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SOME REMARKS ON *G*-METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. We notice some remarks on G-metric spaces and the fixed point results of contractive mappings defined on such spaces. Our results generalize, extend and complement recent fixed point theorems established by Jleli and Samet [M. Jleli and B. Samet, Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory Appl. 2012:210 (2012)].

1. Introduction

Mustafa and Sims [8, 12] introduced G-metric spaces, as a generalization of a metric space. Many fixed point results on such spaces can be found in [2], [3], [6], [10], [13], [14], [15], [16], [18], [19], [20]. There is a close relation between a usual metric space and a G-metric space ([9-14]). Jleli and Samet [5] observed that some fixed point theorems in G-metric space can be concluded by some existing results in quasi-metric and metric space (see [7]). Actually, the authors concluded that $d(x, y) = G(x, y, y)$ forms a quasi-metric. The aim of this paper is to continue the study of the G-metric and quasi metric space. The following definitions and results will be needed in the sequel.

Definition 1.1. [12] Let X be a nonempty set. Suppose that $G : X \times X \times X \to [0,+\infty)$ is a function satisfying the following conditions:

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(1) $G(x, y, z) = 0$ if and only if $x = y = z$;
(2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(4) $G(x, y, z) = G(x, z, y) = G(y, z, x)=...$(symmetry in all three variables);
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G-metric on X and (X,G) is called a G-metric space.

Definition 1.2. [5] A G-metric space (X, G) is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Note that, if (X,G) is symmetric than many fixed point theorems on G-metric spaces are particular cases of existing fixed point theorems in metric spaces (see [5]). Here, we discuss the non-symmetric case.

Definition 1.3. [5] Let X be a nonempty set and $d : X \times X \to [0,\infty)$ be a given function which satisfies
1) $d(x,y)=0$ if and only if $x=y$;
2) $d(x,y) \leq d(x,z) + d(z,y)$ for any points $x,y,z \in X$.

Then d is called a quasi-metric and the pair (X, d) is called a quasi metric space.

Examples 1.4. Any metric space is a quasi-metric space, but the converse is not true.

(i) Let $(X, d) = ([0,+\infty], d)$, where d is given by

$$d(x,y) = \begin{cases} y - x & \text{if } x \leq y, \\ 2(x - y) & \text{if } x>y. \end{cases}.$$

Then (X,d) is a quasi-metric space, which is not a metric space.

(ii) Let $(X, d) = (\mathbf{R}, d)$, where d is given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ |y| & \text{if } x \neq y. \end{cases}.$$

Then (X,d) is a quasi-metric space, which is not a metric space.

(iii) Let $(X, d) = ([0,1], d)$, where d is given by

$$d(x,y) = \begin{cases} |x-y| & \text{if } x, y \in X \text{ and } y \neq 0 \text{ or } x = y = 0, \\ 1 & \text{if } y = 0 \text{ and } 0 < x \leq 1. \end{cases}$$

Then (X,d) is a quasi-metric space, which is not a metric space.

Definition 1.5. [5] Let (X, d) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The
sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

Definition 1.6. [5] Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that
$\{x_n\}$ is left-Cauchy (resp. right-Cauchy) if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m > N$ (resp. $m \geq n > N$).

Definition 1.7. [5] Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that
$\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

Note that a sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 1.8. [5] Let (X, d) be a quasi-metric space.We say that

(1) (X, d) is left-complete if and only if each left-Cauchy sequence in X is convergent;
(2) (X, d) is right-complete if and only if each right-Cauchy sequence in X is convergent;

(3) (X, d) is complete if and only if each Cauchy sequence in X is convergent.

Theorem 1.9. [5] Let (X, G) be a G-metric space. Let $d : X \times X \to [0,\infty)$ be the function defined by $d(x, y) = G(x, y, y)$. Then

(1) (X, d) is a quasi-metric space;
(2) $\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d);
(3) $\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, d);
(4) (X,G) is G-complete if and only if (X, d) is complete.

