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ON ω-INTERPOLATIVE BERINDE WEAK CONTRACTION IN QUASI-PARTIAL B-METRIC SPACE

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Abstract. The aim of this paper is to introduce interpolative weak contraction in the notion of Berinde weak operator in quasi-partial b-metric space and to extend and generalize fixed point results by adopting the condition of ω-admissibility. We also discussed convex contraction mapping and obtained a fixed point result in the setting of Berinde weak operator in quasi-partial b-metric space. Consequently, we present some examples to show the applicability of the concept.

1. Introduction and Preliminaries

In 1922, Banach [1] introduced the highly recognized Banach's contraction principle in the field of non-linear analysis. This result is used to prove the uniqueness of fixed point theorems as well as in Picard theorems. The Banach's contraction in metric space is stated as follow:

Theorem 1.1. *[1] Let us consider (M, d) to be a complete metric space and $T\colon M \to M$ is the given self mapping. Let $\zeta \in (0,1)$ such that*

$$d(T\tau, T\upsilon) \le \zeta d(\tau, \upsilon)$$

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for all $\tau, \upsilon \in M$. Then T has a unique fixed point in M.

In 1992, Matthews [2] brought up the concept of partial metric space. Motivated by which in 1993, Czerwik [3] introduced a more generalized form of Banach fixed point theorem in b-metric space. Many authors including Miculescu et al. [4], Oltra et al [5], and Valero [6] introduced some fixed point results and its topological properties. Later, Karapinar [7] and Shukla [8] introduced quasi-partial metric-space and partial b-metric space respectively. In 1968, Kannan [9] removed the continuity condition from the Banach contraction principle.

Theorem 1.2. *[9] Suppose (M, d) is a complete metric space and $T: M \to M$ is called the Kannan contraction mapping. Let $\zeta \in \left[0, \frac{1}{2}\right)$ such that*

$$d(T\tau, T\upsilon) \leq \zeta[d(\tau, T\tau) + d(\upsilon, T\upsilon)]$$

for all $\tau, \upsilon \in M$. Then T has a unique fixed point in M.

In the year 2004, Berinde [10] brought up the concept of Berinde contraction also known as almost contractions and stated that:

Theorem 1.3. *[10] Let (M, d) is a complete metric space and $T: M \to M$ is almost contraction if there exists $\zeta \in [0, 1)$ and some $R \geq 0$ such that*

$$d(T\tau, T\upsilon) \leq \zeta d(\tau, \upsilon) + Rd(\upsilon, T\tau)$$

for all $\tau, \upsilon \in M$. Hence, T has a unique fixed point in M and is called a weak contraction.

Additionally, Berinde obtained some fixed point theorems that serves as the most important results in literature. Some results are the generalization of Banach, Kannan, Chatterjea, and C̀iric̀ etc. Others can be found in [11–13]. Recently, Turkoglu [14] formulated a fixed point theorem consisting of four mappings using the concept of Berinde contraction in partial metric spaces.

In 2018, the idea of interpolative Kannan-type contraction was introduced by Karapinar [15] which is proved to be a very notable outcome for solving mathematical analysis.

Theorem 1.4. *[15] Suppose a self mapping $T: M \to M$ in complete metric space (M, d) is an interpolative Kannan-type contraction, i.e. there exist $\zeta \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(T\tau, T\upsilon) \leq \zeta[d(\tau, T\upsilon)]^{\alpha}[d(\upsilon, T\upsilon)]^{1-\alpha}$$

for all $\tau, \upsilon \in M$ with $\tau \neq T\tau$. Then T has a fixed point in M.

Taking this approach forward Ampadu [16] introduced the concept of interpolative Berinde weak contraction in metric space and expanded its approach in cone metric and partial metric space as well.

Definition 1.1. *[16] Let us consider (M, d) is a metric space. Let $T : M \to M$ is an interpolative Berinde weak operator if it satisfies,*

$$d(T\tau, T\upsilon) \leq \zeta[d(\tau, \upsilon)]^{\alpha}.[d(\tau, T\tau)]^{1-\alpha} \tag{1.1}$$

for all $\tau, \upsilon \in M$ and $\tau, \upsilon \notin Fix(T)$, where, $\zeta \in [0, 1)$ and $\alpha \in (0, 1)$.

Remark 1.1. *The $Fix(T)$ denotes the set of all fixed points of T.*

The idea of ω-orbital admissible maps was introduced by Popescu [17] as a refining of α-admissible maps of Samet et al [18]. Further, Karapinar proposed that

Definition 1.2. *[19] Let $\omega : M \times M \to [0, \infty)$ and M is non-empty subset and $T : M \times M$ be a self mapping is called ω-orbital admissible if for all $\upsilon \in M$, we have*

$$\omega(\upsilon, T\upsilon) \geq 1 \text{ implies } \omega(T\upsilon, T\upsilon^2) \geq 1. \tag{1.2}$$

Recently in 2015, Gupta and Gautam [20, 21] brought up the concept of quasi-partial b-metric space and its related topological properties. Several authors [22–29] have made valuable contributions in this research field.

