



STOCHASTIC CHEMOTAXIS MODEL WITH FRACTIONAL DERIVATIVE DRIVEN BY MULTIPLICATIVE NOISE

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ABSTRACT. We introduce stochastic model of chemotaxis by fractional Derivative generalizing the deterministic Keller Segel model. These models include fluctuations which are important in systems with small particle numbers or close to a critical point. In this work, we study of nonlinear stochastic chemotaxis model with Dirichlet boundary conditions, fractional Derivative and disturbed by multiplicative noise. The required results prove the existence and uniqueness of mild solution to time and space-fractional, for this we use analysis techniques and fractional calculus and semigroup theory, also studying the regularity properties of mild solution for this model.

1. INTRODUCTION

In this study, we consider on the following generalized SKSM with time-space fractional derivative on a bounded domain $D \subset \mathbb{R}^d (1 \leq d \leq 3)$:

$$(1.1) \quad \begin{cases} {}^c D_t^\beta u + (-\Delta)^{\frac{\alpha}{2}} u - \nabla(u \nabla c) &= g(u) \dot{W}(t), & (t, x) \in [0, T] \times D, \\ {}^c D_t^\beta c + (-\Delta)^{\frac{\alpha}{2}} c - c \nabla c &= f(c) \dot{W}(t), & (t, x) \in [0, T] \times D, \end{cases}$$

with subject to the initial conditions:

$$(1.2) \quad \begin{cases} u(0, x) &= u_0(x), & x \in D, \\ c(0, x) &= c_0(x), & x \in D, \end{cases}$$

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and the Dirichlet boundary conditions:

$$(1.3) \quad \begin{cases} u(t, x) |_{\partial D} = 0, & t \in [0, T], \\ c(t, x) |_{\partial D} = 0, & t \in [0, T], \end{cases}$$

where $u = u(t, x)$ denotes the population density of biological individuals, $c = c(t, x)$ denotes the concentration of chemical substance, and $\nabla(u\nabla c)$ is called a chemotactic term that is used to model the fact that cells are attracted by chemical stimulus. In which the terms $g(u)\dot{W}(t) = g(u)\frac{dW(t)}{dt}$, and $f(c)\dot{W}(t) = f(c)\frac{dW(t)}{dt}$ They describe the case-dependent random noise, where $W(t)_{t \in [0, T]}$ is \mathcal{F}_t - adapted Wiener process defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the expectation \mathbb{E} and associate with the normal filtration $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$, The operator $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in (1, 2)$ stands for the fractional power of the Laplacian (see [1]). We denote by ${}^c D_t^\beta$ the Caputo derivative of order β , which is defined by (see [17])

$$(1.4) \quad \begin{cases} {}^c D_t^\beta u(t, x) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(s, x)}{\partial s} \frac{ds}{(t-s)^\beta}, & 0 < \beta < 1, \\ \frac{\partial u(s, x)}{\partial s}, & \beta = 1, \end{cases} \\ {}^c D_t^\beta c(t, x) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial c(s, x)}{\partial s} \frac{ds}{(t-s)^\beta}, & 0 < \beta < 1, \\ \frac{\partial c(s, x)}{\partial s}, & \beta = 1, \end{cases} \end{cases}$$

where $\Gamma(\cdot)$ stands for the gamma function

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt.$$

The rest of the paper is organized as follows. In Section 2, we will introduce some notations and preliminaries, which play a crucial role in our theorem analysis. In Section 3, the existence and uniqueness of mild solution to the problem of time-space fractional (2.1) and in Section 4, the spatial and temporal regularity properties of mild solution to this time-space fractional (2.1) are proved. In Section 5, the existence and uniqueness of mild solution to the problem of time-space fractional (2.6). Finally, the spatial and temporal regularity properties of mild solution to this time-space fractional (2.6) are proved. We use stochastic analysis techniques, fractional calculus and semigroup theory.

Next, we mention some Notations and preliminaries the task at work.

2. NOTATIONS AND PRELIMINARIES

Denote the basic functional space $L^p(D), 1 \leq p < \infty$ and $H^s(D)$ by the usual Lebesgue and Sobolev spaces, respectively. We assume that A is the negative Laplacian $-\Delta$ in a bounded domain D with zero Dirichlet boundary conditions in a Hilbert space $H = L^2(D)$, which are given by

$$A = -\Delta, \quad D(A) = H_0^1(D) \cap H^2(D).$$

Since the operator A is self-adjoint on H with discrete spectral, i.e., there exists the eigenvectors e_n with corresponding eigenvalues λ_n such that

$$Ae_n = \lambda_n e_n, e_n = \sqrt{2} \sin(n\pi), \lambda_n = \pi^2 n^2, n \in N^+.$$

For any $s > 0$, let \dot{H}^s be the domain of the fractional power $A^{\frac{s}{2}} = (-\Delta)^{\frac{s}{2}}$, which can be defined by

$$A^{\frac{s}{2}} e_n = \lambda_n^{\frac{s}{2}}, \quad n = 1, 2, \dots,$$

and

$$\dot{H}^s = D(A^{\frac{s}{2}}) = \{v \in L^2(D), s.t. \|v\|_{\dot{H}^s}^2 = \sum_{n=1}^{\infty} \lambda_n^{\frac{s}{2}} v_n^2 < \infty\},$$

where $v_n := \langle v, e_n \rangle$ with the inner product $\langle \cdot, \cdot \rangle$ in $L^2(D)$. We denote that

$\|v\|_{\dot{H}^s} = \|A^{\frac{s}{2}} v\|$, and the corresponding dual space \dot{H}^{-s} with the inverse operator $A^{-\frac{s}{2}}$. We also denote A_s for $A^{\frac{s}{2}}$ and the bilinear operators $B(u, c) = \nabla(u \nabla c)$, and $D(B) = H_0^1$ and $L(c, v) = c \nabla v$, and $D(L) = H_0^1$ with a slight abuse of notation $L(c, c) = L(c)$. Then the eqs (1.1) and (1.3) can be rewritten as the following abstract formulation:

$$(2.1) \quad \begin{cases} {}^c D^\beta u(t) &= -A_\alpha u(t) + B(u(t), c(t)) + g(u(t)) \frac{dW(t)}{dt}, \quad t > 0, \\ u(0) &= u_0, \end{cases}$$

and

$$(2.2) \quad \begin{cases} {}^c D^\beta c(t) &= -A_\alpha c(t) + L(c(t)) + f(c(t)) \frac{dW(t)}{dt}, \quad t > 0, \\ c(0) &= c_0, \end{cases}$$

where $\{W(t)\}_{t \geq 0}$ is a \mathbb{Q} -Wiener process with linear bounded covariance operator \mathbb{Q} such that $Tr(\mathbb{Q}) < \infty$. Further, there exists the eigenvalues λ_n and corresponding eigenfunctions e_n satisfy $\mathbb{Q} e_n = \lambda_n e_n, n = 1, 2, \dots$, then the Wiener process is given by

$$W(t) = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \beta_n(t) e_n,$$

in which $\{\beta_n\}_{n \geq 1}$ is a sequence of real-valued standard Brownian motions. Let $L_0^2 = L^2(\mathbb{Q}^{\frac{1}{2}}(H), H)$ denote the space of Hillbert-Schmidt operators from $\mathbb{Q}^{\frac{1}{2}}(H)$ to H with the norm

$$\|\Phi\|_{L_0^2} = \|\Phi \mathbb{Q}^{\frac{1}{2}}\|_{H^s} = \left(\sum_{n=1}^{\infty} \Phi \mathbb{Q}^{\frac{1}{2}} e_n \right)^{\frac{1}{2}},$$

i.e., $L_0^2 = \{\Phi \in L(H) : \sum_{n=1}^{\infty} \|\Phi \mathbb{Q}^{\frac{1}{2}}\|^2 < \infty\}$, where $L(H)$ is the space of bounded linear operators from H to H . For an arbitrary Banach space B , we denote $\|\cdot\|_{L^p(\Omega; B)}$ by the norm in $L^p(\Omega, \mathcal{F}, \mathbb{P}; B)$, which defined as

$$\|v\|_{L^p(\Omega; B)} = (\mathbb{E}[\|v\|_B^p])^{\frac{1}{p}}, \quad \forall v \in L^p(\Omega, \mathcal{F}, \mathbb{P}; B),$$

for any $p \geq 2$. We shall also need the following result with respect to the fractional operator A_α (see Ref. [18]).

Lemma 2.1. For any $\alpha > 0$, an analytic semigroup $S_\alpha(t) = e^{-tA_\alpha}$, $t \geq 0$ is generated by the operator $-A_\alpha$ on L^p , and for any $\nu \geq 0$, there exists a constant $C_{\alpha\nu}$ dependent on α and ν such that

$$(2.3) \quad \|A_\nu S_\alpha(t)\|_{\mathcal{L}(L^p)} \leq C_{\alpha\nu} t^{-\frac{\nu}{\alpha}}, \quad t > 0,$$

in which $\mathcal{L}(B)$ denotes the Banach space of all linear bounded operators from B to itself.

Next, we will introduce the following lemma to estimate the stochastic integrals, which contains the Burkholder-Davis-Gundy's inequality.

Lemma 2.2. ([8]) For any $0 \leq t_1 < t_2 \leq T$ and $p \geq 2$, and for any predictable stochastic process $v : [0, T] \times \Omega \rightarrow L^2_0$, which satisfies

$$\mathbb{E}\left[\left(\int_0^T \|v(s)\|_{L^2_0}^2 ds\right)^{\frac{p}{2}}\right] < \infty,$$

then we have

$$(2.4) \quad \mathbb{E}\left[\left\|\int_{t_1}^{t_2} v(s)dW(s)\right\|^p\right] \leq C(p)\mathbb{E}\left[\left(\int_{t_1}^{t_2} \|v(s)\|_{L^2_0}^2 ds\right)^{\frac{p}{2}}\right],$$

where $C(p) = \left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}} \left(\frac{p}{p-1}\right)^{p(\frac{p}{2}-1)}$ is a constant.

Now, we give the following definition of mild solution for our time-space fractional stochastic Keller-Segel model.

Definition 2.1. A \mathcal{F}_t adapted process $(u(t), c(t))_{t \in [0, T]}$ is called a mild solution (1.1), if $(u(t), c(t))_{t \in [0, T]} \in \mathbb{C}([0, T]; \dot{H}^\nu)$ P -a. e., and it holds,

$$(2.5) \quad \begin{aligned} u(t) &= E_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s)B(u(s), c(s)) ds \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s)g(u(s))dW(s), \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} c(t) &= E_\beta(t)c_0 + \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s)L(c(s))ds \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s)f(c(s))dW(s), \end{aligned}$$

respectively for a. s. $\omega \in \Omega$, where the generalized Mittag-Leffler operators $E_\beta(t)$ and $E_{\beta\beta}(t)$ are defined as

$$E_\beta(t) = \int_0^\infty M_\beta(\theta)S_\alpha(t^\beta\theta)d\theta,$$

and

$$E_{\beta\beta}(t) = \int_0^\infty \beta\theta M_\beta(\theta)S_\alpha(t^\beta\theta)d\theta,$$

which contain the Mainardi's Wright-type function with $\beta \in (0, 1)$ given by

$$M_\beta(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^n}{n! \Gamma(1 - \beta(1 + n))},$$

in which the Mainardi function $M_\beta(\theta)$ act as a bridge between the classical integral-order and fractional derivatives of differential equations, for more details see [19,20]. Here, the derivation of mild solution (2.5) and (2.6) can be found in Appendix (7) and Appendix (8) (respectively).