Lemma 1.10. [4, 17] Let (X, d) be a quasi metric space and let $\{y_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(y_{n+1}, y_n) = 0.$$

If $\{y_n\}$ is not a left-Cauchy sequence (resp. right-Cauchy) in (X, d), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ (resp. $m(k) > n(k) > k$) and the following four sequences tend to $\varepsilon+$ when $k \to\infty$:

$$d(y_{m(k)}, y_{n(k)}),\ d(y_{m(k)}, y_{n(k)+1}),\ d(y_{m(k)-1}, y_{n(k)}),\ d(y_{m(k)-1}, y_{n(k)+1}).$$
$$(\text{resp. } d(y_{n(k)}, y_{m(k)}),\ d(y_{n(k)+1}, y_{m(k)}),\ d(y_{n(k)}, y_{m(k)-1}),\ d(y_{n(k)+1}, y_{m(k)-1})).$$

Proposition 1.11. [1] Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Theorem 1.12. [5] Let (X,G) be a G-metric space. Let $d : X \times X \to [0,\infty)$ be the function defined by $d(x, y) = G(x, y, y)$. Then

(1) (X, d) is a quasi-metric space;

(2) $\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d);
(3)$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, d);
(4) (X,G) is G-complete if and only if (X, d) is complete.

Remark 1.13. [5] Every quasi-metric induces a metric. If (X, d) is a quasi-metric space, then the function $\delta : X \times X \to [0,\infty)$ defined by $\delta(x, y) = \max\{d(x, y), d(y, x)\}$is a metric on X.

2. Main results

Definition 2.1. The two classes of following mappings are defined as

$\Psi = \{\psi \mid \psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and $\psi^{-1}(\{0\}) =\{0\}\}$,
$\Phi = \{\phi \mid \phi : [0,\infty) \to [0,\infty)$ is continuous and $\phi^{-1}(\{0\}) =\{0\}\}$.

Our first main result is inspired from Theorem 3.2. [5] and is more general than it.

Theorem 2.2. Let (X, d) be a complete quasi-metric space and $f, g : X \to X$ be two mappings
satisfying

$$\psi (d(fx, fy)) \leq \psi(d(gx, gy)) - \phi(d(gx, gy)) \tag{2.1}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$. If the range of g contains the range of f and $f(X)$ or $g(X)$ is a closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let us prove first that the point of coincidence of f and g is unique (if it exists). Suppose that w_1 and w_2 are two distinct point of coincidence of f and g. From this follows that there exist two points u_1 and u_2 such that $fu_1 = gu_1 = w_1$ and $fu_2 = gu_2 = w_2$.Then (2.1) implies that

$$\psi\,(d\,(w_1,w_2)) = \psi(d\,(fu_1, fu_2)) \le \psi((d\,(gu_1, gu_2))) - \phi(d(gu_1, gu_2))$$
$$= \psi((d(w_1,w_2))) - \phi\,(d(w_1,w_2)) < \psi((d(w_1,w_2)))\,,$$

a contradiction.

Now, let x_0 be an arbitrary point in X. Choose a point $x_1 \in X$ such that $fx_0 = gx_1$. This can be done, since the range of g contains the range of f. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $fx_n = gx_{n+1}$. Consider the two possible cases.

Suppose that $gx_n = gx_{n+1}$ for some $n \in \mathbf{N}$. Hence, $gx_n = fx_n$ is a point of coincidence and then the proof is finished. Thus, suppose that $gx_n \ne gx_{n+1}$ for any $n \ge 0$. In this case, we have

$$\psi\,(d(gx_{n+1}, gx_n)) = \psi\,(d(fx_n, fx_{n-1})) \le \psi\,(d(gx_n, gx_{n-1})) - \phi(d\,(gx_n, gx_{n-1})) < \psi\,(d\,(gx_n, gx_{n-1})) \qquad (2.2)$$