Definition 1.3. *[20] A quasi-partial b-metric on a non-empty set M is a function $qp_b : M \times M \to R^+$ such that for some real number $s \geq 1$ and for all $\tau, \upsilon, \chi \in M$ following condition hold:*

(QPb_1) $qp_b(\tau, \tau) = qp_b(\tau, \upsilon) = qp_b(\upsilon, \upsilon)$ *implies* $\tau = \upsilon$,

(QPb_2) $qp_b(\tau, \tau) \leq qp_b(\tau, \upsilon)$,

(QPb_3) $qp_b(\tau, \tau) \leq qp_b(\upsilon, \tau)$,

(QPb_4) $qp_b(\tau, \upsilon) \leq s[qp_b(\tau, \chi) + qp_b(\chi, \upsilon)] - qp_b(\chi, \chi)$.

A quasi-partial b-metric space is a pair (M, qp_b) such that M is a nonempty set and qp_b is a quasi-partial b-metric on M. The number s is called the coefficient of (M, qp_b).

Example 1.1. *Let $M = [0, \infty)$. Define $qp_b = ln(k) |\tau - \upsilon| + \tau$, where $k \geq 1$*

Here

$$qp_b(\tau, \tau) = qp_b(\tau, \upsilon) = qp_b(\upsilon, \upsilon)$$

$$\implies ln(k) |\tau - \tau| + \tau = ln(k) |\tau - \upsilon| + \tau = ln(k) |\upsilon - \upsilon| + \upsilon.$$

Therefore, (QPb_1) holds for any $(\tau, \upsilon \in M \times M)$.

Now as $ln(k) |\tau - \upsilon| \geq |\tau - \upsilon| \geq 0$ then $qp_b(\tau, \tau) = \tau \leq qp_b(\tau, \upsilon)$

and

$$\begin{aligned} qp_b(\tau,\tau) &= \tau \\ &= |\tau - \upsilon + \upsilon| \\ &\le |\tau - \upsilon| + \upsilon \\ &\le ln(k)\,|\tau - \upsilon| + \upsilon \\ &\le ln(k)\,|\upsilon - \tau| + \upsilon \\ &\le qp_b(\upsilon,\tau). \end{aligned}$$

Therefore, QPb_2 *and* QPb_3 *both holds.*

Now, $qp_b(\tau,\upsilon) + qp_b(\chi,\chi) = ln(k)\,|\tau-\upsilon| + \tau + \chi$.

Since

$$\begin{aligned} |\tau - \upsilon| &\ge 0 \\ ln(k)\,|\tau - \upsilon| &\ge |\tau - \upsilon| \\ k(|\tau - \chi| + |\chi - \upsilon|) &\ge 0. \end{aligned}$$

Since, $ln(k)$ *is an increasing function. Therefore,*

$$\begin{aligned} qp_b(\tau,\upsilon) + qp_b(\chi,\chi) &= ln(k)\,|\tau - \upsilon| + \tau + \chi \\ &\le ln(k)(|\tau - \chi| + |\chi - \upsilon|) + \tau + \chi \\ &\le k(|\tau - \chi| + |\chi - \upsilon|) + \tau + \chi \\ &\le (k)ln(k)\,|\tau - \chi| + (k)ln(k)\,|\chi - \upsilon| + \tau + \chi \\ &\le (k)(ln(k)\,|\tau - \chi| + |\chi - \upsilon|) + \tau + \chi \\ &\le s(qp_b(\tau,\chi) + qp_b(\upsilon,\chi)). \end{aligned}$$

for all $\tau, \upsilon, \chi \in M$ *and* $s \ge k$.

So (M, qp_b) *is a quasi-partial b-metric space with* $s \ge 1$.

Remark 1.2. *Note that a metric space is included in the class of quasi-partial b-metric space. In fact, the notions of convergent sequence, Cauchy sequence and complete space are defined as in metric spaces.*

Miculescu and Mihail [4] (Lemma 2. 2.) obtain the following result for b-metric spaces.

Lemma 1.1. *Every sequence $\{x_n\}$ of elements from a b-metric space (X, d, b), having the property that there exists $\lambda \in [0, 1)$ such that*

$$d(x_{n+2}, x_{n+1}) \leq \lambda d(x_{n+1}, x_n), \tag{1.3}$$

for any $n \in \mathbb{N}$, is Cauchy.

Remark 1.3. *Note that Lemma 1.1 holds in quasi-partial b-metric space (see proof of Lemma 2. 2 in [4]).*

2. Main results

In this section, we introduce interpolative Berinde weak contractions in quasi-partial b-metric space and adopted the condition of ω-admissibility to obtain a fixed point.

Definition 2.1. *Let (M, qp_b) be a complete quasi-partial b-metric space. We say that self-mapping $T : M \to M$ is an interpolative Berinde weak operator if there exist $\zeta \in \left[0, \frac{1}{s}\right)$ and $\alpha \in (0, 1)$ such that*

$$qp_b(T\tau, T\upsilon) \leq \zeta[qp_b(\tau, \upsilon)]^{\alpha} \left[\frac{1}{s^2} qp_b(\tau, T\tau)\right]^{1-\alpha}, \tag{2.1}$$

for all $\tau, \upsilon \in M \backslash Fix(T)$.