Lemma 2.3. [2] For any $\beta \in (0, 1)$ and $-1 < \varepsilon < \infty$, it is not difficult to verity that

$$(2.7) \quad M_\beta(\theta) \geq 0, \text{ and } \int_0^\infty \theta^\varepsilon M_\beta(\theta) d\theta = \frac{\Gamma(1 + \varepsilon)}{\Gamma(1 + \beta\varepsilon)},$$

for all $\theta \geq 0$.

Theorem 2.1. For any $t > 0$, $E_\beta(t)$ and $E_{\beta\beta}(t)$ are linear and bounded operators. Moreover, for $0 \leq \nu < \alpha < 2$, there exist constants $C_\alpha = C(\alpha, \beta, \nu) > 0$ and $C_\beta = C(\alpha, \beta, \nu) > 0$ such that

$$(2.8) \quad \| E_\beta(t)v \|_{\dot{H}^\nu} \leq C_\alpha t^{-\frac{\beta\nu}{\alpha}} \| v \|, \quad \| E_{\beta\beta}(t)v \|_{\dot{H}^\nu} \leq C_\beta t^{-\frac{\beta\nu}{\alpha}} \| v \|.$$

Proof. For $t > 0$ and $0 \leq \nu < \alpha < 2$, by means of the Lemma (2.1) and Lemma (2.3), we have

$$\begin{aligned} \| E_\beta(t)v \|_{\dot{H}^\nu} &\leq \int_0^\infty M_\beta(\theta) \| A_\nu S_\alpha(t^\beta\theta)v \| d\theta \\ &\leq \int_0^\infty C_{\alpha\nu} t^{-\frac{\beta\nu}{\alpha}} \theta^{-\frac{\nu}{\alpha}} M_\beta(\theta) \| v \| d\theta \\ &= \frac{C_{\alpha\nu} \Gamma(1 - \frac{\nu}{\alpha})}{\Gamma(1 - \frac{\beta\nu}{\alpha})} t^{-\frac{\beta\nu}{\alpha}} \| v \| \quad v \in L^2(D), \end{aligned}$$

and

$$\begin{aligned} \| E_{\beta\beta}(t)v \|_{\dot{H}^\nu} &\leq \int_0^\infty \beta\theta M_\beta(\theta) \| A_\nu S_\alpha(t^\beta\theta)v \| d\theta \\ &\leq \int_0^\infty C_{\alpha\nu} \beta t^{-\frac{\beta\nu}{\alpha}} \theta^{1 - \frac{\nu}{\alpha}} M_\beta(\theta) \| v \| do \\ &= \frac{C_{\alpha\nu} \beta \Gamma(2 - \frac{\nu}{\alpha})}{\Gamma(1 + \beta(1 - \frac{\nu}{\alpha}))} t^{-\frac{\beta\nu}{\alpha}} \| v \|, \quad v \in L^2(D), \end{aligned}$$

which imply that the estimates (2.8) hold, so it is easy to know that $E_\beta(t)$ and $E_{\beta\beta}(t)$ are linear and bounded operators. □

Theorem 2.2. For any $t > 0$, the operators $E_\beta(t)$ and $E_{\beta\beta}(t)$ are strongly continuous. Moreover, for any $0 \leq t_1 < t_2 \leq T$ and for $0 < \nu < \alpha < 2$, there exist constants $C_{\alpha\nu} = C(\alpha, \beta, \nu) > 0$ and $C_{\beta\nu} = C(\alpha, \beta, \nu) > 0$ such that

$$(2.9) \quad \| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} \leq C_{\alpha\nu} (t_2 - t_1)^{\frac{\beta\nu}{\alpha}} \| v \|,$$

and

$$(2.10) \quad \| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} \leq C_{\beta\nu}(t_2 - t_1)^{\frac{\beta\nu}{\alpha}} \| v \| .$$

Proof. for any $0 \leq t_1 < t_2 \leq T$, it is easy to deduce that

$$(2.11) \quad \begin{aligned} \int_{t_1}^{t_2} \frac{dS_\alpha(t^\beta\theta)}{dt} dt &= S_\alpha(t_2^\beta\theta) - S_\alpha(t_1^\beta\theta) \\ &= - \int_{t_1}^{t_2} \beta t^{\beta-1} \theta A_\alpha S_\alpha(t^\beta\theta) dt, \end{aligned}$$

for $0 < \nu < \alpha < 2$, making use of the above expression, the Lemma (2.1) and Lemma (2.3), we can arrive at

$$\begin{aligned} \| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} &= \| A_\nu(E_\beta(t_2) - E_\beta(t_1))v \| \\ &= \left\| \int_0^\infty M_\beta(\theta) A_\nu((S_\alpha(t_2^\beta\theta) - S_\alpha(t_1^\beta\theta))v) d\theta \right\| \\ &\leq \int_0^\infty \beta \theta M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \| A_{\alpha+\nu} S_\alpha(t^\beta\theta)v \|_{L^2} dt d\theta \\ &\leq \int_0^\infty C_{\alpha\nu} \beta \theta^{-\frac{\nu}{\alpha}} M_\beta(\theta) \left(\int_{t_1}^{t_2} t^{-\frac{\beta\nu}{\alpha}-1} dt \right) \| v \| d\theta \\ &= \frac{\alpha C_{\alpha\nu} \Gamma(1-\frac{\nu}{\alpha})}{\nu \Gamma(1-\frac{\beta\nu}{\alpha})} (t_2^{\frac{-\beta\nu}{\alpha}} - t_1^{\frac{-\beta\nu}{\alpha}}) \| v \| \\ &\leq \frac{\alpha C_{\alpha\nu} \Gamma(1-\frac{\nu}{\alpha})}{\nu T^{\frac{2\beta\nu}{\alpha}} \Gamma(1-\frac{\beta\nu}{\alpha})} (t_2 - t_1)^{\frac{\beta\nu}{\alpha}} \| v \|, \quad v \in L^2(D), \end{aligned}$$

and

$$\begin{aligned} \| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} &= \| A_\nu(E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \| \\ &= \left\| \int_0^\infty \beta \theta M_\beta(\theta) A_\nu(S_\alpha(t_2^\beta\theta) - S_\alpha(t_1^\beta\theta))v d\theta \right\| \\ &\leq \int_0^\infty \beta^2 \theta^2 M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \| A_{\alpha+\nu} S_\alpha(t^\beta\theta)v \|_{L^2} dt d\theta \\ &\leq \int_0^\infty C_{\alpha\nu} \beta^2 \theta^{1-\frac{\nu}{\alpha}} M_\beta(\theta) \left(\int_{t_1}^{t_2} t^{-\frac{\beta\nu}{\alpha}-1} dt \right) \| v \| d\theta \\ &= \frac{\alpha C_{\alpha\nu} \Gamma(2-\frac{\nu}{\alpha})}{\nu \Gamma(1+\beta(1-\frac{\nu}{\alpha}))} (t_2^{\frac{-\beta\nu}{\alpha}} - t_1^{\frac{-\beta\nu}{\alpha}}) \| v \| \\ &\leq \frac{\alpha C_{\alpha\nu} \Gamma(1-\frac{\nu}{\alpha})}{\nu T_0^{\frac{2\beta\nu}{\alpha}} \Gamma(1+\beta(1-\frac{\nu}{\alpha}))} (t_2 - t_1)^{\frac{\beta\nu}{\alpha}} \| v \|, \quad v \in L^2(D), \end{aligned}$$

It is obviously to see that the term $\| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} \rightarrow 0$ and $\| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} \rightarrow 0$ as $t_1 \rightarrow t_2$. Which mean that the operators $E_\beta(t)$ and $E_{\beta\beta}(t)$ are strongly continuous. □

Remark 2.1. Assume $\nu = 0$ in theorem (2.2), then there exist constants $C_\alpha = C(\alpha, \beta) > 0$ and $C_\beta = C(\alpha, \beta) > 0$ such that

$$(2.12) \quad \| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} \leq C_\alpha(t_2 - t_1) \| v \|,$$

and

$$(2.13) \quad \| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} \leq C_\beta(t_2 - t_1) \| v \| .$$

Proof. for any $0 < T_0 \leq t_1 < t_2 \leq T$, the same as the proof of Theorem (2.2), we get

$$\begin{aligned} \| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} &= \left\| \int_0^\infty M_\beta(\theta) ((S_\alpha(t_2^\beta \theta) - S_\alpha(t_1^\beta \theta))v) d\theta \right\|_{L^2} \\ &\leq \int_0^\infty \beta \theta M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \| A_\alpha S_\alpha(t^\beta \theta)v \|_{L^2} dt d\theta \\ &\leq \int_0^\infty C_{\alpha\alpha} \beta \theta M_\beta(\theta) \left(\int_{t_1}^{t_2} t^{-1} dt \right) \| v \|_{L^2} d\theta \\ &\leq C_{\alpha\alpha} \beta (\ln t_2 - \ln t_1) \| v \| \\ &= \frac{C_{\alpha\alpha} \beta}{T_0} (t_2 - t_1) \| v \|, \quad v \in L^2(D), \end{aligned}$$

and

$$\begin{aligned} \| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} &= \left\| \int_0^\infty \beta \theta M_{\beta\beta}(\theta) ((S_\alpha(t_2^\beta \theta) - S_\alpha(t_1^\beta \theta))v) d\theta \right\|_{L^2} \\ &\leq \int_0^\infty \beta^2 \theta^2 M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \| A_\alpha S_\alpha(t^\beta \theta)v \|_{L^2} dt d\theta \\ &\leq \int_0^\infty C_{\alpha\alpha} \beta^2 \theta M_\beta(\theta) \left(\int_{t_1}^{t_2} t^{-1} dt \right) \| v \| d\theta \\ &\leq \frac{C_{\alpha\alpha} \beta^2 \Gamma(2)}{\Gamma(1+\beta)} (\ln t_2 - \ln t_1) \| v \| \\ &\leq \frac{C_{\alpha\alpha} \beta^2 \Gamma(2)}{T_0 \Gamma(1+\beta)} (t_2 - t_1) \| v \|, \quad v \in L^2(D). \end{aligned}$$

This completes the proof. □

3. EXISTENCE AND UNIQUENESS OF MILD SOLUTION

Our main purpose of this section is to prove the existence and uniqueness of mild solution to the problem (2.1). To do this, the following assumptions are imposed.

3.1. **Assumption .** The measurable function $g : \Omega \times H \rightarrow L_0^2$ satisfies the following global Lipschitz and growth conditions:

$$(3.1) \quad \|g(v)\|_{L_0^2} \leq C \|v\|, \quad \|g(u) - g(v)\|_{L_0^2} \leq C \|u - v\|,$$

for all $u, v \in H$.

