
\end{aligned}$$

$ii)$ For $f \in L^2(\mathbb{R}^2, \mathbb{H})$, the first order derivative over $f(u,v)$ w.r.t. v is given by

$$\begin{aligned}
&F_{\alpha,\beta}\left[\frac{\partial}{\partial v} f(u,v)\right]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\frac{\partial}{\partial v} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv\\
&= \int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\left\{\left[f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}\right] - \int_{-\infty}^{\infty} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}\left(-\mathbf{j}w_2^{\frac{1}{\beta}}\right) dv\right\} du\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv\left(\mathbf{j}w_2^{\frac{1}{\beta}}\right)\\
&= F_{\alpha,\beta}\left[f(u,v)\right]\left(\mathbf{j}w_2^{\frac{1}{\beta}}\right).
\end{aligned}$$

$iii)$ For $f \in L^2(\mathbb{R}^2, \mathbb{H})$, the second order derivative over $f(u,v)$ w.r.t. u, v is given by

$$\begin{aligned}
&F_{\alpha,\beta}\left[\frac{\partial^2}{\partial u\partial v} f(u,v)\right]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\frac{\partial^2}{\partial u\partial v} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv\\
&= \int_{-\infty}^{\infty}\left\{\left[e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\frac{\partial}{\partial v} f(u,v)\right] - \int_{-\infty}^{\infty} -\mathbf{i}w_1^{\frac{1}{\alpha}} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\frac{\partial}{\partial v} f(u,v) du\right\} e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dv\\
&= \left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\frac{\partial}{\partial v} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv
\end{aligned}$$

$$= \left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right) \int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \left\{ \left[f(u,v)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} \right] - \int_{-\infty}^{\infty} f(u,v)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} \left(-\mathbf{j}w_2^{\frac{1}{\beta}}\right) dv \right\} du$$

$$= \left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv \left(\mathbf{j}w_2^{\frac{1}{\beta}}\right)$$

$$= \left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right) F_{\alpha,\beta}\left[f(u,v)\right] \left(\mathbf{j}w_2^{\frac{1}{\beta}}\right).$$

$iv)$ By using mathematical induction for $n=1$ by (3.6), we get

$$F_{\alpha,\beta}\left[\frac{\partial}{\partial u} f(u,v)\right] = \mathbf{i}w_1^{\frac{1}{\alpha}} F_{\alpha,\beta}\left[f(u,v)\right].$$

For $n=2$, the result holds true.

$$F_{\alpha,\beta}\left[\frac{\partial^2}{\partial u^2} f(u,v)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \frac{\partial^2}{\partial u^2} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv$$

$$= \int_{-\infty}^{\infty} \left\{ \left[e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \frac{\partial}{\partial u} f(u,v) \right] - \int_{-\infty}^{\infty} -\mathbf{i}w_1^{\frac{1}{\alpha}} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \frac{\partial}{\partial u} f(u,v) du \right\} e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dv$$

$$= \mathbf{i}w_1^{\frac{1}{\alpha}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \frac{\partial}{\partial u} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv$$

$$= \mathbf{i}w_1^{\frac{1}{\alpha}} \int_{-\infty}^{\infty} \left\{ \left[e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v) \right] - \int_{-\infty}^{\infty} -\mathbf{i}w_1^{\frac{1}{\alpha}} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v) du \right\} e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dv$$

$$= \left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv$$

$$= \left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right)^2 F_{\alpha,\beta}\left[f(u,v)\right].$$

For $n=k-1$,

$$F_{\alpha,\beta}\left[\frac{\partial^{k-1}}{\partial u^{k-1}} f(u,v)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \frac{\partial^{k-1}}{\partial u^{k-1}} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv$$

$$= \int_{-\infty}^{\infty} \left\{ \left[e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \frac{\partial^{k-2}}{\partial u^{k-2}} f(u,v) \right] - \int_{-\infty}^{\infty} -\mathbf{i}w_1^{\frac{1}{\alpha}} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \frac{\partial^{k-2}}{\partial u^{k-2}} f(u,v) du \right\} e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dv$$

$$= \mathbf{i}w_1^{\frac{1}{\alpha}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} \frac{\partial^{k-2}}{\partial u^{k-2}} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v} dudv.$$