Theorem 2.1. *Let (M, qp_b) be a complete quasi-partial b-metric space and T be an interpolative Berinde weak operator. Then T has a fixed point in M.*

Proof. Let $\tau_0 \in (M, qp_b)$. Consider a constructive sequence τ_n by $\tau_{n+1} = T^n(\tau_0)$ for all $n \in \mathbb{N} \cup \{0\}$. We assume that $\tau_n = \tau_{n+1}$. Indeed if there exist n_0 such that $\tau_{n_0} = \tau_{n_0+1} = T\tau_{n_0}$, then, τ_{n_0} forms a fixed point. Thus, we have $qp_b(\tau_n, T\tau_n) = qp_b(\tau_n, \tau_{n+1}) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Let $\tau = \tau_{n+1}, \upsilon = \tau_{n+2}$

$$\begin{aligned}
qp_b(\tau_{n+1}, \tau_{n+2}) &= qp_b(T\tau_n, T\tau_{n+1}) \\
&\leq \zeta[qp_b(\tau_n, \tau_{n+1})]^{\alpha}. \left[\frac{1}{s^2} qp_b(\tau_n, \tau_{n+1})\right]^{1-\alpha} \\
&\leq \zeta[qp_b(\tau_n, \tau_{n+1})]^{\alpha}. \left[\frac{1}{s^2} qp_b(\tau_{n-1}, \tau_{n+1})\right]^{1-\alpha} \\
&\leq \zeta[qp_b(\tau_n, \tau_{n+1})]^{\alpha}. \left[\frac{1}{s^2} [sqp_b(\tau_{n-1}, \tau_n) + qp_b(\tau_n, \tau_{n+1}) - qp_b(\tau_n, \tau_n)\right]^{1-\alpha} \\
&\leq \zeta[qp_b(\tau_n, \tau_{n+1})]^{\alpha}. \left[\frac{1}{s} qp_b(\tau_{n-1}, \tau_n) + qp_b(\tau_n, \tau_{n+1})\right]^{1-\alpha} \qquad (2.2)
\end{aligned}$$

By induction, for all $n \in \mathbb{N} \cup \{0\}$ we get

$$qp_b(\tau_n, \tau_{n+1}) \leq \zeta^n qp_b(\tau_0, \tau_1).$$

On generalising the inequality,

$$qp_b(\tau_n, \tau_{n+1}) = \frac{s^{n-1}\tau^n}{(1-s\tau)^n} \text{ and } qp_b(\tau_{n+1}, \tau_n) = \frac{s^{n-1}\tau^n}{(1-s\tau)^n}.$$

Now, we shall show that τ_n is Cauchy sequence. Let $n, k \in \mathbb{N}$

$$\begin{aligned} qp_b(\tau_n, \tau_{n+k}) &\leq sqp_b(\tau_n, \tau_{n+1}) + s^2 qp_b(\tau_{n+1}, \tau_{n+2}) + \ldots + s^k qp_b(\tau_{n+k-1}, \tau_{n+k}) \\ &\leq \frac{s.s^{n-1}.\zeta^n.\tau^n}{(1-s\tau)^n} + \frac{s^2.s^n.\zeta^{n+1}.\tau^{n+1}}{(1-s\tau)^{n+1}} + \ldots + \frac{s^{n-k}.s^{n-2}.\zeta^{n-1}.\tau^{n-1}}{(1-s\tau)^{n-1}} \\ &\leq \frac{s^n.\tau^n}{(1-s\tau)^n} + \frac{s^{n+2}.\tau^{n+1}}{(1-s\tau)^{n+1}} + \ldots + \frac{s^{2n-k-2}.\tau^{n-1}}{(1-s\tau)^{n-1}} \\ &\leq \frac{s^n.\tau^n}{(1-s\tau)^n}\left[1 + \frac{s^2.\tau}{(1-s\tau)} + \ldots + \frac{s^{2n-k-2}.\tau^{n-k-1}}{(1-s\tau)^{n-k-1}}\right] \end{aligned}$$

The inequality, $0 \leq \tau \leq \frac{1}{s^2(s+1)}$ then $\frac{s^2.\tau}{(1-s\tau)} \leq 1$.

$$\begin{aligned} qp_b(\tau_n, \tau_{n+k}) &\leq \frac{\left(\frac{s\tau}{1-s\tau}\right)^n \left\{1 - \left(\frac{s^2.\tau}{1-s\tau}\right)^{n-k}\right\}}{\left(1 - \frac{s^2.\tau}{1-s\tau}\right)} \\ &\leq \left(\frac{s\tau}{a-s\tau}\right)^n \frac{(1-s\tau)}{1-s\tau-s^2\tau}. \end{aligned} \tag{2.3}$$

Therefore, we claim that τ_n is a Cauchy sequence in (M, qp_b). Let $m, n \in \mathbb{N}$. By triangle inequality in (2.2), we deduce that

$$qp_b(\tau_{n+m}, \tau_{n+m+k}) \leq \left(\frac{s\tau}{a-s\tau}\right)^n \frac{(1-s\tau)}{1-s\tau-s^2\tau}$$

Since, $\frac{s\tau}{1-s\tau} \leq 1$ and taking $n \to \infty$ in (2.3) and using $\lim_{n\to\infty} \tau(t^n) = 0$ for $t \geq 0$, we get

$$qp_b(\tau_{n+m}, \tau_{n+m+k}) \leq \left(\frac{s\tau}{a-s\tau}\right)^n \frac{(1-s\tau)}{1-s\tau-s^2\tau}$$

Therefore,

$$\lim_{n\to\infty} qp_b(\tau_n, \tau_{n+k}) = \lim_{m\to\infty, n\to\infty} qp_b(\tau_{n+m}, \tau_{n+m+k}) = 0 \tag{2.4}$$