3.2. **Assumption .** Let C, C_1 are a positive real number, then the bounded bilinear operator $B : L_0^2(D) \rightarrow H^{-1}(D)$ satisfies the following properties:

$$\begin{aligned} \|B(u, c)\|_{\dot{H}^{-1}} &\leq C \|u\| \|c\| \\ &\leq C_1 \|u\|^2, \end{aligned}$$

and

$$(3.2) \quad \|B(u, c) - B(v, c)\|_{\dot{H}^{-1}} \leq CC_1 (\|u\| + \|v\|) \|u - v\|,$$

where C_1 depend a norm the c in $L_0^2(D)$, and for all $u, v, c \in L_0^2(D)$.

3.3. **Assumption .** Assume that the initial value $u_0 : \Omega \rightarrow \dot{H}^\nu$ is a \mathcal{F}_0 -measurable random variable, it holds that

$$(3.3) \quad \|u_0\|_{L^p(\Omega, \dot{H}^\nu)} < \infty,$$

for any $0 \leq \nu < \alpha < 2$.

Theorem 3.1. *Let Assumption (3.1) to (3.3) be satisfied for some $p \geq 2$, then there exists a unique mild solution $(u(t))_{t \in [0, T]}$ in the space $L^p(\Omega, \dot{H}^\nu)$ with $0 \leq \nu < \alpha < 2$.*

Proof. We fix an $\omega \in \Omega$ and use the standard Picard's iteration argument to prove the existence of mild solution. To begin with, the sequence of stochastic process $\{u(t)\}_{n \geq 0}$ is constructed as

$$(3.4) \quad \begin{cases} u_{n+1}(t) = E_\beta(t)u_0 + N_1(u_n(t)) + N_2(u_n(t)), \\ u_0(t) = u_0, \end{cases}$$

where

$$(3.5) \quad \begin{cases} N_1(u_n(t)) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(t-s) B(u_n(s), c(s)) ds, \\ N_2(u_n(t)) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(t-s) g(u_n(s)) dW(s). \end{cases}$$

The proof will be split into three steps.

Step1 For each $n \geq 0$, we show that

$$\sup E[\| u_n(t) \|_{\dot{H}^\nu}^p] < \infty,$$

Note that

$$(3.6) \quad \begin{aligned} E[\| u_{n+1}(t) \|_{\dot{H}^\nu}^p] &\leq 3^{p-1} E[\| E_\beta(t) u_0 \|_{\dot{H}^\nu}^p] + 3^{p-1} E[\| N_1(u_n(t)) \|_{\dot{H}^\nu}^p] \\ &+ 3^{p-1} E[\| N_2(u_n(t)) \|_{\dot{H}^\nu}^p]. \end{aligned}$$

The application of the Lemma (2.1) gives

$$(3.7) \quad \begin{aligned} E[\| E_\beta(t) u_0 \|_{\dot{H}^\nu}^p] &\leq E[\int_0^\infty M_\beta(\theta) (\| A_\nu S_\alpha(t^\beta \theta) u_0 \|^2)^{\frac{1}{2}} d\theta] \\ &= E[\int_0^\infty M_\beta(\theta) (\sum_{n=1}^\infty \langle A_\alpha e^{-t^\beta \theta A_\alpha} u_0, e_n \rangle^2)^{\frac{1}{2}} d\theta] \\ &= E[\int_0^\infty M_\beta(\theta) (\sum_{n=1}^\infty \langle A_\alpha u_0, e^{-t^\beta \theta \lambda^{\frac{\alpha}{2}}} e_n \rangle^2)^{\frac{1}{2}} d\theta] \\ &\leq E[\int_0^\infty M_\beta(\theta) \| u_0 \|_{\dot{H}^\nu} d\theta] = E[\| u_0 \|_{\dot{H}^\nu}] \end{aligned}$$

Applying the following Hölder inequality to the second term of the right-hand side of (3.6)

$$(3.8) \quad \begin{aligned} E[\| N_1(u_n(t)) \|_{\dot{H}^\nu}^p] &\leq E[(\int_0^t \| (t-s)^{\beta-1} A_1 E_{\beta,\beta}(t-s) A_{\nu-1} B(u_n(s), c(s)) \|^2 ds)^p] \\ &\leq C_\beta^p (\int_0^t (t-s)^{\frac{p(\beta-1-\frac{\beta}{\alpha})}{p-1}} ds)^{p-1} \int_0^t E[\| A_{\nu-1} B(u_n(s), c(s)) \|^p] ds \\ &\leq K_1 \int_0^t E[\| u_n(s) \|_{\dot{H}^\nu}^p] ds, \end{aligned}$$

where $K_1 = C_\beta^p C^p C_1^p [\frac{p-1}{p\beta(1-\frac{1}{\alpha p \beta \alpha})-1}]^{p-1} T^{p\beta(1-\frac{1}{\alpha})-1} (\max_{t \in [0, T]} E[\| u_n(t) \|_{\dot{H}^\nu}^p])$. Making use of the Hölder inequality and Lemma (2.2) to the third term of the right-hand side of (3.6), we get

$$(3.9) \quad \begin{aligned} E[\| N_2(u_n(t)) \|_{\dot{H}^\nu}^p] &\leq C(p) E[(\int_0^t \| (t-s)^{\beta-1} E_{\beta,\beta}(t-s) \|^2 A_\nu g(u_n(s)) \|_{L^2}^2 ds)^{\frac{p}{2}}] \\ &\leq K_2 \int_0^t E[\| u_n(s) \|_{\dot{H}^\nu}^2] ds, \end{aligned}$$

where $K_2 = C(p) C_\beta^p C^p C_1^p [\frac{p-2}{p(2\beta-1)-2}]^{\frac{p-2}{2}} T^{\frac{p(2\beta-1)-2}{2}}$.

Using the above estimates (3.6) and (3.9), we have

$$E[\| u_{n+1}(t) \|_{\dot{H}^\nu}^p] \leq 3^{p-1}E[\| u_0 \|_{\dot{H}^\nu}^p] + 3^{p-1}(K_1 + K_2) \int_0^t E[\| u_n(s) \|_{\dot{H}^\nu}^p] ds$$

By means of the extension of Gronwall’s lemma, it holds that

$$\sup_{t \in [0, T]} E[\| u_{n+1}(t) \|_{\dot{H}^\nu}^p] < \infty, \text{ for each } n \geq 0.$$

Step2 Show that the sequence $\{u_n(t)\}_{n \geq 0}$ is a Cauchy sequence in the space $L^p(\Omega; \dot{H}^\nu)$. For any $n \geq m \geq 1$, applying the similar arguments employed to obtain (3.8) and (3.9), we get

$$\begin{aligned} E[\| u_n(t) - u_m(t) \|_{\dot{H}^\nu}^p] &\leq 2^{p-1}E[\| N_1(u_{n-1}(t)) - N_1(u_{m-1}(t)) \|_{\dot{H}^\nu}^p] \\ (3.10) \qquad \qquad \qquad &+ 2^{p-1}E[\| N_2(u_{n-1}(t)) - N_2(u_{m-1}(t)) \|_{\dot{H}^\nu}^p] \\ &\leq K \int_0^t E[\| u_{n-1}(s) - u_{m-1}(s) \|_{\dot{H}^\nu}^p] ds, \end{aligned}$$

in wich

$$\begin{aligned} K &= 2^{p-1} \{C_\beta^p C^p C_1^p\} \left[\frac{p-1}{p(\beta-\frac{\beta}{\nu})-1} \right]^{p-1} T^{p(\beta-\frac{\beta}{\alpha})} \left(\max_{t \in [0, T]} E[\| u_{n-1}(t) \|_{\dot{H}^\nu}^p] \right) \\ &+ \max_{t \in [0, T]} E[\| u_{m-1}(t) \|_{\dot{H}^\nu}^p] + C(p) C_\beta^p C^p C_1^p \left[\frac{p-2}{p(2\beta-1)-2} \right]^{\frac{p-2}{2}} T^{\frac{p(2\beta-1)-2}{2}} \}. \end{aligned}$$

A direct application of Gronwall’s lemma yields

$$\sup_{t \in [0, T]} E[\| u_n(t) - u_m(t) \|_{\dot{H}^\nu}^p] = 0,$$

for all $T > 0$. Taking limits to the stochastic sequence $\{u_n(t)\}_{n \geq 0}$ in (3.4) as $n \rightarrow \infty$, we finish the proof of the existence of mild solution to (2.1).

Step3 We show the uniqueness of mild solution. Assume u and v are two mild solutions of the problem (2.1), using the similar calculations as in **Step 2**, we can obtain

$$(3.11) \qquad \qquad \qquad \sup_{t \in [0, T]} E[\| u(t) - v(t) \|_{\dot{H}^\nu}^p] = 0,$$

for all $T > 0$, which implies that $u = v$, it follows that the uniqueness of mild solution. Obviously, when $\nu = 0$, the above three steps still work. Thus the proof of Theorem 3.1 is completed. □

4. REGULARITY OF MILD SOLUTION

In this section, we will prove the spatial and temporal regularity properties of mild solution to time-space fractional SKSM based on the analytic semigroup.

Theorem 4.1. *Let Assumptions (3.1) to (3.3) hold with $1 \leq \nu < \alpha < 2$ and $p \geq 2$ let $u(t)$ be a unique mild solution of the problem (2.1) with $\mathbb{P}(u(t) \in \dot{H}^\nu) = 1$ for any $t \in [0, T]$, then there exists a constant C such that*

$$(4.1) \quad \sup_{t \in [0, T]} \|u(t)\|_{L^p(\Omega; \dot{H}^\nu)} \leq C(\|u_0\|_{L^p(\Omega; H)} + \sup_{t \in [0, T]} \|u(t)\|_{L^p(\Omega; \dot{H}^1)}).$$

Proof. For any $0 \leq t \leq T$ and $1 \leq \nu < \alpha < 2$, we have

$$\begin{aligned} \|u(t)\|_{L^p(\Omega; \dot{H}^\nu)} &= (E[\|u(t)\|_{\dot{H}^\nu}^p])^{\frac{1}{p}} = \|A_\nu u(t)\|_{L^p(\Omega; H)} \\ &\leq \|A_\nu E_\beta(t)u_0\|_{L^p(\Omega; H)} \\ (4.2) \quad &+ \|A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(t-s) B(u(s), c(s)) ds\|_{L^p(\Omega; H)} \\ &+ \|A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(t-s) g(u(s)) dW(s)\|_{L^p(\Omega; H)} \\ &= I + II + III. \end{aligned}$$

Using Theorem (2.1), the first term can be estimated by

$$(4.3) \quad I = \|A_\nu E_\beta(t)u_0\|_{L^p(\Omega; H)} \leq C_\alpha t^{-\frac{\beta\nu}{\alpha}} \|u_0\|_{L^p(\Omega; H)} < \infty.$$

It is easy to know that

$$(4.4) \quad \int_0^T C_\alpha t^{-\frac{\beta\nu}{\alpha}} \|u_0\|_{L^p(\Omega; H)} dt = \frac{\alpha C_\alpha}{\alpha - \beta\nu} T^{\frac{\alpha - \beta\nu}{\alpha}} \|u_0\|_{L^p(\Omega; H)}$$

The application of Theorem (2.1) and Assumptions (3.2), we get

$$\begin{aligned} (II)^p &\leq E[(\|A_\nu \int_0^t (t-s)^{\beta-1} A_\nu E_{\beta, \beta}(t-s) B(u(s), c(s))\| ds)^p] \\ (4.5) \quad &\leq C_\beta^p (\int_0^t (t-s)^{\frac{p[\beta-1-\frac{\beta(\nu+1)]}{p-1}]} ds)^{p-1} \int_0^t E[\|A_{-1} B(u(s), c(s))\|_{\dot{H}^1}^p] ds \\ &\leq C_2 \sup_{t \in [0, T]} E[\|u(s)\|_{\dot{H}^1}^p], \end{aligned}$$

where $C_2 = C_\beta^p C^p C_1^p \left\{ \frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}] - 1} \right\}^{p-1} T^{p[\beta - \frac{\beta(\nu+1)}{\alpha}] - 1} + \left(\max_{t \in [0, T]} E[\|u(t)\|_{\dot{H}^1}] \right)$.