On repeating the integration by parts, we get

$$F_{\alpha,\beta}\left[\frac{\partial^{k-1}}{\partial u^{k-1}}f(u,v)\right]=\left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right)^{k-1}F_{\alpha,\beta}\left[f(u,v)\right].$$

By method of mathematical induction, the result is true for all $n=k$.

$$F_{\alpha,\beta}\left[\frac{\partial^{k}}{\partial u^{k}}f(u,v)\right]=\left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right)^{k}F_{\alpha,\beta}\left[f(u,v)\right].$$

Thus, it is true for all n.

Similarly, $v)$ and $vi)$ can be proved. ☐

Property 3.5 (Power of u,v)**.** *For* $f\in L^2(\mathbb{R}^2,\mathbb{H})$

$$i)\ F_{\alpha,\beta}\left[uf(u,v)\right]=\left(\boldsymbol{i}\frac{\alpha}{w_1^{\frac{1-\alpha}{\alpha}}}\right)\frac{\partial}{\partial w_1}F_{\alpha,\beta}\left[f(u,v)\right]. \tag{3.12}$$

$$ii)\ F_{\alpha,\beta}\left[vf(u,v)\right]=\frac{\partial}{\partial w_2}F_{\alpha,\beta}\left[f(u,v)\right]\left(\boldsymbol{j}\frac{\beta}{w_2^{\frac{1-\beta}{\beta}}}\right). \tag{3.13}$$

Proof. For $f\in L^2(\mathbb{R}^2,\mathbb{H})$ and using (3.1), we get

$$\begin{aligned}
F_{\alpha,\beta}\left[uf(u,v)\right]&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}uf\left(u,v\right)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\mathbf{i}\frac{\alpha}{w_1^{\frac{1-\alpha}{\alpha}}}\right)\frac{\partial}{\partial w_1}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}f\left(u,v\right)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\\
&=\left(\mathbf{i}\frac{\alpha}{w_1^{\frac{1-\alpha}{\alpha}}}\right)\frac{\partial}{\partial w_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}f\left(u,v\right)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\\
&=\left(\mathbf{i}\frac{\alpha}{w_1^{\frac{1-\alpha}{\alpha}}}\right)\frac{\partial}{\partial w_1}F_{\alpha,\beta}\left[f(u,v)\right].
\end{aligned}$$

$$\begin{aligned}
F_{\alpha,\beta}\left[vf(u,v)\right]&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}vf\left(u,v\right)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}f\left(u,v\right)\frac{\partial}{\partial w_2}e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}\left(\mathbf{j}\frac{\beta}{w_2^{\frac{1-\beta}{\beta}}}\right)dudv\\
&=\frac{\partial}{\partial w_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}f\left(u,v\right)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\left(\mathbf{j}\frac{\beta}{w_2^{\frac{1-\beta}{\beta}}}\right)\\
&=\frac{\partial}{\partial w_2}F_{\alpha,\beta}\left[f(u,v)\right]\left(\mathbf{j}\frac{\beta}{w_2^{\frac{1-\beta}{\beta}}}\right).
\end{aligned}$$

Hence the proof. ☐

Property 3.6 (Power of $\mathbf{i},\mathbf{j}$)**.** *For* $f \in L^2(\mathbb{R}^2, \mathbb{H})$*;* $m, n \in \mathbb{N}$

$$F_{\alpha,\beta}\left[\boldsymbol{i}^m f(u,v)\boldsymbol{j}^n\right] = \boldsymbol{i}^m F_{\alpha,\beta}\left[f(u,v)\right]\boldsymbol{j}^n. \tag{3.14}$$

Proof. For $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and using (3.1), we get

$$\begin{aligned} F_{\alpha,\beta}\left[\mathbf{i}^m f(u,v)\mathbf{j}^n\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u}\mathbf{i}^m f(u,v)\mathbf{j}^n e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv \\ &= \mathbf{i}^m \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}u} f(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}v}dudv\mathbf{j}^n \\ &= \mathbf{i}^m F_{\alpha,\beta}\left[f(u,v)\right]\mathbf{j}^n. \end{aligned}$$