Since M is complete, so there exist $\chi \in M$ such that

$$\lim_{n\to\infty} \tau_n = \chi.$$

Suppose $\tau_n \neq T\tau_n$ for each $n \geq 0$

$$\begin{aligned} qp_b(\tau_{n+1}, T\chi) &= qp_b(T\tau_n, T\chi) \\ &\leq \zeta[qp_b(\tau_n, \chi)]^{\alpha}.\left[\frac{1}{s^2} qp_b(\tau_n, T\chi)\right]^{1-\alpha} \end{aligned}$$

Since $\frac{1}{s} \leq 1$, this implies $\frac{1}{s^2} \leq 1$ therefore

$$
\begin{aligned}
&\leq \zeta[qp_b(\tau_n, \chi)]^{\alpha}. \left[\frac{1}{s^2} qp_b(\tau_n, T\chi)\right]^{1-\alpha} \\
&\leq \zeta[qp_b(\tau_n, \chi)]^{\alpha}.[qp_b(\tau_n, T\chi)]^{1-\alpha}. \qquad (2.5)
\end{aligned}
$$

Letting $n \to \infty$ in (2.5), we get

$$qp_b(\chi, T\chi) = 0$$

which is a contradiction. Thus $T\chi = \chi$. □

Definition 2.2. *[7] Let $T\colon M \to M$ be a map and $\alpha\colon M \times M \to R$ be a function. Then T is said to be α-admissible if*

$$\alpha(\tau, \upsilon) \geq 1 \textit{ implies } \alpha(T\tau, T\upsilon) \geq 1. \qquad (2.6)$$

We introduce ω-admissible interpolative Berinde weak contraction in quasi-partial b-metric space engrossed by Gupta and Gautam [20].

Definition 2.3. *Let (M, qp_b) be a complete quasi-partial b-metric space. The map $T\colon M \to M$ is said to be an ω-interpolative Berinde weak contraction if there exists ζ, $\omega\colon M \times M \to [0, \infty)$ and $\alpha \in (0, 1)$ such that*

$$\omega(\tau, \upsilon) qp_b(T\tau, T\upsilon) \leq \zeta[qp_b(\tau, \upsilon)]^{\alpha} \left[\frac{1}{s^2} qp_b(\tau, T\tau)\right]^{1-\alpha}, \qquad (2.7)$$

for all $\tau, \upsilon \in M \backslash Fix(T)$.

Theorem 2.2. *Let us consider (M, qp_b) to be a complete quasi-partial b-metric space with self mapping $T\colon M \to M$ is ω-orbital admissibile and forms an ω-interpolative Berinde weak contraction on a complete quasi-partial b-metric space (M, qp_b). If there exists $\tau_0 \in M$ such that $\omega(\tau_0, T\tau_1) \geq 1$, then T posses a fixed point in M.*

Proof. Let $\tau \in M$ be a point such that $\tau_{n+1} = T^n(\tau_0)$ for all $n \in \mathbb{N} \cup \{0\}$. If we have, $\tau_{n_0} = \tau_{n_0+1}$ then τ_n is a fixed point of T which ends the proof otherwise $\tau_n \neq \tau_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. We have $\omega(\tau_0, \tau_1) \geq 1$. Since T is ω - orbital admissible,

$$\omega(\tau_1, \tau_2) = \omega(T\tau_0, T\tau_1) \geq 1 \qquad (2.8)$$

continuing $\omega(\tau_n, \tau_{n+1}) \geq 1$.

Let $\tau = \tau_{n+1}$, $\upsilon = \tau_{n+2}$, we have

$$qp_b(\tau_n, \tau_{n+1}) \leq \omega(\tau_{n+1}, \tau_{n+2}) qp_b(T\tau_{n+1}, T\tau_{n+2})$$

$$\begin{aligned}
qp_b(\tau_{n+1}, \tau_{n+2}) &= qp_b(T\tau_n, T\tau_{n+1}) \\
&\leq \zeta [qp_b(\tau_n, \tau_{n+1})]^{\alpha}. \left[\frac{1}{s^2} qp_b(\tau_n, \tau_{n+1})\right]^{1-\alpha} \\
&\leq \zeta [qp_b(\tau_n, \tau_{n+1})]^{\alpha}. \left[\frac{1}{s^2} qp_b(\tau_{n-1}, \tau_{n+1})\right]^{1-\alpha} \\
&\leq \zeta [qp_b(\tau_n, \tau_{n+1})]^{\alpha}. \left[\frac{1}{s^2} [sqp_b(\tau_{n-1}, \tau_n) + qp_b(\tau_n, \tau_{n+1}) - qp_b(\tau_n, \tau_n)\right]^{1-\alpha} \\
&\leq \zeta [qp_b(\tau_n, \tau_{n+1})]^{\alpha}. \left[\frac{1}{s} qp_b(\tau_{n-1}, \tau_n) + qp_b(\tau_n, \tau_{n+1})\right]^{1-\alpha} \qquad (2.9)
\end{aligned}$$

By induction, for all $n \in \mathbb{N} \cup \{0\}$ we get

$$\omega(\tau_{n+1}, \tau_{n+2}) qp_b(\tau_n, \tau_{n+1}) \leq \zeta^n qp_b(\tau_0, \tau_1).$$