Now, according to the properties of function ψ it follows that the sequence $\{d(gx_{n+1}, gx_n)\}$ is a decreasing sequence of positive numbers. Therefore, $d(gx_{n+1}, gx_n) \to r \ge 0$ when $n \to \infty$. Passing to the limit in (2.2) when $n \to \infty$, we obtain that $\psi\,(r) \le \psi\,(r) - \phi(r)$ and $r = 0$ by the properties of functions $\psi \in \Psi$, $\phi \in \Phi$. Thus, we have

$$\lim_{n\to\infty} d(gx_{n+1}, gx_n) = 0.$$

Using the same technique, we also have

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0.$$

We next prove that $\{gx_n\}$ is a Cauchy sequence in the quasi-metric space (X, d), that is, $\{gx_n\}$ is a left-Cauchy and right-Cauchy. It is sufficient to show that $\{gx_n\}$ is a left-Cauchy (resp. right-Cauchy) sequence. Suppose that is not a case. Then using Lemma 1.8. we get that there exists $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers and sequences

$$d(y_{m(k)}, y_{n(k)}),\ d(y_{m(k)}, y_{n(k)+1}),\ d(y_{m(k)-1}, y_{n(k)}),\ d(y_{m(k)-1}, y_{n(k)+1})$$

all tend to $\varepsilon+$ when $k \to \infty$. Applying condition (2.1) to elements $x=x_{m(k)}$ and $y=x_{n(k)+1}$ and putting $y_n=fx_n=gx_{n+1}$ for each $n\geq 0$, we get that

$$\psi(d(fx_{m(k)}, fx_{n(k)+1})) = \psi(d(y_{m(k)}, y_{n(k)+1})) \leq \psi(d(gx_{m(k)}, gx_{n(k)+1})) - \phi(d(gx_{m(k)}, gx_{n(k)+1})) \tag{2.3}$$

$$= \psi(d(y_{m(k)-1}, y_{n(k)})) - \phi(d(y_{m(k)-1}, y_{n(k)}))$$

Letting $k \to \infty$ in (2.3) we obtain $\psi(\varepsilon)\leq\psi(\varepsilon)- \phi(\varepsilon)$ which is a contradiction if $\varepsilon>0$.

This shows that $\{gx_n\}$ is a left-Cauchy sequence in the quasi-metric space (X, d). Similarly, we can show that $\{gx_n\}$ is a right-Cauchy sequence in the quasi-metric space (X, d).

Since $g(X)$ is closed in (X, d), there exists $u\in X$ such that

$$\lim_{n\to\infty} d(gx_n, gu) = \lim_{n\to\infty} d(gu, gx_n) = 0. \tag{2.4}$$

Now, we have

$$\psi(d(fx_n, fu)) \leq \psi(d(gx_n, gu)) - \phi(d(gx_n, gu))$$

or

$$\psi(d(gx_{n+1}, fu)) \leq \psi(d(gx_n, gu)) - \phi(d(gx_n, gu)). \tag{2.5}$$

Letting $n\to\infty$ in (2.5) we have $\lim_{n\to\infty} d(gx_n, fu)=0$. Similarly, we have $\lim_{n\to\infty} d(fu, gx_n) = 0$. It follows from (2.4) and (2.5) that $fu= gu$. Hence, f and g have a (unique) point of coincidence. By the Proposition 1.9. f and g have the unique common fixed point. □

Corollary 2.3. Let (X, d) be a complete quasi-metric space and $f: X \to X$ be a mapping satisfying

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{2.6}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$. Then f has a unique common fixed point.

Proof: Follows from the Theorem 2.2. by taking $g=i_X$ (the identity map).
□

Corollary 2.4. Let (X, d) be a complete quasi-metric space and $f, g : X \to X$ be two mappings
satisfying

$$d(fx, fy)) \leq kd(gx, gy)) \tag{2.7}$$

for all $x, y \in X$ and for $k \in [0,1)$. If the range of g contains the range of f and $f(X)$ or $g(X)$ is a closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof: Follows by taking $\psi(t)= t$ and $\phi(t)=(1-k)t$ in Theorem 2.2. This is an extension of a well-known Jungck theorem on a quasi-metric space.
□

Corollary 2.5. Let (X, d) be a complete quasi-metric space and $f: X \to X$ be a mapping satisfying

$$d(fx, fy)) \leq kd(x, y)) \tag{2.8}$$

for all $x, y \in X$ and for $k \in [0,1)$. Then f has a unique common fixed point.

Proof: Follows from the Corollary 2.4. by taking $g=i_X$ (the identity map). This is an extension of a Banach contraction principle on a quasi-metric space. □

The next theorem is inspired from Theorem 3.3. [5] and is more general than it.