By means of Theorem (2.1), Assumptions (3.1) and Lemma (2.2), we can deduce

$$\begin{aligned}
 (III)^p &\leq C(p) E\left[\|A_\nu \int_0^t (t-s)^{\beta-1} A_{\nu-1} E_{\beta, \beta}(t-s) \|^2 \|A_1 g(u(s))\|_{L_2^0}^2 ds\right]^{\frac{p}{2}} \\
 (4.6) \quad &\leq C(p) C_\beta^p \left(\int_0^t (t-s)^{\frac{2p[\beta-1 - \frac{\beta(\nu-1)}{\alpha}]}{p-2}} ds\right)^{\frac{p-2}{2}} \int_0^t E \|A_1 g(u(s))\|_{L_2^0}^p ds \\
 &\leq C_3 \sup_{t \in [0, T]} E[\|u(s)\|_{\dot{H}^1}^p],
 \end{aligned}$$

where $C_3 = C(p) C_\beta^p C^p C_1^p \left[\frac{p-2}{p[2\beta-1 - \frac{\beta(\nu-1)}{\alpha}] - 2} \right]^{\frac{p-2}{2}} T^{\frac{p[2\beta-1 - \frac{\beta(\nu-1)}{\alpha}] - 2}{2}}$.

Thus, we conclude the proof of Theorem (4.2) by combining with the estimates (4.2)- (4.6). □

Next, we will devote to the temporal regularity of the mild solution.

Theorem 4.2. *Let Assumptions (3.1) to (3.3) hold with $0 < \nu < \alpha < 2$ and $p \geq 2$ for any $0 \leq t_1 < t_2 \leq T$, the unique mild solution $u(t)$ to the problem (2.1) is Hölder continuous with respect to the norm $\|\cdot\|_{L^p(\Omega; \dot{H}^\nu)}$ and satisfies*

$$(4.7) \quad \|u(t_2) - u(t_1)\|_{L^p(\Omega; \dot{H}^\nu)} \leq C(t_2 - t_1)^\gamma.$$

Proof. For any $0 \leq t_1 < t_2 \leq T$, for the mild solution (2.5), we have

$$\begin{aligned}
 (4.8) \quad u(t_2) - u(t_1) &= E_\beta(t_2)u_0 - E_\beta(t_1)u_0 \\
 &+ \int_0^{t_2} (t_2 - s)^{\beta-1} E_{\beta, \beta}(t_2 - s) B(u(s), c(s)) ds \\
 &- \int_0^{t_1} (t_1 - s)^{\beta-1} E_{\beta, \beta}(t_1 - s) B(u(s), c(s)) ds \\
 &+ \int_0^{t_2} (t_2 - s)^{\beta-1\beta-1} E_{\beta, \beta}(t_2 - s) g(u(s)) dW(s) \\
 &- \int_0^{t_1} (t_2 - s)^{\beta-1\beta-1} E_{\beta, \beta}(t_1 - s) G(u(s)) dW(s) \\
 &= I_1 + I_2 + I_3,
 \end{aligned}$$

where

$$I_1 = E_\beta(t_2)u_0 - E_\beta(t_1)u_0,$$

and

$$\begin{aligned}
 I_2 &= \int_0^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s) B(u(s), c(s)) ds \\
 &\quad - \int_0^{t_1} (t_1 - s)^{\beta-1} E_{\beta,\beta}(t_1 - s) B(u(s), c(s)) ds \\
 &= \int_0^{t_1} (t_1 - s)^{\beta-1} [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] B(u(s), c(s)) ds \\
 (4.9) \quad &\quad + \int_0^{t_1} [(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}] E_{\beta,\beta}(t_2 - s) B(u(s), c(s)) ds \\
 &\quad + \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s) B(u(s), c(s)) ds \\
 &= I_{21} + I_{22} + I_{23},
 \end{aligned}$$

and

$$\begin{aligned}
 I_3 &= \int_0^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s) g(u(s)) dW(s) \\
 &\quad - \int_0^{t_1} (t_1 - s)^{\beta-1} E_{\beta,\beta}(t_1 - s) g(u(s)) dW(s) \\
 &= \int_0^{t_1} (t_1 - s)^{\beta-1} [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] g(u(s)) dW(s) \\
 (4.10) \quad &\quad + \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] E_{\beta,\beta}(t_2 - s) g(u(s)) dW(s) \\
 &\quad + \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s) g(u(s)) dW(s) \\
 &= I_{31} + I_{32} + I_{33}.
 \end{aligned}$$

For any $0 < \nu < \alpha < 2$ and $p \geq 2$, by virtue of Theorem (2.2), it follows that

$$\begin{aligned}
 E[\| I_1 \|_{H^\nu}^p] &= E[\| A_\nu [E_\beta(t_2) - E_\beta(t_1)] u_0 \|^p] \\
 (4.11) \quad &\leq C_{\alpha,\nu}^p (t_2 - t_1)^{\frac{p\beta\nu}{\alpha}} E[\| u_0 \|^p].
 \end{aligned}$$

For the first term I_{21} in (4.9), applying the Assumption (3.2) and Theorem (2.2) and Hölder’s inequality, we have

$$\begin{aligned}
 E[\| I_{21} \|_{H^\nu}^p] &= E[\| \int_0^{t_1} (t_1 - s)^{\beta-1} A_\nu [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] B(u(s), c(s)) ds \|^p] \\
 (4.12) \quad &\leq C_{\beta\nu}^p (t_2 - t_1)^{\frac{p\beta(\nu+1)}{\alpha}} (\int_0^{t_1} (t_1 s)^{\frac{p(\beta-1)}{p-1}} ds)^{p-1} \int_0^{t_1} E[\| A_{-1} B(u(s), c(s)) \|_{H^1}^p] ds \\
 &\leq C^p C_1^p C_{\beta\nu}^p T^p (\frac{p-1}{p\beta-1})^{p-1} (\sup_{t \in [0, T]} E[\| u(s) \|_{H^1}^{2p}]) (t_2 - t_1)^{\frac{p\beta(\nu+1)}{\alpha}}.
 \end{aligned}$$

Using the Assumptions (3.2), Theorem 2.1 and Hölder’s inequality, we get

$$\begin{aligned}
 E[\| I_{22} \|_{H^\nu}^p] &= E[\| \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] A_\nu E_{\beta,\beta}(t_2 - s) B(u(s), c(s)) ds \|^p] \\
 (4.13) \quad &\leq C_\beta^p (\int_0^{t_1} \{ [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] \times (t_2 - s)^{-\frac{\beta(\nu+1)}{\alpha}} \}^{\frac{p}{p-1}} ds)^{p-1} \\
 &\times \int_0^{t_1} E[\| A_1 B(u(s), c(s)) \|_{H^1}^p] ds \\
 &\leq C^p C_1^p C_\beta^p T \{ \frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}] - 1} \}^{p-1} (\sup_{t \in [0, T]} E[\| u(s) \|_{H^1}^{2p}]) (t_2 - t_1)^{\frac{p\beta(\alpha - \nu - 1) - \alpha}{\alpha}},
 \end{aligned}$$

and

$$\begin{aligned}
 E[\| I_{23} \|_{H^\nu}^p] &= E[\| \int_{t_1}^{t_2} (t_2 s)^{\beta-1} A_\nu E_{\beta\beta}(t_2 - s) B(u(s), c(s)) ds \|^p] \\
 (4.14) \quad &\leq C_\beta^p (\int_{t_1}^{t_2} [(t_2 - s)^{\beta-1 - \frac{\beta(\nu+1)}{\alpha}}]^{\frac{p}{p-1}} ds)^{p-1} \\
 &\times \int_{t_1}^{t_2} E[\| A_1 B(u(s), c(s)) \|_{H^1}^p] ds \\
 &\leq C^p C_1^p C_\beta^p \{ \frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}] - 1} \}^{p-1} (\sup_{t \in [0, T]} E[\| u(s) \|_{H^1}^{2p}]) (t_2 - t_1)^{\frac{p\beta(\alpha - \nu - 1)}{\alpha}}.
 \end{aligned}$$

Next, by following the similar arguments as in the proof of (4.12)- (4.14) and using the Lemma (2.2), there holds

$$\begin{aligned}
 E[\| I_{31} \|_{\dot{H}^\nu}^p] &= E[\| \int_0^{t_1} (t_1 - s)^{\beta-1} A_\nu [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] g(u(s)) dW(s) \|^p] \\
 &\leq C(p) E[\int_0^{t_1} \| (t_1 - s)^{\beta-1} A_\nu [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] \|^2 \| g(u(s)) \|_{L_0^2}^2 ds]^{\frac{p}{2}} \\
 &\leq C(p) C_{\beta\nu}^p (t_2 - t_1)^{\frac{p\beta\nu}{\alpha}} (\int_0^{t_1} (t_1 - s)^{\frac{2p(\beta-1)}{p-2}} ds)^{\frac{p-2}{2}} \int_0^{t_1} E \| g(u(s)) \|_{L_0^2}^p ds \\
 &\leq C(p) C_{\beta\nu}^p T^{\frac{2p\beta-p-1}{2}} (\frac{p-1}{2p\beta-p-2})^{p-1} (\sup_{t \in [0, T]} E[\| u(t) \|^p]) (t_1 - t_2)^{\frac{p\beta\nu}{\alpha}},
 \end{aligned}$$

and

$$\begin{aligned}
 E[\| I_{32} \|] &= E[\| \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] A_\nu E_{\beta\beta}(t_2 - s) g(u(s)) dW(s) \|^p] \\
 &\leq C(p) E[\int_0^{t_1} \| ((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}) A_\nu E_{\beta\beta}(t_2 - s) \|^2 \| g(u(s)) \|_{L_0^2}^2 ds] \\
 (4.15) \quad &\leq C(p) C_\beta^p \int_0^{t_1} \{ [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] \times (t_2 - s)^{-\frac{\beta\nu}{2}} \}^{\frac{2p}{p-2}} ds^{\frac{p-2}{2}} \\
 &\times \int_0^{t_1} E \| g(u(s)) \|_{L_0^2}^p ds \\
 &\leq C(p) C_\beta^p C^p T^{\frac{\alpha(p-2)}{2p\beta(\alpha-\nu)-(p+2)\alpha}} (\sup_{t \in [0, T]} E[\| u(t) \|^p]) (t_2 - t_1)^{\frac{2p\beta(\alpha-\nu)-(p+2)\alpha}{2\alpha}},
 \end{aligned}$$