Hence the proof. □

Definition 3.2. *The convolution for the quaternion valued functions* $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ *is defined [14] by*

$$f * g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)g(x-u, y-v)dudv. \tag{3.15}$$

Theorem 3.7 (Convolution theorem)**.** *For* $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$*;*

$$F_{\alpha,\beta}\left[f * g\right] = F_{\alpha,\beta}\left[f\right] F_{\alpha,\beta}\left[g\right]. \tag{3.16}$$

Proof. For $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$; $X = (x_1, x_2),\ Y = (y_1, y_2)$ and $Z = (z_1, z_2)$;

$$\begin{aligned} F_{\alpha,\beta}\left[f * g\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}x_1}\left(f * g\right)(X)\, e^{-\mathbf{j}w_2^{\frac{1}{\beta}}x_2}dX \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}x_1}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(Y)g(X-Y)dY\right] e^{-\mathbf{j}w_2^{\frac{1}{\beta}}x_2}dX \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}x_1} f(Y)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(X-Y)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}x_2}dX\right]dY. \end{aligned}$$

Substituting $Z = X - Y$, we get

$$\begin{aligned} F_{\alpha,\beta}\left[f * g\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}(y_1+z_1)} f(Y)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(Z)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}(y_2+z_2)}dZ\right]dY \\ &= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}y_1} f(Y)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}y_2}dY\right)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}z_1} g(Z)e^{-\mathbf{j}w_2^{\frac{1}{\beta}}z_2}dZ\right) \\ &= F_{\alpha,\beta}\left[f\right] F_{\alpha,\beta}\left[g\right]. \end{aligned}$$

□

Theorem 3.8. *The scalar product of two quaternion-valued functions $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ is given by the scalar product of the corresponding 2D-QFrFTs $\hat{f}$ and $\hat{g}$:*

$$\langle f, g\rangle = \frac{1}{(2\pi)^2\alpha\beta}\left\langle \mathcal{F}_{\alpha,\beta}(w_1, w_2), \mathcal{G}_{\alpha,\beta}(w_1, w_2)\right\rangle . \tag{3.17}$$

Proof. For $f,\ g \in L^2(\mathbb{R}^2, \mathbb{H})$ and using (2.8), we get

$$\begin{aligned}
\langle f,\ g\rangle &= \int_{\infty}^{\infty}\int_{\infty}^{\infty}\left\langle f(u,v)\overline{g(u,v)}\right\rangle dudv \\
&= \int_{\infty}^{\infty}\int_{\infty}^{\infty}\left\langle \frac{1}{(2\pi)^2\alpha\beta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathbf{i}w_1^{\frac{1}{\alpha}}u} w_1^{\frac{1-\alpha}{\alpha}}\hat{f}_{\alpha,\beta}(w_1,w_2)\right. \\
&\qquad \left.\times\, e^{\mathbf{j}w_2^{\frac{1}{\beta}}v} w_2^{\frac{1-\beta}{\beta}} dw_1 dw_2 \overline{g(u,v)}\right\rangle dudv
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^2\alpha\beta}\int_{\infty}^{\infty}\int_{\infty}^{\infty}\left\langle \hat{f}_{\alpha,\beta}(w_1,w_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathbf{i}w_1^{\frac{1}{\alpha}}u} w_1^{\frac{1-\alpha}{\alpha}} e^{\mathbf{j}w_2^{\frac{1}{\beta}}v} w_2^{\frac{1-\beta}{\beta}}\right. \\
&\qquad \left.\times\, \overline{g(u,v)}dudv\right\rangle dw_1 dw_2 \\
&= \frac{1}{(2\pi)^2\alpha\beta}\int_{\infty}^{\infty}\int_{\infty}^{\infty}\left\langle \hat{f}_{\alpha,\beta}(w_1,w_2) w_1^{\frac{1-\alpha}{\alpha}} w_2^{\frac{1-\beta}{\beta}}\right. \\
&\qquad \left.\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathbf{j}w_2^{\frac{1}{\beta}}v}\overline{g(u,v)} e^{\mathbf{i}w_1^{\frac{1}{\alpha}}u} dudv\right\rangle dw_1 dw_2 \\
&= \frac{1}{(2\pi)^2\alpha\beta}\int_{\infty}^{\infty}\int_{\infty}^{\infty}\left\langle \hat{f}_{\alpha,\beta}(w_1,w_2) w_1^{\frac{1-\alpha}{\alpha}} w_2^{\frac{1-\beta}{\beta}}\right. \\
&\qquad \left.\times\, \overline{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}v} g(u,v) e^{-\mathbf{j}w_2^{\frac{1}{\beta}}u} dudv}\right\rangle dw_1 dw_2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^2\alpha\beta}\int_{\infty}^{\infty}\int_{\infty}^{\infty}\left\langle w_1^{\frac{1-\alpha}{\alpha}}\hat{f}_{\alpha,\beta}(w_1,w_2) w_2^{\frac{1-\beta}{\beta}}\overline{\hat{g}_{\alpha,\beta}(w_1,w_2)}\right\rangle dw_1 dw_2 \\
&= \frac{1}{(2\pi)^2\alpha\beta}\left\langle \mathcal{F}_{\alpha,\beta}(w_1,w_2), \mathcal{G}_{\alpha,\beta}(w_1,w_2)\right\rangle
\end{aligned}$$