On generalising the inequality,

$$qp_b(\tau_n, \tau_{n+1}) = \frac{s^{n-1}\tau^n}{(1-s\tau)^n} \text{ and } qp_b(\tau_{n+1}, \tau_n) = \frac{s^{n-1}\tau^n}{(1-s\tau)^n}.$$

Now we shall show that τ_n is Cauchy sequence. Let $n, k \in \mathbb{N}$

$$\begin{aligned}
qp_b(\tau_n, \tau_{n+k}) &\leq sqp_b(\tau_n, \tau_{n+1}) + s^2 qp_b(\tau_{n+1}, \tau_{n+2}) + \ldots + s^{n-k} qp_b(\tau_{n+k-1}, \tau_{n+k}) \\
&\leq \frac{s.s^{n-1}.\zeta^n.\tau^n}{(1-s\tau)^n} + \frac{s^2.s^n.\zeta^{n+1}.\tau^{n+1}}{(1-s\tau)^{n+1}} + \ldots + \frac{s^{n-k}.s^{n-2}.\zeta^{n-1}.\tau^{n-1}}{(1-s\tau)^{n-1}} \\
&\leq \frac{s^n.\tau^n}{(1-s\tau)^n} + \frac{s^{n+2}.\tau^{n+1}}{(1-s\tau)^{n+1}} + \ldots + \frac{s^{2n-k-2}.\tau^{n-1}}{(1-s\tau)^{n-1}} \\
&\leq \frac{s^n.\tau^n}{(1-s\tau)^n} \left[1 + \frac{s^2.\tau}{(1-s\tau)} + \ldots + \frac{s^{2n-k-2}.\tau^{n-k-1}}{(1-s\tau)^{n-k-1}}\right]
\end{aligned}$$

The inequality, $0 \leq \tau \leq \frac{1}{s^2(s+1)}$ then $\frac{s^2.\tau}{(1-s\tau)} \leq 1$

$$\begin{aligned}
qp_b(\tau_n, \tau_{n+k}) &\leq \frac{\left(\frac{s\tau}{1-s\tau}\right)^n \left\{1 - \left(\frac{s^2.\tau}{1-s\tau}\right)^{n-k}\right\}}{\left(1 - \frac{s^2.\tau}{1-s\tau}\right)} \\
&\leq \left(\frac{s\tau}{a-s\tau}\right)^n \frac{(1-s\tau)}{1-s\tau-s^2\tau}. \qquad (2.10)
\end{aligned}$$

Therefore, we claim that τ_n is a Cauchy sequence in (M, qp_b). Let $m, n \in \mathbb{N}$. On account of the triangle inequality in (2.9), we deduce that

$$qp_b(\tau_{n+m}, \tau_{n+m+k}) \leq \left(\frac{s\tau}{a-s\tau}\right)^n \frac{(1-s\tau)}{1-s\tau-s^2\tau}$$

Since, $\frac{s\tau}{1-s\tau} \leq 1$, and taking $n \to \infty$ in (2.10) and using $\lim_{n\to\infty} \tau(t^n) = 0$ for $t \geq 0$, we get

$$\lim_{n\to\infty} qp_b(\tau_n, \tau_{n+k}) = \lim_{m\to\infty, n\to\infty} qp_b(\tau_{n+m}, \tau_{n+m+k}) = 0$$

Since M is complete, so there exists $\chi \in M$ such that

$$\lim_{n\to\infty} \tau_n = \chi.$$

Suppose $\tau_n \neq T\tau_n$ for each $n \geq 0$

$$\begin{aligned} qp_b(\tau_{n+1}, T\chi) &= qp_b(T\tau_n, T\chi) \\ &\leq \zeta[qp_b(\tau_n, \chi)]^{\alpha}.[\frac{1}{s^2} qp_b(\tau_n, T\chi)]^{1-\alpha}. \end{aligned}$$

Since $\frac{1}{s} \leq 1$, this implies $\frac{1}{s^2} \leq 1$, therefore

$$\leq \zeta[qp_b(\tau_n, \chi)]^{\alpha}.[qp_b(\tau_n, T\chi)]^{1-\alpha}. \tag{2.11}$$

Letting $n \to \infty$ in (2.11), we get

$$qp_b(\chi, T\chi) = 0$$

which is a contradiction. Thus $T\chi = \chi$. □

Corollary 2.1. *Let (M, qp_b) be quasi-partial b-metric space. Let $T\colon M \to M$ be the mapping, such that*

$$\omega(\tau, \upsilon) qp_b(T\tau, T\upsilon) \leq \zeta[qp_b(\tau, T\upsilon)]^{\alpha}. \left[\frac{1}{s^2} qp_b(\tau, T\tau)\right]^{1-\alpha}$$

for all $\tau, \upsilon \in M \backslash Fix(T)$ with $\tau \preceq \upsilon, s \geq 1$ and $\alpha \in (0,1)$.

Assume that:

(1) *T is non-decreasing with respect to $\preceq$,*

(2) *there exists $\tau_0 \in X$ such that $\tau_0 \preceq T\tau_0$,*

(3) *T is continuous.*

Then, T has a fixed point in M.