and

$$\begin{aligned}
 E[\| I_{33} \|] &= E[\| \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} A_\nu E_{\beta\beta}(t_2 - s) g(u(s)) dW(s) \|^p] \\
 &\leq C(p) E[\int_{t_1}^{t_2} \| (t_2 - s)^{\beta-1} A_\nu E_{\beta\beta}(t_2 - s) \|^2 \| g(u(s)) \|_{L_0^2}^2 ds]^{\frac{p}{2}} \\
 (4.16) \quad &\leq C(p) C_\beta^p (\int_{t_1}^{t_2} [(t_2 - s)^{\beta-1-\frac{\beta\nu}{\alpha}}]^{\frac{2p}{p-2}} ds)^{\frac{p-2}{2}} \int_{t_1}^{t_2} E \| g(u) \|_{L_0^2}^p ds \\
 &\leq C(p) C_\beta^p C^p [\frac{\alpha(p-2)}{2p\beta(\alpha-\nu)-(p+2)\alpha}]^{\frac{p-2}{2}} (\sup_{t \in [0, T]} E[\| u(t) \|^p]) (t_2 - t_1)^{\frac{2p\beta(\alpha-\nu)-p\alpha}{2\alpha}}.
 \end{aligned}$$

□

Taking expectation on the both side of (4.8), and in view of the estimates (4.11)- (4.16), we conclude that

$$(4.17) \quad \| u(t_2) - u(t_1) \|_{L^p(\Omega; \dot{H}^\nu)} \leq C(t_2 - t_1)^\gamma,$$

in which we tak $\gamma = \min\{\frac{\beta\nu}{\alpha}, \frac{p\beta(\alpha-\nu-1)-\alpha}{p\alpha}, \frac{2p\beta(\alpha-\nu)-(p+2)\alpha}{2p\alpha}\}$, where $0 < t_2 - t_1 < 1$.

Otherwise, if $t_2 - t_1 \geq 1$ then we set $\gamma = \max\{\frac{\beta(\nu+1)}{\alpha}, \frac{\beta(\alpha-\nu-1)}{\alpha}, \frac{2p\beta(\alpha-\nu)-p\alpha}{2p\alpha}\}$. This completes the proof of Theorem (4.2)

5. EXISTENCE AND UNIQUENESS OF MILD SOLUTION

Our main purpose of this section is to prove the existence and uniqueness of mild solution to the problem (2.6). To do this, the following assumptions are imposed.

5.1. Assumption. The measurable function $f : \Omega \times H \rightarrow L_0^2$ satisfies the following global Lipschitz and growth conditions:

$$(5.1) \quad \| f(v) \|_{L_0^2} \leq C \| v \|, \quad \| f(u) - f(v) \|_{L_0^2} \leq C \| u - v \|,$$

for all $u, v \in H$.

5.2. Assumption. Let C , is a positive real number, then the bounded bilinear operator $L : L_0^2(D) \rightarrow H^{-1}(D)$ satisfies the following properties:

$$(5.2) \quad \| L(c) \|_{\dot{H}^{-1}} \leq C \| c \|^2,$$

and

$$(5.3) \quad \| L(c) - L(v) \|_{\dot{H}^{-1}} \leq C(\| c \| + \| v \|) \| c - v \|,$$

and for all $v, c \in L_0^2(D)$.

5.3. Assumption. Assume that the initial value $c_0 : \Omega \rightarrow \dot{H}^\nu$ is a \mathcal{F}_0 -measurable random variable, it holds that

$$(5.4) \quad \| c_0 \|_{L^p(\Omega, \dot{H}^\nu)} < \infty,$$

for any $0 \leq \nu < \alpha < 2$.

Theorem 5.1. *Let Assumption (5.1) to (5.3) be satisfied for some $p \geq 2$, then there exists a unique mild solution $(c(t))_{t \in [0, T]}$ in the space $L^p(\Omega, \dot{H}^\nu)$ with $0 \leq \nu < \alpha < 2$.*

Proof. We fix an $\omega \in \Omega$ and use the standard Picard's iteration argument to prove the existence of mild solution. To begin with, the sequence of stochastic process $\{c_n(t)\}_{n \geq 0}$ is constructed as

$$(5.5) \quad \begin{cases} c_{n+1}(t) &= E_\beta(t)c_0 + N_1(c_n(t)) + N_2(c_n(t)), \\ c_0(t) &= c_0, \end{cases}$$

where

$$(5.6) \quad \begin{cases} N_1(c_n(t)) &= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(t-s)L(c_n(s))ds, \\ N_2(c_n(t)) &= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(t-s)f(c_n(s))dW(s). \end{cases}$$

The proof will be split into three steps.

Step1 For each $n \geq 0$, we show that

$$\sup E[\| c_n(t) \|_{\dot{H}^\nu}^p] < \infty,$$

note that

$$(5.7) \quad \begin{aligned} E[\| c_{n+1}(t) \|_{\dot{H}^\nu}^p] &\leq 3^{p-1}E[\| E_\beta(t)c_0 \|_{\dot{H}^\nu}^p] + 3^{p-1}E[\| N_1(c_n(t)) \|_{\dot{H}^\nu}^p] \\ &+ 3^{p-1}E[\| N_2(c_n(t)) \|_{\dot{H}^\nu}^p]. \end{aligned}$$

The application of the Lemma (2.1) gives

$$(5.8) \quad \begin{aligned} E[\| E_\beta(t)c_0 \|_{\dot{H}^\nu}^p] &\leq E[\int_0^\infty M_\beta(\theta)(\| A_\nu S_\alpha(t^\beta\theta)c_0 \|^2)^{\frac{1}{2}} d\theta] \\ &= E[\int_0^\infty M_\beta(\theta)(\sum_{n=1}^\infty \langle A_\alpha e^{-t^\beta\theta A_\alpha} c_0, e_n \rangle^2)^{\frac{1}{2}} d\theta] \\ &= E[\int_0^\infty M_\beta(\theta)(\sum_{n=1}^\infty \langle A_\alpha u_0, e^{-t^\beta\theta\lambda^{\frac{\alpha}{2}}} e_n \rangle^2)^{\frac{1}{2}} d\theta] \\ &\leq E[\int_0^\infty M_\beta(\theta) \| c_0 \|_{\dot{H}^\nu} d\theta] = E[\| c_0 \|_{\dot{H}^\nu}]. \end{aligned}$$

Applying the following Hölder inequality to the second term of the right-hand side of (5.7)

$$(5.9) \quad \begin{aligned} E[\| N_1(c_n(t)) \|_{\dot{H}^\nu}^p] &\leq E[(\int_0^t \|(t-s)^{\beta-1} A_1 E_{\beta,\beta}(t-s) A_{\nu-1} L(c_n(s))\| ds)^p] \\ &\leq C_\beta^p (\int_0^t (t-s)^{\frac{p(\beta-1-\frac{\beta}{\alpha})}{p-1}} ds)^{p-1} \int_0^t E[\| A_{\nu-1} L(c_n(s)) \|^p] ds \\ &\leq K_1 \int_0^t E[\| c_n(s) \|_{\dot{H}^\nu}^p] ds, \end{aligned}$$

where $K_1 = C_\beta^p C^p [\frac{p-1}{p(\beta-\frac{1}{\alpha})-1}]^{p-1} T^{p\beta(1-\frac{1}{\alpha})-1} (\max_{t \in [0, T]} E[\|c_n(t)\|_{\dot{H}^\nu}^p])$.

Making use of the Hölder inequality and Lemma (2.2) to the third term of the right-hand side of (5.7), we get

$$\begin{aligned}
 E[\|N_2(c_n(t))\|_{\dot{H}^\nu}^p] &\leq C(p) E[\int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(t-s) \|^2 A_\nu f(c_n(s))\|_{L_0^2}^2 ds]^{\frac{p}{2}} \\
 (5.10) \qquad \qquad \qquad &\leq K_2 \int_0^t E[\|c_n(s)\|_{\dot{H}^\nu}^2] ds,
 \end{aligned}$$

where $K_2 = C(p) C_\beta^p C^p [\frac{p-2}{p(2\beta-1)-2}]^{\frac{p-2}{2}} T^{\frac{p(2\beta-1)-2}{2}}$.

Using the above estimates (5.7)- (5.10), we have

$$E[\|c_{n+1}(t)\|_{\dot{H}^\nu}^p] \leq 3^{p-1} E[\|c_0\|_{\dot{H}^\nu}^p] + 3^{p-1} (K_1 + K_2) \int_0^t E[\|c_n(s)\|_{\dot{H}^\nu}^p] ds$$

. By means of the extension of Gronwall’s lemma, it holds that

$$\sup_{t \in [0, T]} E[\|c_{n+1}(t)\|_{\dot{H}^\nu}^p] < \infty, \text{ for each } n \geq 0.$$

Step1: Show that the sequence $\{c_n(t)\}_{n \geq 0}$ is a Cauchy sequence in the space $L^p(\Omega; \dot{H}^\nu)$. For any $n \geq m \geq 1$, applying the similar arguments employed to obtain (5.9) and (5.10), we get

$$\begin{aligned}
 E[\|c_n(t) - c_m(t)\|_{\dot{H}^\nu}^p] &\leq 2^{p-1} E[\|N_1(c_{n-1}(t)) - N_1(c_{m-1}(t))\|_{\dot{H}^\nu}^p] \\
 (5.11) \qquad \qquad \qquad &+ 2^{p-1} E[\|N_2(c_{n-1}(t)) - N_2(c_{m-1}(t))\|_{\dot{H}^\nu}^p] \\
 &\leq K \int_0^t E[\|c_{n-1}(s) - c_{m-1}(s)\|_{\dot{H}^\nu}^p] ds,
 \end{aligned}$$

in wich

$$\begin{aligned}
 K &= 2^{p-1} \{C_\beta^p C^p\} [\frac{p-1}{p(\beta-\frac{1}{\nu})-1}]^{p-1} T^{p(\beta-\frac{\beta}{\alpha})} (\max_{t \in [0, T]} E[\|c_{n-1}(t)\|_{\dot{H}^\nu}^p]) \\
 (5.12) \qquad \qquad \qquad &+ \max_{t \in [0, T]} E[\|c_{m-1}(t)\|_{\dot{H}^\nu}^p] + C(p) C_\beta^p C^p [\frac{p-2}{p(2\beta-1)-2}]^{\frac{p-2}{2}} T^{\frac{p(2\beta-1)-2}{2}} \}.
 \end{aligned}$$

A direct application of Gronwall’s lemma yields

$$\sup_{t \in [0, T]} E[\|c_n(t) - c_m(t)\|_{\dot{H}^\nu}^p] = 0, \text{ for all } T > 0.$$

Taking limits to the stochastic sequence $\{c_n(t)\}_{n \geq 0}$ in (5.5) as $n \rightarrow \infty$, we finish the proof of the existence of mild solution to (2.6).