where

$\mathcal{F}_{\alpha,\beta}(w_1, w_2) = w_1^{\frac{1-\alpha}{\alpha}}\hat{f}_{\alpha,\beta}(w_1, w_2)$;

$\mathcal{G}_{\alpha,\beta}(w_1, w_2) = w_2^{\frac{1-\beta}{\beta}}\overline{\hat{g}_{\alpha,\beta}(w_1, w_2)}$.

Thus, the theorem holds true. $\square$

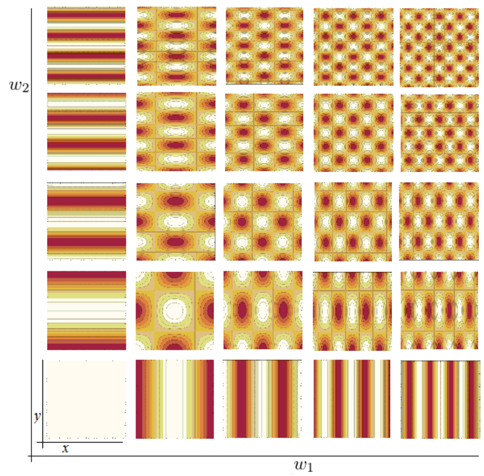

FIGURE 1. Kernel of 2D-QFrFT at $\alpha = 1$ and $\beta = 1$.

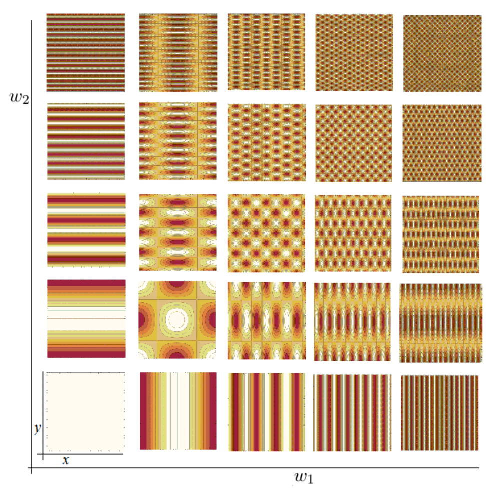

FIGURE 2. Kernel of 2D-QFrFT at $\alpha = 1/2$ and $\beta = 1/2$.

Figure 1 shows the kernel of 2D-QFrFT for various values of w_1, w_2 at order $\alpha = 1$ and $\beta = 1$ which is a particular case of the study developed in this paper. Figure 2 shows the kernel of 2D-QFrFT for various values of w_1, w_2 at order $\alpha = 1/2$ and $\beta = 1/2$. For both the figures the range of x and y is between -3 and 3. The 2D-QFrFT is superior in disparity estimation and analyzing genuine 2D texture as compared to other fractional Fourier transforms and [7].