Proof. It suffices to take, in Theorem 2.1,

$$\omega(\tau, \upsilon) = \begin{cases} 1 & \text{if } (\tau \preceq \upsilon) \text{ or } (\upsilon \preceq \tau), \\ 0 & \text{otherwise.} \end{cases}$$ □

Corollary 2.2. *Assume that the subsets B_1 and B_2 of a quasi-partial b-metric space (M, qp_b) are closed. Suppose that $T : B_1 \cap B_2 \to B_1 \cap B_2$ satisfies,*

$$\omega(\tau, \upsilon) qp_b(T\tau, T\upsilon) \leq \zeta[qp_b(\tau, T\upsilon)]^{\alpha}. \left[\frac{1}{s^2} qp_b(\tau, T\tau)\right]^{1-\alpha}$$

for all $\tau \in B_1$ *and* $\upsilon \in B_2$*, such that* $\tau, \upsilon \notin Fix(T)$*, where* $\alpha \in (0,1), s \geq 1$*, If* $T(B_1) \subseteq B_2$ *and* $T(B_2) \subseteq B_1$*, then there exists a fixed point of* T *in* $B_1 \cap B_2$*.*

Proof. It suffices to take, in Theorem 2.1,

$$\omega(\tau, \upsilon) = \begin{cases} 1 & \text{if } (\tau \preceq \upsilon) \text{ or } (\upsilon \preceq \tau), \\ 0 & \text{otherwise.} \end{cases}$$

□

Example 2.1. *Let a set* $M = [0,4]$ *with* $qp_b(\tau, \upsilon) = Log(k)|\tau - \upsilon| + \tau$*. Let* T *be a self-mapping on* M *shown as*

$$T\tau = \begin{cases} \frac{10}{3}, & if\ \tau \in [3,4], \\ \frac{3}{4}, & if\ \tau \in [0,3]. \end{cases}$$

We illustrate the self-mappings of T *in the Fig1.*

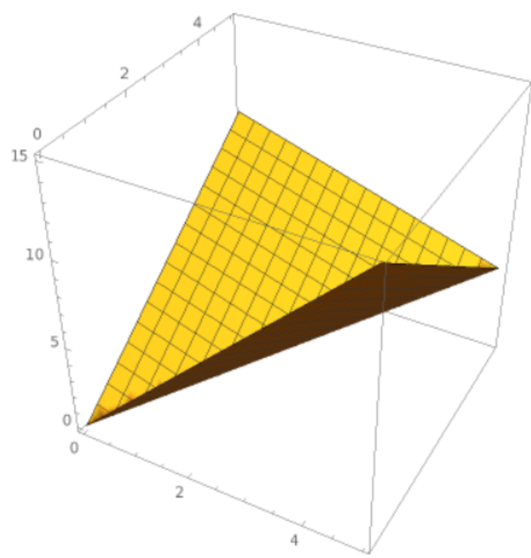

Figure 1. $\frac{10}{3}$ and $\frac{3}{4}$ are the fixed points of T.

Take,

$$\omega(\tau, \upsilon) = \begin{cases} 1, & if\ \ \tau, \upsilon \in [3,4], \\ 0 & otherwise. \end{cases}$$

Let $\tau, \upsilon \in M$ *such that* $\tau \neq T\tau, \upsilon \neq T\upsilon$ *and* $\omega(\tau, \upsilon) \geq 1$*. Then* $\tau, \upsilon \in [3,4]$ *and* $\tau, \upsilon \notin \frac{10}{3}$*. We have* $T\tau = T\upsilon = \frac{10}{3}$*. For* $\tau_0 = 4$*, we have*

$$\omega(4, T4) = \omega(4, \frac{10}{3}) = 1$$

Now, let $\tau, \upsilon \in M$ *be such that* $\omega(\tau, \upsilon) \geq 1$*. This shows that* $\tau, \upsilon \in [3,4]$*, so* $T\tau = T\upsilon$ *in* $[3,4]$*. Hence,* $\omega(T\tau, T\upsilon) \geq 1$*, that is,* T *is* ω*-orbital admissible. Suppose* τ_n *to be a sequence in* M *such that* $\omega(\tau_n, \tau_{n+1}) \geq 1$

for each $n \in \mathbb{N}$. Then, $\tau_n \subset [3,4]$. If $\tau_n \to w$ as $n \to \infty$, we have $|\tau_n - w| \to 0$ as $n \to \infty$. Hence, $w \in [3,4]$ and so, $\omega(\tau_n, w) = 1$. Therefore, Theorem 2.1 holds. So, $\frac{10}{3}$ and $\frac{3}{4}$ are the two fixed points of T.

In 2016, Berinde and Fukhar-ud-din [30] modified the concept of convex metric space and applied it to obtain fixed point results of quasi-contractive operators. Motivated by this, Ampadu [16] introduced convex interpolative Berinde weak operator in metric spaces. Forging this approach, we introduce the following result in quasi-partial b-metric space.

Definition 2.4. *Let (M, qp_b) be a complete quasi-partial b-metric space with continuous mapping $T\colon M \to M$ is convex interpolative Berinde weak operator if the following is true for all $\tau, \upsilon \in X, \tau, \upsilon \notin Fix(T), Fix(T^2)$.*

$$qp_b(T^2\tau, T^2\upsilon) \leq \zeta_1 qp_b(\tau,\upsilon)^{\frac{1}{2}} + \zeta_2 qp_b(T\tau, T\upsilon)^{\frac{1}{2}} qp_b(T\tau, T^2\tau)^{\frac{1}{2}}, \tag{2.12}$$

where $\zeta_1, \zeta_2 \in \left[0, \frac{1}{s}\right)$ with $\zeta_1 + \zeta_2 \leq \frac{1}{s}$ for $s \geq 1$.