Step 3: We show the uniqueness of mild solution. Assume c and v are two mild solutions of the problem

(2.6), using the similar calculations as in

Step 2, we can obtain

$$(5.13) \quad \sup_{t \in [0, T]} E[\| c(t) - v(t) \|_{\dot{H}^\nu}^p] = 0,$$

for all $T > 0$, which implies that $c = v$, it follows that the uniqueness of mild solution. Obviously, when $\nu = 0$, the above three steps still work. Thus the proof of Theorem (6.1) is completed. \square

6. REGULARITY OF MILD SOLUTION

In this section, we will prove the spatial and temporal regularity properties of mild solution to time-space fractional SKSM based on the analytic semigroup.

Theorem 6.1. *Let Assumptions (5.1) to (5.3) hold with $1 \leq \nu < \alpha < 2$ and $p \geq 2$, let $c(t)$ be a unique mild solution of the problem (2.6) with $\mathbb{P}(c(t) \in \dot{H}^\nu) = 1$ for any $t \in [0, T]$, then there exists a constant C such that*

$$(6.1) \quad \sup_{t \in [0, T]} \| c(t) \|_{L^p(\Omega; \dot{H}^\nu)} \leq C(\| c_0 \|_{L^p(\Omega; H)} + \sup_{t \in [0, T]} \| c(t) \|_{L^p(\Omega; \dot{H}^1)}).$$

Proof. For any $0 \leq t \leq T$ and $1 \leq \nu < \alpha < 2$, we have

$$\begin{aligned} \| c(t) \|_{L^p(\Omega; \dot{H}^\nu)} &= (E[\| c(t) \|_{\dot{H}^\nu}^p])^{\frac{1}{p}} = \| A_\nu c(t) \|_{L^p(\Omega; H)} \\ &\leq \| A_\nu E_\beta(t) c_0 \|_{L^p(\Omega; H)} \\ &+ \| A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(t-s) L(c(s)) ds \|_{L^p(\Omega; H)} \\ &+ \| A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(t-s) f(c(s)) dW(s) \|_{L^p(\Omega; H)} \\ &= I + II + III. \end{aligned} \tag{6.2}$$

Using Theorem (2.1), the first term can be estimated by

$$(6.3) \quad I = \| A_\nu E_\beta(t) c_0 \|_{L^p(\Omega; H)} \leq C_\alpha t^{-\frac{\beta\nu}{\alpha}} \| c_0 \|_{L^p(\Omega; H)} < \infty.$$

It is easy to know that

$$(6.4) \quad \int_0^T C_\alpha t^{-\frac{\beta\nu}{\alpha}} \| c_0 \|_{L^p(\Omega; H)} dt = \frac{\alpha C_\alpha}{\alpha - \beta\nu} T^{\frac{\alpha - \beta\nu}{\alpha}} \| c_0 \|_{L^p(\Omega; H)}.$$

The application of Theorem (2.1) and Assumptions (5.2), we get

$$\begin{aligned}
 (II)^p &\leq E[(\| A_\nu \int_0^t (t-s)^{\beta-1} A_\nu E_{\beta,\beta}(t-s)L(c(s)) \| ds)^p] \\
 (6.5) \quad &\leq C_\beta^p (\int_0^t (t-s)^{\frac{p[\beta-1-\frac{\beta(\nu+1)]}{p-1}]} ds)^{p-1} \int_0^t E[\| A_{-1}L(c(s)) \|_{\dot{H}^1}^p] ds \\
 &\leq C_2 \sup_{t \in [0,T]} E[\| c(s) \|_{\dot{H}^1}^p],
 \end{aligned}$$

where $C_2 = C_\beta^p C^p \{ \frac{p-1}{p[\beta-\frac{\beta(\nu+1)]-1}]} \}^{p-1} T^{p[\beta-\frac{\beta(\nu+1)]-1} + (\max_{t \in [0,T]} E[\| c(t) \|_{\dot{H}^1}])$.

By means of Theorem (2.1), Assumptions (5.1) and Lemma (2.2), we can deduce

$$\begin{aligned}
 (III)^p &\leq C(p) E[(\| A_\nu \int_0^t \| (t-s)^{\beta-1} A_{\nu-1} E_{\beta,\beta}(t-s) \|^2 \| A_1 f(c(s)) \|_{L_2^0}^2 ds)^{\frac{p}{2}}] \\
 (6.6) \quad &\leq C(p) C_\beta^p (\int_0^t (t-s)^{\frac{2p[\beta-1-\frac{\beta(\nu-1)]}{p-2}]} ds)^{\frac{p-2}{2}} \int_0^t E \| A_1 f(c(s)) \|_{L_2^0}^p ds \\
 &\leq C_3 \sup_{t \in [0,T]} E[\| c(s) \|_{\dot{H}^1}^p],
 \end{aligned}$$

where $C_3 = C(p) C_\beta^p C^p [\frac{p-2}{p[2\beta-1-\frac{\beta(\nu-1)]-2}]}]^{\frac{p-2}{2}} T^{\frac{p[2\beta-1-\frac{\beta(\nu-1)]-2}{2}}$.

Thus, we conclude the proof of Theorem (6.1) by combining with the estimates (6.2)- (6.6). □

Next, we will devote to the temporal regularity of the mild solution.

Theorem 6.2. *Let Assumptions (5.1) to (5.3) hold with $0 < \nu < \alpha < 2$ and $p \geq 2$, for any $0 \leq t_1 < t_2 \leq T$, the unique mild solution $c(t)$ to the problem (2.6) is Hölder continuous with respect to the norm $\| \cdot \|_{L^p(\Omega; \dot{H}^\nu)}$ and satisfies*

$$(6.7) \quad \| c(t_2) - c(t_1) \|_{L^p(\Omega; \dot{H}^\nu)} \leq C(t_2 - t_1)^\gamma.$$

Proof. For any $0 \leq t_1 < t_2 \leq T$, for the mild solution (2.6), we have

$$\begin{aligned}
 c(t_2) - c(t_1) &= E_\beta(t_2)c_0 - E_\beta(t_1)c_0 \\
 &+ \int_0^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s)L(c(s))ds \\
 &- \int_0^{t_1} (t_1 - s)^{\beta-1} E_{\beta,\beta}(t_1 - s)L(c(s))ds \\
 &+ \int_0^{t_2} (t_2 - s)^{\beta-1\beta-1} E_{\beta,\beta}(t_2 - s)f(c(s))dW(s) \\
 &- \int_0^{t_1} (t_2 - s)^{\beta-1\beta-1} E_{\beta,\beta}(t_1 - s)G(u(s))dW(s) \\
 &= I_1 + I_2 + I_3,
 \end{aligned}
 \tag{6.8}$$

where

$$I_1 = E_\beta(t_2)c_0 - E_\beta(t_1)c_0,$$

and

$$\begin{aligned}
 I_2 &= \int_0^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s)L(c(s))ds \\
 &- \int_0^{t_1} (t_1 - s)^{\beta-1} E_{\beta,\beta}(t_1 - s)L(c(s))ds \\
 &= \int_0^{t_1} (t_1 - s)^{\beta-1} [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)]L(c(s))ds \\
 &+ \int_0^{t_1} [(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}] E_{\beta,\beta}(t_2 - s)L(c(s))ds \\
 &+ \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s)L(c(s))ds \\
 &= I_{21} + I_{22} + I_{23},
 \end{aligned}
 \tag{6.9}$$

and

$$\begin{aligned}
 I_3 &= \int_0^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s) f(c(s)) dW(s) \\
 &\quad - \int_0^{t_1} (t_1 - s)^{\beta-1} E_{\beta,\beta}(t_1 - s) f(c(s)) dW(s) \\
 &= \int_0^{t_1} (t_1 - s)^{\beta-1} [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] f(c(s)) dW(s) \\
 (6.10) \quad &\quad + \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] E_{\beta,\beta}(t_2 - s) g(u(s)) dW(s) \\
 &\quad + \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} E_{\beta,\beta}(t_2 - s) f(c(s)) dW(s) \\
 &= I_{31} + I_{32} + I_{33}.
 \end{aligned}$$

For any $0 < \nu < \alpha < 2$ and $p \geq 2$, by virtue of Theorem (2.2), it follows that

$$\begin{aligned}
 E[\| I_1 \|_{\dot{H}^\nu}^p] &= E[\| A_\nu [E_\beta(t_2) - E_\beta(t_1)] c_0 \|]^p \\
 (6.11) \quad &\leq C_{\alpha\nu}^p (t_2 - t_1)^{\frac{p\beta\nu}{\alpha}} E[\| c_0 \|]^p.
 \end{aligned}$$

For the first term I_{21} in (6.9), applying the Assumption (5.2) and Theorem (6.2) and Hölder’s inequality, we have

$$\begin{aligned}
 E[\| I_{21} \|_{\dot{H}^\nu}^p] &= E[\| \int_0^{t_1} (t_1 - s)^{\beta-1} A_\nu [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] L(c(s)) ds \|^p] \\
 (6.12) \quad &\leq C_{\beta\nu}^p (t_2 - t_1)^{\frac{p\beta(\nu+1)}{\alpha}} (\int_0^{t_1} (t_1 - s)^{\frac{p(\beta-1)}{p-1}} ds)^{p-1} \int_0^{t_1} E[\| A_{-1} L(c(s)) \|_{\dot{H}^1}^p] ds \\
 &\leq C^p C_{\beta\nu}^p T^p (\frac{p-1}{p\beta-1})^{p-1} (\sup_{t \in [0, T]} E[\| c(s) \|_{\dot{H}^1}^{2p}]) (t_2 - t_1)^{\frac{p\beta(\nu+1)}{\alpha}}.
 \end{aligned}$$

Using the Assumptions (5.2), Theorem (6.2) and Hölder’s inequality, we get

$$\begin{aligned}
 E[\| I_{22} \|_{\dot{H}^\nu}^p] &= E[\| \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] A_\nu E_{\beta,\beta}(t_2 - s) L(c(s)) ds \|]^p \\
 &\leq C_\beta^p (\int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] \\
 &\quad \times (t_2 - s)^{-\frac{\beta(\nu+1)}{\alpha}} \}^{\frac{p}{p-1}} ds)^{p-1} \times \int_0^{t_1} E[\| A_1 L(c(s)) \|_{\dot{H}^1}^p] ds \\
 &\leq C^p C_\beta^p T \{ \frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}] - 1} \}^{p-1} (\sup_{t \in [0, T]} E[\| c(s) \|_{\dot{H}^1}^{2p}]) (t_2 - t_1)^{\frac{p\beta(\alpha - \nu - 1) - \alpha}{\alpha}},
 \end{aligned}
 \tag{6.13}$$

and

$$\begin{aligned}
 E[\| I_{23} \|_{\dot{H}^\nu}^p] &= E[\| \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} A_\nu E_{\beta,\beta}(t_2 - s) L(c(s)) ds \|]^p \\
 &\leq C_\beta^p (\int_{t_1}^{t_2} [(t_2 - s)^{\beta-1 - \frac{\beta(\nu+1)}{\alpha}}] \}^{\frac{p}{p-1}} ds)^{p-1} \\
 &\quad \times \int_{t_1}^{t_2} E[\| A_1 L(c(s)) \|_{\dot{H}^1}^p] ds \\
 &\leq C^p C_\beta^p \{ \frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}] - 1} \}^{p-1} (\sup_{t \in [0, T]} E[\| c(s) \|_{\dot{H}^1}^{2p}]) (t_2 - t_1)^{\frac{p\beta(\alpha - \nu - 1)}{\alpha}}.
 \end{aligned}
 \tag{6.14}$$