Theorem 2.6. Let (X, G) be a G-complete metric space and $f, g : X \to X$ be two mappings satisfying

$$\psi(G(fx, fy, fy)) \leq \psi(G(gx, gy, gy)) - \phi(G(gx, gy, gy)) \qquad (2.9)$$

for all $x, y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$. If the range of g contains the range of f and $f(X)$ or $g(X)$ is a closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof: Let us consider the quasi-metric $d(x, y) = G(x, y, y)$ for all $x, y \in X$. From (2.9) follows

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \phi(d(gx, gy)),$$

for all $x, y \in X$. Result follows from the Theorem 2.2.
□

Corollary 2.7. Let (X, G) be a G-complete metric space and $f, g : X \to X$ be two mappings satisfying

$$\psi(G(fx, fy, fy)) \leq \psi(G(x, y, y)) - \phi(G(x, y, y)) \qquad (2.10)$$

for all $x, y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$. If $f(X)$ is a closed subset of X, then f has a unique common fixed point.

Proof: Follows from the Theorem 2.5. by taking $g=i_X$ (the identity map).
□

From Corollary 2.6. follows Theorem 3.3. from [5] when we consider ψ as an identical map.

Theorem 2.8. Let (X, G) be a G-complete metric space and f: $X \to X$ be a mapping satisfying

$$G(fx, fy, fz) \leq kG(x, y, z) \tag{2.11}$$

for all $x, y \in X$, where $k \in [0,1)$. Then f has a unique fixed point.

Proof: If we take $y=x$ than from (2.11) follows

$$G(fx, fx, fz) \leq kG(x, x, z) \tag{2.12}$$

If we take $y=z$ than from (2.11) follows

$$G(fx, fz, fz) \leq kG(x, z, z). \tag{2.13}$$

From (2.12) and (2.13) we have

$$G(fx, fx, fz) + G(fx, fz, fz) \leq kG(x, x, z) + kG(x, z, z).$$

Let us consider the metric $d_G(x, y) = G(x, y, y) + G(x, x, y)$ for all $x, y \in X$. From (2.12) follows

$$d_G(fx, fz) \leq kd_G(x, z)$$

for all $x, y \in X$. Now, f has a unique fixed point by Banach theorem.
□

Most theorems from the Section 3 in [2] can be proved by using previous method. In that section, proofs are long and not as simple as the proof of the Theorem 2.8. For example, we prove the next theorem.

The next theorem is inspired from Theorem 3.1. [2] and is more general than it.

Theorem 2.9. Let (X,G) be a G-metric space. Let $T : X \to X$ and $g : X \to X$ be two mappings
such that

$$G(Tx, Ty, Tz) \leq kG(gx, gy, gz) \tag{2.14}$$

for all x, y, z. Assume that T and g satisfy the following conditions:

(A1) $T(X) \subset g(X)$,
(A2) $g(X)$ is G-complete,
(A3) T and g are weakly compatible

If $k \in [0,1)$, then there is a unique $x \in X$ such that $gx = Tx = x$.

Proof: Like in the Theorem 2.8., let us consider the metric $d_G(x,y) = G(x,y,y) + G(x,x,y)$ for all $x,y \in X$. From (2.14) follows

$$d_G(Tx,Tz) \leq kd_G(gx,gz)$$

for all $x,y \in X$. Now, f has a unique fixed point by Banach contraction principle.

The next theorem is inspired from Theorem 3.7. [2]. If we assume that function ϕ is a superadditive function than the proof is the same as in Theorem 2.8.

Theorem 2.10. Let $(X, \preceq)$ be an ordered set endowed with a G-metric and T: $X \to X$ and g: $X \to X$ be given mappings. Suppose that the following conditions hold:

(i) (X,G) is G-complete;
(ii) T is G-continuous;
(iii) T is g-nondecreasing;
(iv) there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$;

(v) $T(X) \subset g(X)$ and g is G-continuous and commutes with T;
(vi) there exists a superadditive function $\phi\in\Phi$ (where Φ is defined in the Definition 2.1.) such that for all $x, y, z \in X$ with $gx \succeq gy \succeq gz$,

$$G(Tx, Ty, Tz) \leq \phi(G(gx, gy, gz)). \tag{2.15}$$

Then T and g have a coincidence point, that is, there exists $w\in X$ such that $gw= Tw$.

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