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ON MOCANU-TYPE FUNCTIONS WITH GENERALIZED BOUNDED VARIATIONS

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Abstract. The main focus of this article is the study of classes $M_{\mu}^{\delta}(\varphi, \mathcal{H})$ and $\mathcal{Q}_{\mu}^{\delta}(\varphi, g_1, \mathcal{H})$. We present various inclusion relationships and some applications of our investigations are considered. Also, we include radius problem.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. If f and g are analytic in $\mathcal{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in $\mathcal{U}$ such that $f(z) = g(w(z))$.

The convolution or Hadamard product of two functions $f, g \in \mathcal{A}$ is denoted by $f * g$ and is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}. \tag{1.2}$$

Analytic functions p in the class $\mathcal{P}[A, B]$ can be defined by using subordination as follows [3].

Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. Then $p \in \mathcal{P}[A, B]$, if and only if,

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \qquad -1 \leq B < A \leq 1, \, z \in \mathcal{U}.$$

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For $k \geqslant 0$, the conic domains Ω_k, defined as;

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

The domains Ω_k $(k = 0)$ represents right half plane, Ω_k $(0 < k < 1)$ represents hyperbola, Ω_k $(k = 1)$ represents a parabola and Ω_k $(k > 1)$ represents an ellipse. The extremal functions for these conic regions are given as;

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0 \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1 \\ 1 + \frac{2}{1-k^2}\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & 0 < k < 1 \\ 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.3}$$

where $u(z) = \frac{z-\sqrt{t}}{z-\sqrt{tz}}$, $t \in (0,1)$, $z \in \mathcal{U}$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. See [4,5] for more information. These conic regions are being studied by several authors, see [6,9,12].

In 2017, Dziok and Noor [2] introduced and studied the concepts of some general classes given as below.

Definition 1.1. *Let $\mu \geq 0$, $\Phi = (\phi_1(z), \phi_2(z))$ and $\mathcal{H} = (h_1(z), h_2(z))$ where $h_i \in \mathcal{A}$ with $h_i(0) = 1$, $(i = 1, 2)$. Then*

$$\mathcal{P}_\mu(\mathcal{H}) = \{\mu q_1 + (1-\mu)\, q_2 : q_1 \in \mathcal{P}(h_1),\ q_2 \in \mathcal{P}(h_2)\},$$

where

$$\mathcal{P}(h) = \{q \in \mathcal{A} : q \prec h \text{ with } q(0) = 1\}.$$

Some special cases:

(i) $\mathcal{P}_\mu(h) = \mathcal{P}_\mu((h,h))$. If $\mu = \frac{m}{4} + \frac{1}{2}$, $(m \geq 2)$, then $\mathcal{P}_\mu(h) = \mathcal{P}_m(h)$.

(ii) If $\mu = \frac{m}{4} + \frac{1}{2}$, $(m \geq 2)$, and $h(z) = \frac{1+(1-2\rho)z}{1-z}$, then $\mathcal{P}_\mu(h) = \mathcal{P}_m(\rho)$, this class was introduced by Padmanabhan et al. [13].

(iii) If $\mu = \frac{m}{4} + \frac{1}{2}$, $(m \geq 2)$ and $h(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, then $\mathcal{P}_\mu(h) = \mathcal{P}_m[A, B]$, this class was introduced by Noor [10]. Moreover, for $A = 1$ and $B = -1$ we have $\mathcal{P}_\mu(h) = \mathcal{P}_m$; see [14].

(iv) If $\mu = \frac{m}{4} + \frac{1}{2}$, $(m \geq 2)$ and $h(z) = p_\kappa(z)$ $(\kappa \geq 0)$, then $\mathcal{P}_\mu(h) = \mathcal{P}_m(p_\kappa)$, this class was defined by Noor et al. [11].

Definition 1.2. *Let $f \in \mathcal{A}$ and $\delta \geq 0$. Then $f \in M_\mu^\delta(\Phi, \xi, \mathcal{H})$ if and only if $J_\delta(f((z))) \in \mathcal{P}_\mu(\mathcal{H})$, where*

$$J_\delta(f((z))) = (1-\delta)\frac{(\xi * \phi_2) * f}{(\xi * \phi_1) * f} + \delta\frac{\phi_2 * f}{\phi_1 * f}.$$

If $\xi_1(z) = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n$, $\phi_1(z) = z\varphi'(z)$ and $\phi_2(z) = z\phi_1'(z)$, then we have the following special cases.