Theorem 2.3. *Let (M, qp_b) be a complete quasi-partial b-metric space with self mapping $T\colon M \to M$ be a convex interpolative Berinde weak operator. If (M, qp_b) is complete, then the fixed point exists.*

Proof. Let τ_n be a sequence in M such that $\tau_{n+1} = T\tau_n = T^2\tau_{n-1}$, for all positive integers n. Now, we observe that

$$\begin{aligned}
qp_b(\tau_{n+1}, \tau_{n+2}) &= qp_b(T^2\tau_{n-1}, T^2\tau_n) \\
&\leq \zeta_1 qp_b(\tau_n, \tau_{n-1})^{\frac{1}{2}} qp_b(\tau_{n-1}, T\tau_{n-1})^{\frac{1}{2}} + \\
&\quad\ \zeta_2 qp_b(T\tau_n, T\tau_{n-1})^{\frac{1}{2}} qp_b(T\tau_{n-1}, T^2\tau_{n-1})^{\frac{1}{2}} \\
&= \zeta_1 qp_b(\tau_n, \tau_{n-1})^{\frac{1}{2}} qp_b(\tau_{n-1}, \tau_n)^{\frac{1}{2}} + \\
&\quad\ \zeta_2 qp_b(\tau_n, \tau_{n+1})^{\frac{1}{2}} qp_b(\tau_n, \tau_{n+1})^{\frac{1}{2}} \\
&= \zeta_1 qp_b(\tau_n, \tau_{n-1}) + \zeta_2 qp_b(\tau_n, \tau_{n+1}) \\
&\leq (\zeta_1 + \zeta_2) max\{qp_b(\tau_n, \tau_{n-1}), qp_b(\tau_n, \tau_{n-1})\} \\
&= (\zeta_1 + \zeta_2) qp_b(\tau_n, \tau_{n+1}).
\end{aligned}$$

From the above, we deduce that

$$qp_b(\tau_{n+1}, \tau_{n+2}) \leq h qp_b(\tau_n, \tau_{n+1}) \implies qp_b(\tau_{n+1}, \tau_{n+2}) \leq \frac{1}{s} qp_b(\tau_n, \tau_{n+1}).$$

where $h := \zeta_1 + \zeta_2 \leq \frac{1}{s}$. By induction, the following is clear for all $n \in \mathbb{N} \bigcup 0$

$$qp_b(\tau_n, \tau_{n+1}) \leq h^n qp_b(\tau_0, \tau_1).$$

Now, we shall show that τ_n is a Cauchy sequence. For this, let $n, m \in \mathbb{N}$ with $m \geq n$, and we have

$$\begin{aligned}
qp_b(\tau_m, \tau_n) &\leq qp_b(\tau_m, \tau_{m-1}) + qp_b(\tau_{m-1}, \tau_{m-2}) + \ldots + qp_b(\tau_{n+1}, \tau_n) \\
&\leq sqp_b(\tau_m, \tau_{m-1}) + s^2 qp_b(\tau_{m-1}, \tau_{m-2}) + \ldots + s^m qp_b(\tau_{n+1}, \tau_n) \\
&\leq \left[s(\zeta_1 + \zeta_2)^n + s^2(\zeta_1 + \zeta_2)^{n+1} + \ldots + s^m(\zeta_1 + \zeta_2)^{m+n-1}\right] qp_b(\tau_0, \tau_1) \\
&\leq \left[s(h)^n + s^2(h)^{n+1} + \ldots + s^m(h)^{m+n-1}\right] qp_b(\tau_0, \tau_1) \\
&\leq s^m \sum_{i=n}^{m+n-1} h^i qp_b(\tau_0, \tau_1) \\
&\leq s^m \sum_{i=n}^{\infty} h^i qp_b(\tau_0, \tau_1).
\end{aligned}$$

Now, letting $m, n \to \infty$ in the above inequality it follows that τ_n is a Cauchy sequence and since M is complete $\chi \in M$ such that

$$\lim_{n \to \infty} \tau_n = \chi.$$

Suppose $qp_b(\chi, T\chi) = 0$, but $qp_b(\chi, T\chi) \geq 0$. Therefore we observe that

$$\begin{aligned}
0 &\geq qp_b(\chi, T^2\chi) \\
&\geq qp_b(\chi, \tau_{n+1}) + qp_b(\tau_{n+1}, T^2\chi) \\
&= qp_b(\chi, \tau_{n+1}) + qp_b(T^2\tau_{n-1}, T^2\chi) \\
&\geq qp_b(\chi, \tau_{n+1}) + \zeta_1 qp_b(\tau_{n-1}, \chi)^{\frac{1}{2}} qp_b(\tau_{n-1}, T\tau_{n-1})^{\frac{1}{2}} \\
&\quad + \zeta_2 qp_b(T\tau_{n-1}, T\chi)^{\frac{1}{2}} qp_b(T\tau_{n-1}, T^2\tau_{n-1})^{\frac{1}{2}}.
\end{aligned}$$

Taking $n \to \infty$ in the above inequality we find that $qp_b(\chi, T\chi) = 0$, which is a contradiction. Thus $T\chi = \chi$. □

3. Conclusion

The main contribution of this paper is to prove the existence of fixed points via interpolative Berinde weak contraction. The interpolative weak contraction is extended to adapt various nonlinear self mappings leading to achieve best topological and geometrical results. One of the major applications of Berinde weak contraction is that it is used to solve multivalued mappings, that is, it can be used to get more than one fixed point. Also, it is used to solve initial value problems in ordinary differential equations and integral equations. Weak contraction merged large amount of contractive operators and formulated fixed points by the means of Picard iteration.