Next, by following the similar arguments as in the proof of (6.12)- (6.14) and using the Lemma (2.2), there holds

$$\begin{aligned}
 E[\| I_{31} \|_{\dot{H}^\nu}^p] &= E[\| \int_0^{t_1} (t_1 - s)^{\beta-1} A_\nu [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] f(u(s)) dW(s) \|]^p \\
 &\leq C(p) E[\int_0^{t_1} \| (t_1 - s)^{\beta-1} A_\nu [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] \|^2 \| f(c(s)) \|_{L^2}^2 ds] \\
 &\leq C(p) C_{\beta\nu}^p (t_2 - t_1)^{\frac{p\beta\nu}{\alpha}} (\int_0^{t_1} (t_1 - s)^{\frac{2p(\beta-1)}{p-2}} ds)^{\frac{p-2}{2}} \int_0^{t_1} E \| f(c(s)) \|_{L^2}^p ds \\
 &\leq C(p) C_{\beta\nu}^p T^{\frac{2p\beta - p - 1}{2}} (\frac{p-1}{2p\beta - p - 2})^{p-1} (\sup_{t \in [0, T]} E[\| c(t) \|]^p) (t_1 - t_2)^{\frac{p\beta\nu}{\alpha}},
 \end{aligned}$$

and

$$\begin{aligned}
 E[\| I_{32} \|] &= E[\| \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] A_\nu E_{\beta\beta}(t_2 - s) f(c(s)) dW(s) \|^p] \\
 &\leq C(p) E[\int_0^{t_1} \| ((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}) A_\nu E_{\beta\beta}(t_2 - s) \|^2 \| f(c(s)) \|_{L^2_0}^2 ds] \\
 (6.15) \quad &\leq C(p) C_\beta^p (\int_0^{t_1} \{ [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] \times (t_2 - s)^{-\frac{\beta\nu}{2}} \}^{\frac{2p}{p-2}} ds)^{\frac{p-2}{2}} \\
 &\times \int_0^{t_1} E \| f(c(s)) \|_{L^2_0}^p ds \\
 &\leq C(p) C_\beta^p C^\rho T [\frac{\alpha(p-2)}{2p\beta(\alpha-\nu)-(p+2)\alpha}]^{\frac{p-2}{2}} (\sup_{t \in [0, T]} E[\| c(t) \|^p]) (t_2 - t_1)^{\frac{2p\beta(\alpha-\nu)-(p+2)\alpha}{2\alpha}},
 \end{aligned}$$

and

$$\begin{aligned}
 E[\| I_{33} \|] &= E[\| \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} A_\nu E_{\beta\beta}(t_2 - s) f(c(s)) dW(s) \|^p] \\
 &\leq C(p) E[\int_{t_1}^{t_2} (t_2 - s)^{\beta-1} A_\nu E_{\beta\beta}(t_2 - s) \|^2 \| f(c(s)) \|_{L^2_0}^2 ds]^{\frac{p}{2}} \\
 (6.16) \quad &\leq C(p) C_\beta^p (\int_{t_1}^{t_2} [(t_2 - s)^{\beta-1-\frac{\beta\nu}{\alpha}}]^{\frac{2p}{p-2}} ds)^{\frac{p-2}{2}} \int_{t_1}^{t_2} E \| f(u) \|_{L^2_0}^p ds \\
 &\leq C(p) C_\beta^p C^\rho [\frac{\alpha(p-2)}{2p\beta(\alpha-\nu)-(p+2)\alpha}]^{\frac{p-2}{2}} (\sup_{t \in [0, T]} E[\| c(t) \|^p]) (t_2 - t_1)^{\frac{2p\beta(\alpha-\nu)-p\alpha}{2\alpha}}
 \end{aligned}$$

Taking expectation on the both side of (6.8), and in view of the estimates (6.11)- (6.16), we conclude that

$$(6.17) \quad \| c(t_2) - c(t_1) \|_{L^p(\Omega; \dot{H}^\nu)} \leq C(t_2 - t_1)^\gamma,$$

in which we tak $\gamma = \min\{\frac{\beta\nu}{\alpha}, \frac{p\beta(\alpha-\nu-1)-\alpha}{p\alpha}, \frac{2p\beta(\alpha-\nu)-(p+2)\alpha}{2p\alpha}\}$, whene $0 < t_2 - t_1 < 1$.

Otherwise, if $t_2 - t_1 \geq 1$ then we set $\gamma = \max\{\frac{\beta(\nu+1)}{\alpha}, \frac{\beta(\alpha-\nu-1)}{\alpha}, \frac{2p\beta(\alpha-\nu)-p\alpha}{2p\alpha}\}$. □

This completes the proof of Theorem (6.2)

7. APPENDIX A

Considering the following abstract formulation of time-space fractional stochastic of equation (2.1)

$$(7.1) \quad \begin{cases} {}^c D_t^\beta u(t) &= -A_\alpha u(t) + B(u(t), c(t)) + g(u(t)) \frac{dW(t)}{dt}, \quad t > 0, \\ u(0) &= u_0. \end{cases}$$

We derive the mild solution to (7.1) by means of Laplace transform, which denoted by $\hat{\cdot}$. Let $\lambda > 0$, and we define that

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda s} u(s) ds, \quad \hat{B}(\lambda) = \int_0^\infty e^{-\lambda s} B(u(s), c(s)) ds,$$

and

$$\hat{G}(\lambda) = \int_0^\infty e^{-\lambda s} [g(u(s)) \frac{dW(s)}{ds}] ds = \int_0^\infty e^{-\lambda s} g(u(s)) dW(s).$$

Upon Laplace transform, using the formula ${}^c \hat{D}_t^\beta u(\lambda) = \lambda^\beta \hat{u} - \lambda^{\beta-1} u_0$. Then applying the Laplace transform to (7.1), we obtain

$$(7.2) \quad \begin{aligned} \hat{u}(\lambda) &= \frac{1}{\lambda} u_0 + \frac{1}{\lambda^\beta} (-A_\alpha) \hat{u}(\lambda) + \frac{1}{\lambda^\beta} [\hat{B}(\lambda) + \hat{G}(\lambda)] \\ &= \lambda^{\beta-1} (\lambda^\beta I + A_\alpha)^{-1} u_0 + (\lambda^\beta I + A_\alpha)^{-1} [\hat{B}(\lambda) + \hat{G}(\lambda)] \\ &= \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) u_0 ds + \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) [\hat{B}(\lambda) + \hat{G}(\lambda)] ds, \end{aligned}$$

in which I is the identity operator, and $S_\alpha(t) = e^{-tA_\alpha}$ is an analytic semigroup generated by the operator $-A_\alpha$. We introduce the following one-sided stable probability density function:

$$(7.3) \quad W_\beta = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{\beta n - 1} \frac{\Gamma(\beta n + 1)}{n!} \sin(n\pi\beta), \quad \theta \in (0, \infty),$$

whose Laplace transform is given by

$$(7.4) \quad \int_0^\infty e^{-\lambda\theta} W_\beta(\theta) d\theta = e^{-\lambda^\beta}, \quad 0 < \beta < 1.$$

Making use of above expression (7.4), then the terms on the right-hand side of (7.2) can be written as

$$\begin{aligned}
 \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) u_0 ds &= \int_0^\infty \lambda^{\beta-1} e^{-\lambda^\beta t^\beta} S_\alpha(t^\beta) u_0 dt \\
 &= \int_0^\infty \beta (\lambda t)^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) u_0 dt \\
 (7.5) \qquad &= \int_0^\infty \frac{1}{\lambda} \frac{d}{dt} [e^{-(\lambda t)^\beta}] S_\alpha(t^\beta) u_0 dt \\
 &= \int_0^\infty \int_0^\infty \theta W_\beta(\theta) e^{-\lambda t \theta} S_\alpha(t^\beta) u_0 d\theta dt \\
 &= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) u_0 d\theta \right] dt,
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) \hat{B}(\lambda) ds \\
 &= \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) \hat{B}(\lambda) dt \\
 &= \int_0^\infty \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) e^{-\lambda s} t^{\beta-1} B(u(s), c(s)) ds dt \\
 (7.6) \qquad &= \int_0^\infty \int_0^\infty \int_0^\infty \beta W_\beta(\theta) e^{-\lambda t \theta} S_\alpha(t^\beta) e^{-\lambda s} t^{\beta-1} B(u(s), c(s)) d\theta ds dt \\
 &= \int_0^\infty \int_0^\infty \int_0^\infty \beta W_\beta(\theta) e^{-\lambda(t+s)} S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) \frac{t^{\beta-1}}{\theta^\beta} B(u(s), c(s)) d\theta ds dt \\
 &= \int_0^\infty e^{-\lambda t} \left[\beta \int_0^t \int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} B(u(s), c(s)) d\theta ds \right] dt,
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) \hat{G}(\lambda) ds \\
 &= \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) \hat{G}(\lambda) dt \\
 &= \int_0^\infty \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) e^{-\lambda s} g(u(s)) dW(s) dt
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\infty \int_0^\infty \int_0^\infty \beta W_\beta(\theta) e^{-\lambda t \theta} S_\alpha(t^\beta) e^{-\lambda s t^{\beta-1}} g(u(s)) d\theta dW(s) dt \\
 (7.7) \quad &= \int_0^\infty \int_0^\infty \int_0^\infty \beta W_\beta(\theta) e^{-\lambda(t+s)} S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) \frac{t^{\beta-1}}{\theta^\beta} g(u(s)) d\theta dW(s) dt \\
 &= \int_0^\infty e^{-\lambda t} \left[\beta \int_0^t \int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} g(u(s)) d\theta dW(s) \right] dt.
 \end{aligned}$$

Together with (7.2) and (7.5)- (7.7) helps us to get

$$\begin{aligned}
 (7.8) \quad \hat{u}(\lambda) &= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) u_0 d\theta \right] dt \\
 &+ \int_0^\infty e^{-\lambda t} \left[\beta \int_0^t \int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} B(u(s), c(s)) d\theta ds \right] dt \\
 &+ \int_0^\infty e^{-\lambda t} \left[\beta \int_0^t \int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} g(u(s)) d\theta dW(s) \right] dt,
 \end{aligned}$$