$$M^\delta(\Phi, \xi, h) = M_1^\delta(\Phi, \xi, (h,h)),\ \ M_\mu^\delta(\Phi, \mathcal{H}) = M_\mu^\delta(\Phi, \xi_1, \mathcal{H}),$$

$$M_{\mu}^{\delta}(\varphi,\mathcal{H})=M_{\mu}^{\delta}((\phi_2,\phi_1),\mathcal{H}), \tag{1.4}$$

$$S_{\mu}^{*}(\varphi,\mathcal{H})=M_{\mu}^{0}(\varphi,\mathcal{H}), \quad S^{*}(\varphi,h)=M_{1}^{0}(\varphi,h). \tag{1.5}$$

Definition 1.3. *Let $f\in\mathcal{A}$, $\mathcal{G}=(g_1,g_2)$, where $g_i\in\mathcal{A}$ with $g_i(0)=1$ $(i=1,2)$, and $\delta,\vartheta\geq 0$. Then $f\in\mathcal{Q}_{\mu,\vartheta}^{\delta}(\Phi,\xi,\mathcal{G},\mathcal{H})$ if there exists $g\in S_{\vartheta}^{*}(\varphi,\mathcal{G})$ such that*

$$(1-\delta)\frac{(\xi*\phi_2)*f}{(\xi*\phi_1)*g}+\delta\frac{\phi_2*f}{\phi_1*g}\in\mathcal{P}_{\mu}(\mathcal{H}).$$

If $\xi_1(z)=z+\sum\limits_{n=2}^{\infty}\frac{1}{n}z^n$, $\phi_1(z)=z\varphi'(z)$ and $\phi_2(z)=z\phi_1'(z)$, then we have the following special cases.

$$\mathcal{Q}^{\delta}(\Phi,\xi,g_1,h_1)=M_{1,1}^{\delta}(\Phi,\xi,(g_1,g_2),(h_1,h_2)),$$

$$\mathcal{Q}_{\mu,\vartheta}^{\delta}(\Phi,\mathcal{G},\mathcal{H})=M_{\mu,\vartheta}^{\delta}(\Phi,\xi_1,\mathcal{G},\mathcal{H}),$$

$$\mathcal{Q}_{\mu}^{\delta}(\varphi,g_1,H)=\mathcal{Q}_{\mu,1}^{\delta}((\phi_2,\phi_1),(g_1,g_1),H). \tag{1.6}$$

From (1.4), we denote the class $M_{\mu}^{\delta}(\varphi,\mathcal{H})$ of functions $f\in\mathcal{A}$ satisfies $J_{\delta}(f(z))\in\mathcal{P}_{\mu}(\mathcal{H})$, where

$$J_{\delta}(f(z))=(1-\delta)\frac{z(\varphi*f)'}{(\varphi*f)}+\delta\frac{\left(z(\varphi*f)'\right)'}{(\varphi*f)'},$$

and $\mathcal{P}_{\mu}(\mathcal{H})$ is given by Definition 1.1.

Similarly, from (1.6), we denote the class $\mathcal{Q}_{\mu}^{\delta}(\varphi,h,\mathcal{H})$ of functions $f\in\mathcal{A}$ satisfies $J_{\delta}(f(z),g(z))\in\mathcal{P}_{\mu}(\mathcal{H})$, where

$$J_{\delta}(f(z),g(z))=(1-\delta)\frac{z(\varphi*f)'}{(\varphi*g)}+\delta\frac{\left(z(\varphi*f)'\right)'}{(\varphi*g)'},$$

for $g\in S^{*}(\varphi,h)$, the class $S^{*}(\varphi,h)$ is given by (1.5).

2. Preliminary Results

Lemma 2.1. *[2] Let $\mathcal{H}=(h_1,h_2)$, where h_i $(i=1,2)$ are analytic, univalent convex functions with $h_i(0)=1$ $(i=1,2)$ and let $\varkappa:U\to\mathbb{C}$ (set of complex numbers) with $\Re(\varkappa)>0$. If $p(z)$ is analytic, with $p(0)=1$ in $\mathcal{U}$, satisfies*

$$p(z)+\varkappa zp'(z)\in\mathcal{P}_{\mu}(\mathcal{H}),$$

then $p(z)\in\mathcal{P}_{\mu}(\mathcal{H})$.

Lemma 2.2. *[8] Let h be analytic, univalent convex function in $\mathcal{U}$ with $h(0) = 1$ and $Re\,(\gamma h(z) + \sigma) > 0$, $\sigma, \gamma \in \mathbb{C}$ and $\gamma \neq 0$. If $p(z)$ is analytic in $\mathcal{U}$ and $p(0) = h(0)$, then*

$$\left\{p(z) + \frac{zp'(z)}{\gamma p(z) + \sigma}\right\} \prec h(z),$$

implies $p(z) \prec q(z) \prec h(z)$, where $q(z)$ is best dominant and is given as,

$$q(z) = \left[\left\{\int_0^1 \left(\exp \int_t^{tz} \frac{h(u) - 1}{u} du\right) dt\right\}^{-1} - \frac{\sigma}{\gamma}\right].$$

Lemma 2.3. *[15] If $f \in C, g \in S^*$, then for each h analytic in $\mathcal{U}$ with $h(0) = 1$,*

$$\frac{(f * hg)\,(\mathcal{U})}{(f * g)\,(\mathcal{U})} \subset \overline{Co}h(\mathcal{U}),$$

where $\overline{Co}h(\mathcal{U})$ denotes the convex hull of $h(\mathcal{U})$.