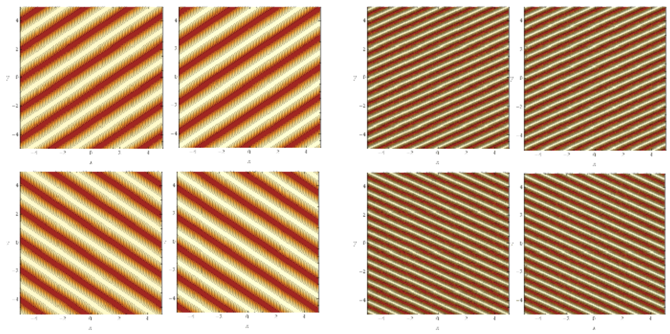

FIGURE 3. 2D-QFrFT Kernel. Left: Top row: $(1+\mathbf{ij})/2$ and $(\mathbf{i}-\mathbf{j})/2$ components. Bottom row: $(1-\mathbf{ij})/2$ and $(\mathbf{i}+\mathbf{j})/2$ components at $\alpha = 1,\ \beta = 1$; Right: Top row: $(1+\mathbf{ij})/2$ and $(\mathbf{i}-\mathbf{j})/2$ components. Bottom row: $(1-\mathbf{ij})/2$ and $(\mathbf{i}+\mathbf{j})/2$ components at $\alpha = 1/2,\ \beta = 1/2$

The components $(1+\mathbf{ij})/2$, $(\mathbf{i}-\mathbf{j})/2$, $(1-\mathbf{ij})/2$ and $(\mathbf{i}+\mathbf{j})/2$ are shown in Figure 3 at $\alpha = 1,\ \beta = 1$ and $\alpha = 1/2,\ \beta = 1/2$ which represents 2D-QFT extended to 2D-QFrFT. We can also observe the scale-invariant feature of 2D-QFrFT.

Example 3.1. *Find the quaternion fractional Fourier transform of the function:*

$$f(x,y) = \begin{cases} 1; & |x| < 1,\ |y| < 1 \\ 0; & otherwise. \end{cases} \tag{3.18}$$

By using (3.1), we get

$$F_{\alpha,\beta}[f(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}x} f(x,y)\, e^{-\mathbf{j}w_2^{\frac{1}{\beta}}y} dxdy.$$

$$F_{\alpha,\beta}[f(x,y)] = \int_{-1}^{1}\int_{-1}^{1} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}x} e^{-\mathbf{j}w_2^{\frac{1}{\beta}}y} dxdy$$

$$F_{\alpha,\beta}[f(x,y)] = 4\frac{\sin w_1^{\frac{1}{\alpha}}}{w_1^{\frac{1}{\alpha}}} \cdot \frac{\sin w_2^{\frac{1}{\beta}}}{w_2^{\frac{1}{\beta}}}. \tag{3.19}$$

The graphical representation of the quaternion fractional Fourier transform of the function (3.18) obtained using $\alpha = 1$ and $\beta = 1$ in (3.19), is now a particular case of (3.1) which is represented in the following figure:

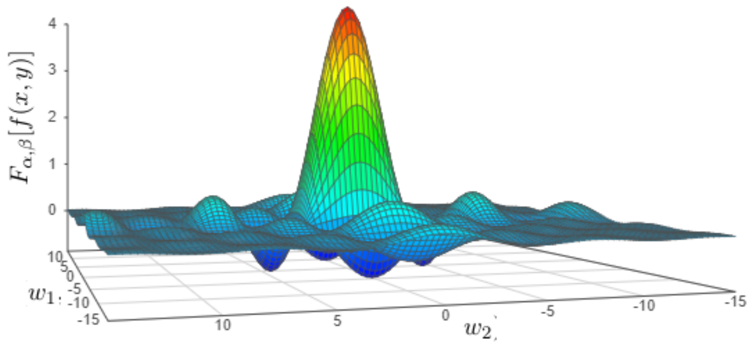

FIGURE 4. Graph of $F_{\alpha,\beta}[f(x,y)]$ with $\alpha = 1$ and $\beta = 1$.