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References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fund. Math. 3 (1922), 133–181.

[2] S. G. Matthews, Partial metric topology, Ann. N. Y. Acad. Sci. 728 (1994), 183-197.

[3] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5-11.

[4] R. Miculescu, A. Mihail, New fixed point theorems for set-valued contractions in b-metric spaces, J. Fixed Point Theory Appl. 19 (2017), 2153-2163.

[5] S. Oltra, O. Valero, Banach's fixed point theorem for partial metric spaces, Rend. Istit. Mat. Univ. Trieste, 36 (2004), 17-26.

[6] O. Valero, On Banach fixed point theorems for partial metric spaces, Appl. Gen. Topol. 6(2) (2005), 229-240.

[7] E. Karapınar, I.M. Erhan, A. Öztürk, Fixed point theorems on quasi-partial metric spaces, Math. Comput. Model. 57 (2013), 2442-2448.

[8] S. Shukla, Partial b-metric spaces and fixed point theorems, Mediterr. J. Math. 11 (2014), 703-711.

[9] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71–76.

[10] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum. 9(1) (2004), 43–53.

[11] M. Abbas, D. Ilic, Common fixed points of generalized almost non expansive mappings, Filomat. 24(3) (2010), 11–18.

[12] V. Berinde, Some remarks on a fixed point theorem for Ćirić type almost contractions, Carpathian J. Math. 25(2) (2009), 157–162.

[13] I. Altun, O. Acar, Fixed point theorems for weak contractions in the sense of Berinde on partial metric spaces, Topol. Appl. 159(10-11) (2012), 2642–2648.

[14] A. D. T. Turkoglu, V. Ozturk, Common fixed point results for four mappings on partial metric spaces, Abstr. Appl. Anal. 2012 (2012), 190862.

[15] E. Karapinar, Revisiting the Kannan type contractions via interpolation, Adv. Theory Nonlinear Anal. Appl. 2 (2018), 85-87.

[16] C. Boateng Ampadu, Some fixed point theory results for the interpolative Berinde weak operator, Earthline J. Math. Sci. 4(2) (2020), 253-271.

[17] O. Popescu, Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces, Fixed Point Theory Appl. 2014 (2014), 190.

[18] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha - \psi$-contractive type mappings, Nonlinear Anal., Theory Meth. Appl. 75 (2012), 2154–2165.

[19] H. Aydi, E. Karapinar, A. F. Roldán López de Hierro, ω-interpolative Ćirić-Reich-Rus-type contractions, Mathematics. 7 (2019), 57.

[20] A. Gupta, P. Gautam, Quasi-partial b-metric spaces and some related fixed point theorems, Fixed Point Theory Appl. 2015 (2015), 18.

[21] A. Gupta, P. Gautam, Topological structure of quasi-partial b-metric space, Int. J. Pure Math Sci. 17 (2016), 8-18.

[22] P. Gautam, V. N. Mishra, K. Negi, Common Fixed point theorems for cyclic Reich-Rus-Ćirić contraction mappings in quasi-partial b-metric space, Ann. Fuzzy Math. Inf. 12(1) (2020), 47-56.

[23] P. Gautam, V. N. Mishra, R. Ali, S. Verma, Interpolative Chatterjea and cyclic Chatterjea contraction on quasi-partial b-metric space, AIMS Math. 6(2) (2021), 1727-1742.

[24] V. N. Mishra, L. M. Sánchez Ruiz., P. Gautam, S. Verma, Interpolative Reich-Rus-Ćirić and Hardy-Rogers contraction on quasi-partial b-metric space and related fixed point results, Mathematics. 8 (2020), 1598.

[25] L. M. Sánchez Ruiz., P. Gautam, S. Verma, Fixed point of interpolative Reich-Rus-Ćirić and contraction mapping on Rectangular quasi-partial b-metric space, Symmetry. 13(1) (2021), Art. ID 32.

[26] P. Gautam, S. Verma, Fixed point via implicit contraction mapping on quasi-partial b-metric space, J. Anal. (2021). https://doi.org/10.1007/s41478-021-00309-6.

[27] P. Gautam, S. Verma, M. De La Sen, S. Sundriyal, Fixed point results for ω-interpolative Chatterjea type contraction in quasi-partial b-metric space, Int. J. Anal. Appl. 19 (2021), 280-287.

[28] P. Gautam, L. M. Sánchez Ruiz, S. Verma, G. Gupta, Common fixed point results on generalized weak compatible mapping in quasi-partial b-metric space, J. Math. 2021 (2021), Art. ID 5526801.

[29] P. Gautam, S. Verma, S. Gulati, *omega*-Interpolative Ćirić-Reich-Rus type contraction on quasi-partial b-metric space, Filomat. Accepted.

[30] H. Fukhar-ud-din, V. Berinde, Iterative methods for the class of quasi-contractive type operators and comparsion of their rate of convergence in convex metric spaces, Filomat. 30(1) (2016), 223-230.