3. Main Results

3.1. **Inclusion Results.**

Theorem 3.1. *Let $\delta \geq 0$, $\varphi \in \mathcal{A}$ and h be any convex univalent function in $\mathcal{U}$. Then*

$$M_1^{\delta}\,(\varphi, h) \subset M_1^{0}\,(\varphi, h)\,.$$

Proof. Let $f \in M_1^{\delta}\,(\varphi, h)$. Then, by definition,

$$(1 - \delta)\,\frac{z\,(\varphi * f)'}{(\varphi * f)} + \delta\frac{\left(z\,(\varphi * f)'\right)'}{(\varphi * f)'} \in \mathcal{P}(h),$$

or

$$(1 - \delta)\,\frac{z\,(\varphi * f)'}{(\varphi * f)} + \delta\frac{\left(z\,(\varphi * f)'\right)'}{(\varphi * f)'} \prec h(z). \tag{3.1}$$

Consider

$$\frac{z\,(\varphi * f)'}{(\varphi * f)} = p(z). \tag{3.2}$$

On logarithmic differentiation of (3.2), we have

$$\frac{\left(z\,(\varphi * f)'\right)'}{(\varphi * f)'} = \frac{z\,(\varphi * f)'}{(\varphi * f)} + \frac{zp'(z)}{p(z)}. \tag{3.3}$$

From (3.2) and (3.3), we get

$$\frac{\left(z\,(\varphi * f)'\right)'}{(\varphi * f)'} = p(z) + \frac{zp'(z)}{p(z)}. \tag{3.4}$$

On making use of (3.2) and (3.4) in (3.1), we obtain

$$(1 - \delta)\,p(z) + \delta\left[p(z) + \frac{zp'(z)}{p(z)}\right] \prec h(z),$$

this implies

$$p(z) + \delta\frac{zp'(z)}{p(z)} \prec h(z).$$

By using Lemma 2.2, we conclude $p(z) \prec h(z)$. Hence $f \in M_1^{0}\,(\varphi, h)$. ∎

Remark 3.1. *Following different choices of φ and h give certain inclusion results for the above theorem.*

(i) $\varphi \in A$, $h(z) = \frac{1+Az}{1+Bz}$, where $-1 \leq B < A \leq 1$.

(ii) $\varphi \in A$, $h(z) = p_k(z)$, where $p_k(z)$ is given by (1.3).

Corollary 3.1. *Let $\delta \geq 1$. Then*

$$M_1^{\delta}(\varphi, h) \subset M_1^{1}(\varphi, h).$$

Proof. Let $f \in M_1^{\delta}(\varphi, h)$. Then , by definition,

$$(1-\delta)\frac{z(\varphi * f)'}{(\varphi * f)} + \delta\frac{\left(z(\varphi * f)'\right)'}{(\varphi * f)'} = s_1(z) \prec h(z),$$

from previous theorem, we can write

$$\frac{z(\varphi * f)'}{(\varphi * f)} = s_2(z) \prec h(z).$$

Now,

$$\begin{aligned}\delta\frac{\left(z(\varphi * f)'\right)'}{(\varphi * f)'} &= \left[(1-\delta)\frac{z(\varphi * f)'}{(\varphi * f)} + \delta\frac{\left(z(\varphi * f)'\right)'}{(\varphi * f)'}\right] + (\delta - 1)\frac{z(\varphi * f)'}{(\varphi * f)} \\ &= s_1(z) + (\delta - 1)s_2(z).\end{aligned}$$

Implies that

$$\frac{\left(z(\varphi * f)'\right)'}{(\varphi * f)'} = \left(1 - \frac{1}{\delta}\right)s_2(z) + \frac{1}{\delta}s_1(z). \tag{3.5}$$

Since $s_1, s_2 \prec h(z)$, (3.5) gives us

$$\frac{\left(z(\varphi * f)'\right)'}{(\varphi * f)'} \prec h(z).$$

Hence $f \in M_1^{\delta}(\varphi, h)$. □

Remark 3.2. *The different choices of φ and h given in Remark 3.1 hold the inclusion result proved in above theorem.*