Now, by means of inverse Laplace transform to (7.8), we have achieved that

$$\begin{aligned}
 (7.9) \quad u(t) &= \int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) u_0 d\theta \\
 &+ \beta \int_0^t \int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} B(u(s), c(s)) d\theta ds \\
 &+ \beta \int_0^t \int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} g(u(s)) d\theta dW(s) \\
 &= \int_0^\infty \frac{1}{\beta} \theta^{-\frac{1}{\beta}-1} W_\beta(\theta^{-\frac{1}{\beta}}) S_\alpha(t^\beta \theta) u_0 d\theta \\
 &+ \int_0^t \int_0^\infty \theta^{-\frac{1}{\beta}} W_\beta(\theta^{-\frac{1}{\beta}}) S_\alpha((t-s)^\beta \theta) (t-s)^{\beta-1} B(u(s), c(s)) d\theta ds \\
 &\quad \int_0^t \int_0^\infty \theta^{-\frac{1}{\beta}} W_\beta(\theta^{-\frac{1}{\beta}}) S_\alpha((t-s)^\beta \theta) (t-s)^{\beta-1} g(u(s)) d\theta dW(s).
 \end{aligned}$$

Here, we also introduce the Mainardi’s Wright-type function

$$\begin{aligned}
 M_\beta(\theta) &= \sum_{n=0}^\infty \frac{(-1)^n \theta^n}{n\theta! \Gamma(1 - \beta(1+n))} \\
 &= \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-1)^{n-1} \theta^{n-1}}{(n-1)!} \Gamma(n\beta) \sin(n\pi\beta),
 \end{aligned}$$

where $0 < \beta < 1$ and $\theta \in (0, \infty)$. Further, the relationships between the probability density function $W_\beta(\theta)$ and Mainardi's Wright-type function $M_\beta(\theta)$ are shown that

$$M_\beta(\theta) = \frac{1}{\beta} \theta^{-\frac{1}{\beta}-1} W_\beta(\theta^{-\frac{1}{\beta}}).$$

We denote the generalized Mittag-Leffler operators $E_\alpha(t)$ and $E_{\beta\beta}(t)$ as

$$E_\alpha(t) = \int_0^\infty M_\beta(\theta) S_\alpha(t^\beta \theta) d\theta,$$

and

$$E_{\beta\beta}(t) = \int_0^\infty \beta \theta M_\beta(\theta) S_\alpha(t^\beta \theta) d\theta.$$

Therefore, the equation (7.9) can be written as

$$\begin{aligned} (7.10) \quad u(t) &= E_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) B(u(s), c(s)) ds \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) g(u(s)) dW(s), \end{aligned}$$

Up to now, we have deduced the mild solution (7.10) to the time-space fractional stochastic equation (2.1).

8. APPENDIX B

Considering the following abstract formulation of time-space fractional stochastic of equation (2.6)

$$(8.1) \quad \begin{cases} {}^c D_t^\beta c(t) &= -A_\alpha c(t) + L(c(t)) + f(c(t)) \frac{dW(t)}{dt}, \quad t > 0, \\ c(0) &= c_0, \end{cases}$$

We derive the mild solution to (8.1) by means of Laplace transform, which denoted by $\hat{\cdot}$. $\lambda > 0$, and we define that

$$\hat{c}(\lambda) = \int_0^\infty e^{-\lambda s} c(s) ds, \quad \hat{L}(\lambda) = \int_0^\infty e^{-\lambda s} L(c(s)) ds,$$

and

$$\hat{H}(\lambda) = \int_0^\infty e^{-\lambda s} [f(c(s)) \frac{dW(s)}{ds}] ds = \int_0^\infty e^{-\lambda s} f(c(s)) dW(s).$$

Upon Laplace transform, using the formula ${}^c \hat{D}_t^\beta c(\lambda) = \lambda^\beta \hat{c} - \lambda^{\beta-1} c_0$. Then applying the Laplace transform to (8.1), we obtain

$$\begin{aligned} (8.2) \quad \hat{c}(\lambda) &= \frac{1}{\lambda} c_0 + \frac{1}{\lambda^\beta} (-A_\alpha) \hat{c}(\lambda) + \frac{1}{\lambda^\beta} [\hat{L}(\lambda) + \hat{H}(\lambda)] \\ &= \lambda^{\beta-1} (\lambda^\beta I + A_\alpha)^{-1} c_0 + (\lambda^\beta I + A_\alpha)^{-1} [\hat{L}(\lambda) + \hat{H}(\lambda)] \\ &= \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) c_0 ds + \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) [\hat{L}(\lambda) + \hat{H}(\lambda)] ds \end{aligned}$$

in which I is the identity operator, and $S_\alpha(t) = e^{-tA_\alpha}$ is an analytic semigroup generated by the operator $-A_\alpha$. We introduce the following one-sided stable probability density function:

$$(8.3) \quad W_\beta = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{\beta n-1} \frac{\Gamma(\beta n + 1)}{n!} \sin(n\pi\beta), \quad \theta \in (0, \infty),$$

whose Laplace transform is given by

$$(8.4) \quad \int_0^\infty e^{-\lambda\theta} W_\beta(\theta) d\theta = e^{-\lambda^\beta}, \quad 0 < \beta < 1.$$

Making use of above expression (8.4), then the terms on the right-hand side of (8.2) can be written as

$$(8.5) \quad \begin{aligned} \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) c_0 ds &= \int_0^\infty \lambda^{\beta-1} e^{-\lambda^\beta t^\beta} S_\alpha(t^\beta) c_0 dt \\ &= \int_0^\infty \beta(\lambda t)^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) c_0 dt \\ &= \int_0^\infty \frac{1}{\lambda} \frac{d}{dt} [e^{-(\lambda t)^\beta}] S_\alpha(t^\beta) c_0 dt \\ &= \int_0^\infty \int_0^\infty \theta W_\beta(\theta) e^{-\lambda t^\beta} S_\alpha(t^\beta) c_0 d\theta dt \\ &= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) c_0 d\theta \right] dt, \end{aligned}$$

and

$$(8.6) \quad \begin{aligned} &\int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) \hat{L}(\lambda) ds \\ &= \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) \hat{L}(\lambda) dt \\ &= \int_0^\infty \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) e^{-\lambda s} t^{\beta-1} L(c(s)) ds dt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \beta W_\beta(\theta) e^{-\lambda t^\beta} S_\alpha(t^\beta) e^{-\lambda s} t^{\beta-1} L(c(s)) d\theta ds dt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \beta W_\beta(\theta) e^{-\lambda(t+s)} S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) \frac{t^{\beta-1}}{\theta^\beta} L(c(s)) d\theta ds dt \\ &= \int_0^\infty e^{-\lambda t} \left[\beta \int_0^t \int_0^\infty W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} L(c(s)) d\theta ds \right] dt, \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^{\infty} e^{-\lambda^\beta s} S_\alpha(s) \hat{H}(\lambda) ds \\
 &= \int_0^{\infty} \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) \hat{H}(\lambda) dt \\
 &= \int_0^{\infty} \int_0^{\infty} \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) e^{-\lambda s} f(c(s)) dW(s) dt \\
 (8.7) \quad &= \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \beta W_\beta(\theta) e^{-\lambda t \theta} S_\alpha(t^\beta) e^{-\lambda s t^{\beta-1}} f(c(s)) d\theta dW(s) dt \\
 &= \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \beta W_\beta(\theta) e^{-\lambda(t+s)} S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) \frac{t^{\beta-1}}{\theta^\beta} f(c(s)) d\theta dW(s) dt \\
 &= \int_0^{\infty} e^{-\lambda t} \left[\beta \int_0^t \int_0^{\infty} W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} f(c(s)) d\theta dW(s) \right] dt.
 \end{aligned}$$

Together with (8.2) and (8.5)- (8.7) helps us to get

$$\begin{aligned}
 \hat{c}(\lambda) &= \int_0^{\infty} e^{-\lambda t} \left[\int_0^{\infty} W_\beta(\theta) S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) c_0 d\theta \right] dt \\
 (8.8) \quad &+ \int_0^{\infty} e^{-\lambda t} \left[\beta \int_0^t \int_0^{\infty} W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} L(c(s)) d\theta ds \right] dt \\
 &+ \int_0^{\infty} e^{-\lambda t} \left[\beta \int_0^t \int_0^{\infty} W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} f(c(s)) d\theta dW(s) \right] dt.
 \end{aligned}$$

Now, by means of inverse Laplace transform to (8.8), we have achieved that

$$\begin{aligned}
 c(t) &= \int_0^{\infty} W_\beta(\theta) S_\alpha\left(\frac{t^\beta}{\theta^\beta}\right) c_0 d\theta \\
 &+ \beta \int_0^t \int_0^{\infty} W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} L(c(s)) d\theta ds \\
 &+ \beta \int_0^t \int_0^{\infty} W_\beta(\theta) S_\alpha\left(\frac{(t-s)^\beta}{\theta^\beta}\right) \frac{(t-s)^{\beta-1}}{\theta^\beta} f(c(s)) d\theta dW(s)
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\infty \frac{1}{\beta} \theta^{-\frac{1}{\beta}-1} W_\beta(\theta^{-\frac{1}{\beta}}) S_\alpha(t^\beta \theta) c_0 d\theta \\
 (8.9) \quad &+ \int_0^t \int_0^\infty \theta^{-\frac{1}{\beta}} W_\beta(\theta^{-\frac{1}{\beta}}) S_\alpha((t-s)^\beta \theta) (t-s)^{\beta-1} L(c(s)) d\theta ds \\
 &\int_0^t \int_0^\infty \theta^{-\frac{1}{\beta}} W_\beta(\theta^{-\frac{1}{\beta}}) S_\alpha((t-s)^\beta \theta) (t-s)^{\beta-1} f(c(s)) d\theta dW(s).
 \end{aligned}$$

Here, we also introduce the Mainardi's Wright-type function

$$\begin{aligned}
 M_\beta(\theta) &= \sum_{n=0}^\infty \frac{(-1)^n \theta^n}{n\theta! \Gamma(1-\beta(1+n))} \\
 &= \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-1)^{n-1} \theta^{n-1}}{(n-1)!} \Gamma(n\beta) \sin(n\pi\beta),
 \end{aligned}$$

where $0 < \beta < 1$ and $\theta \in (0, \infty)$. Further, the relationships between the probability density function $W_\beta(\theta)$ and Mainardi's Wright-type function $M_\beta(\theta)$ are shown that

$$M_\beta(\theta) = \frac{1}{\beta} \theta^{-\frac{1}{\beta}-1} W_\beta(\theta^{-\frac{1}{\beta}}).$$

We denote the generalized Mittag-Leffler operators $E_\alpha(t)$ and $E_{\beta\beta}(t)$ as

$$E_\alpha(t) = \int_0^\infty M_\beta(\theta) S_\alpha(t^\beta \theta) d\theta,$$

and

$$E_{\beta\beta}(t) = \int_0^\infty \beta \theta M_\beta(\theta) S_\alpha(t^\beta \theta) d\theta.$$

Therefore, the equation (7.9) can be written as

$$\begin{aligned}
 (8.10) \quad c(t) &= E_\beta(t) c_0 + \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) L(c(s)) ds \\
 &+ \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) f(c(s)) dW(s).
 \end{aligned}$$

Up to now, we have deduced the mild solution (8.10) to the time-space fractional stochastic equation (2.6).

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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