The graphical representation of the quaternion fractional Fourier transform of the function (3.18) obtained using $\alpha = 1/2$ and $\beta = 1/2$ in (3.19) is represented in the following figure:

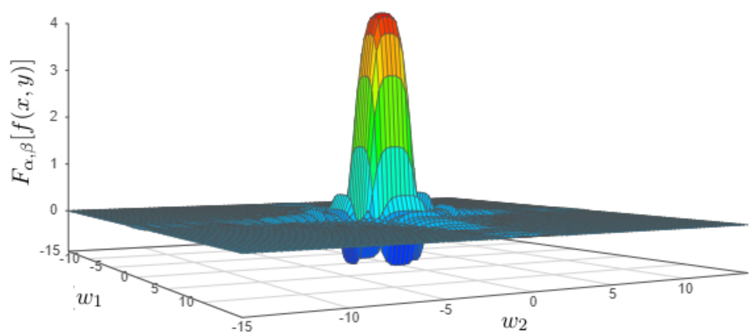

FIGURE 5. Graph of $F_{\alpha,\beta}[f(x,y)]$ with $\alpha = 1/2$ and $\beta = 1/2$.

The graphical representation of the quaternion fractional Fourier transform of the function (3.18) obtained using $\alpha = 1/2$ and $\beta = 1$ in (3.19) is represented in the following figure:

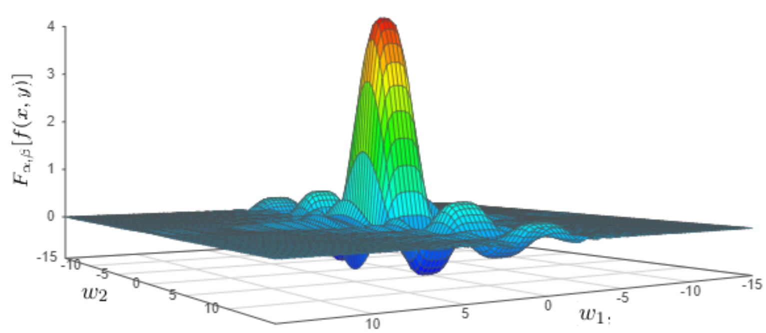

FIGURE 6. Graph of $F_{\alpha,\beta}[f(x,y)]$ with $\alpha = 1/2$ and $\beta = 1$.

Figures 1-3 are plotted using online freeware version of wolframalpha. Figures 4-6 are plotted using online freeware version 3D surface plotter of academo.

4. Application

Let us consider the initial value problem from [2]:

$$\frac{\partial h}{\partial t} - \nabla^2 h = 0, \ on \ \mathbb{R}^{0,\ 2} \times (0, \infty), \tag{4.1}$$

and

$$h(u, v) = f(u, v), \ f \in \mathcal{S}(\mathbb{R}^{0,\ 2};\ \mathbb{H}) \ at \ t = 0, \tag{4.2}$$

where $\mathcal{S}(\mathbb{R}^{0,\ 2};\ \mathbb{H})$ is the quaternion Schwartz space and $\nabla^2 = \dfrac{\partial^2}{\partial^2 u} + \dfrac{\partial^2}{\partial^2 v}$.
Applying the definition of 2D-QFrFT to both sides of (4.1), we get

$$F_{\alpha,\beta}\left[\frac{\partial h}{\partial t}\right] = \left(\mathbf{i}w_1^{\frac{1}{\alpha}}\right)^2 F_{\alpha,\beta}[h] + F_{\alpha,\beta}[h]\left(\mathbf{j}w_2^{\frac{1}{\beta}}\right)^2$$

$$\frac{\partial}{\partial t} F_{\alpha,\beta}\left[h\right] = -\left(w_1^{\frac{2}{\alpha}} + w_2^{\frac{2}{\beta}}\right) F_{\alpha,\beta}[h]. \tag{4.3}$$

The general solution of (4.3) is given by

$$F_{\alpha,\beta}\left[h\right] = Ce^{-\left(w_1^{\frac{2}{\alpha}} + w_2^{\frac{2}{\beta}}\right)t}, \tag{4.4}$$

where C is a quaternion constant.

By using the initial value condition, we get

$$F_{\alpha,\beta}\left[h\right] = e^{-\left(w_1^{\frac{2}{\alpha}} + w_2^{\frac{2}{\beta}}\right)t} F_{\alpha,\beta}\left[f\right]. \tag{4.5}$$

Analogous to [2, equation 6.6], we have

$$\frac{1}{4\pi t} F_{\alpha,\beta}\left[e^{-\left(u^{\frac{2}{\alpha}} + v^{\frac{2}{\beta}}\right)/4t}\right] = e^{-\left(w_1^{\frac{2}{\alpha}} + w_2^{\frac{2}{\beta}}\right)t}. \tag{4.6}$$

Applying the inversion formula of 2D-QFrFT to (4.5), we get

$$\begin{aligned} h &= F_{\alpha,\beta}^{-1}\left[e^{-\left(w_1^{\frac{2}{\alpha}} + w_2^{\frac{2}{\beta}}\right)t} F_{\alpha,\beta}\left[f\right]\right] \\ &= F_{\alpha,\beta}^{-1}\left[\frac{1}{4\pi t} F_{\alpha,\beta}\left[e^{-\left(u^{\frac{2}{\alpha}} + v^{\frac{2}{\beta}}\right)/4t}\right] F_{\alpha,\beta}\left[f\right]\right]. \end{aligned}$$

Using convolution theorem, we have

$$h = K_t * f \tag{4.7}$$

where $K_t = \dfrac{1}{4\pi t} e^{-\left(u^{\frac{2}{\alpha}} + v^{\frac{2}{\beta}}\right)/4t}$.

5. Conclusion

The authors developed a new two-dimensional quaternion fractional Fourier transform in this study. The properties such as linearity, shifting and derivatives of the quaternion-valued function are demonstrated. The convolution theorem and inversion formula are also established. An example is illustrated with graphical representation. In the concluding section, an application related to the two-dimensional quaternion Fourier transform is also demonstrated.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Bahri, A.K. Amir, C. Lande, The quaternion domain Fourier transform and its application in mathematical statistics, IAENG Int. J. Appl. Math. 48 (2018), 1-7.

[2] M. Bahri, R. Ashino, R. Vaillancourt, Continuous quaternion Fourier and wavelet transforms, Int. J. Wavelets Multiresolut. Inf. Process. 12 (2014), 1460003.

[3] T. Bülow, M. Felsberg, G. Sommer, Non-commutative hyper complex Fourier transforms of multidimensional signals, In: G. Sommer (eds) Geometric Computing with Clifford Algebras, Springer, Berlin, Heidelberg, 2001.

[4] D. Cheng, K.I. Kou, Plancherel theorem and quaternion Fourier transform for square integrable functions, Complex Var. Elliptic Equ. 64 (2019), 223–242.

[5] B. Davies, Integral transforms and their Applications, Springer, New York, 1978.

[6] T.A. Ell, Quaternion-Fourier transforms for analysis of two-dimensional linear time-invariant partial differential systems, in: Proceedings of 32nd IEEE Conference on Decision and Control, IEEE, San Antonio, TX, USA, 1993: pp. 1830–1841.

[7] T.A. Ell, S.J. Sangwine, Hypercomplex Fourier Transforms of Color Images, IEEE Trans. Image Process. 16 (2007), 22–35.

[8] X. Guanlei, W. Xiaotong, X. Xiaogang, Fractional quaternion Fourier transform, convolution and correlation, Signal Processing. 88 (2008), 2511–2517.

[9] E.M.S. Hitzer, Quaternion Fourier Transform on Quaternion Fields and Generalizations, Adv. Appl. Clifford Alg. 17 (2007), 497–517.

[10] A.C. Lewis, William Rown Hamilton, Lectures on quaternions (1853), in: Landmark Writings in Western Mathematics 1640-1940, Elsevier, 2005: pp. 460–469.

[11] P. Lian, Uncertainty principle for the quaternion Fourier transform, J. Math. Anal. Appl. 467 (2018), 1258–1269.

[12] L. Romero, R. Cerutti, L. Luque, A new Fractional Fourier Transform and convolutions products, Int. J. Pure Appl. Math. 66 (2011), 397-408.

[13] R. Roopkumar, Quaternionic one-dimensional fractional Fourier transform, Optik, 127 (2016), 11657-11661.

[14] A.H. Zemanian, Generalized integral transformations, John Wiley Sons Inc., New York, 1968.