Theorem 3.2. *Let δ, $\mu \geq 0$, $\varphi \in \mathcal{A}$, $\mathcal{H} = (h_1, h_2)$ where $h_i, h \in \mathcal{A}$ with $h_i(0) = h(0) = 1$ $(i = 1, 2)$. Then*

$$\mathcal{Q}_{\mu}^{\delta}(\varphi, h, \mathcal{H}) \subset \mathcal{Q}_{\mu}^{0}(\varphi, h, \mathcal{H}).$$

Proof. Let $f \in \mathcal{Q}_{\mu}^{\delta}(\varphi, h, \mathcal{H})$. Then, by definition,

$$(1-\delta)\frac{z(\varphi * f)'}{(\varphi * g)} + \delta\frac{\left(z(\varphi * f)'\right)'}{(\varphi * g)'} \in \mathcal{P}_{\mu}(\mathcal{H}), \tag{3.6}$$

for $g \in S^*(\varphi, h)$.

Consider

$$\frac{z(\varphi * f)'}{(\varphi * g)} = p(z), \tag{3.7}$$

where $p(z)$ is analytic with $p(0) = 1$ in $\mathcal{U}$.

On logarithmic differentiation of (3.7), we get

$$\frac{\left(z\left(\varphi * f\right)'\right)'}{\left(\varphi * f\right)'} = \frac{z\left(\varphi * g\right)'}{\left(\varphi * g\right)} + \frac{zp'(z)}{p(z)},$$

$$\frac{\left(z\left(\varphi * f\right)'\right)'}{\left(\varphi * g\right)'} = \frac{z\left(\varphi * f\right)'}{\left(\varphi * g\right)'}\left[\frac{z\left(\varphi * g\right)'}{\left(\varphi * g\right)} + \frac{zp'(z)}{\frac{z(\varphi * f)'}{(\varphi * g)}}\right],$$

this implies

$$\frac{\left(z\left(\varphi * f\right)'\right)'}{\left(\varphi * g\right)'} = \frac{z\left(\varphi * f\right)'}{\left(\varphi * g\right)} + \frac{zp'(z)}{\frac{z(\varphi * g)'}{(\varphi * g)}}. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\frac{\left(z\left(\varphi * f\right)'\right)'}{\left(\varphi * g\right)'} = p(z) + \frac{zp'(z)}{p_0(z)}; \text{ with } p_0(z) = \frac{z\left(\varphi * g\right)'}{\left(\varphi * g\right)}. \tag{3.9}$$

Now, from (3.6), (3.7) and (3.9), we obtain

$$(1-\delta)\, p(z) + \delta\left(p(z) + \frac{zp'(z)}{p_0(z)}\right) \in \mathcal{P}_\mu(\mathcal{H}),$$

or equivalently,

$$p(z) + \frac{\delta}{p_0(z)} zp'(z) \in \mathcal{P}_\mu(\mathcal{H}).$$

If $g \in S^*(\varphi, h)$, then $\frac{z(\varphi * g)'}{(\varphi * g)} \prec h(z)$; $h \in \mathcal{P}$. This implies $\Re(p_0(z)) > 0$ in $\mathcal{U}$. Thus, by Lemma 2.1, we conclude $p(z) \in \mathcal{P}_\mu(\mathcal{H})$. Consequently, $\frac{z(\varphi * f)'}{(\varphi * g)} \in \mathcal{P}_\mu(\mathcal{H})$. Hence, $f \in \mathcal{Q}_\mu^0(\varphi, h, \mathcal{H})$. □

Remark 3.3. *It is easy to see that the inclusion in Theorem 3.2 is true for different choices of φ, h and $\mathcal{H} = (h_1, h_2)$ given as following.*

(i) $\varphi \in A$, $h_1(z) = \frac{1+Az}{1+Bz} = h_2(z)$, where $-1 \leq B < A \leq 1$.

(ii) $\varphi \in A$, $h_1(z) = p_k(z) = h_2(z)$, where $p_k(z)$ is given by (1.3).

(iii) $\varphi \in A$, $h_1(z) = \frac{1+Az}{1+Bz}$, $\quad h_2(z) = p_k(z)$.

(iv) $\varphi \in A$, $h_1(z) = p_k(z)$, $\ h_2(z) = \frac{1+Az}{1+Bz}$.

Corollary 3.2. *Let $\delta \geq 1$. Then*

$$\mathcal{Q}_\mu^\delta(\varphi, h, \mathcal{H}) \subset \mathcal{Q}_\mu^1(\varphi, h, \mathcal{H}).$$

Proof. Let $f \in \mathcal{Q}_\mu^\delta(\varphi, h, \mathcal{H})$. Then, by definition,

$$(1-\delta)\frac{z\left(\varphi * f\right)'}{\left(\varphi * g\right)} + \delta\frac{\left(z\left(\varphi * f\right)'\right)'}{\left(\varphi * g\right)'} = p_1(z) \in \mathcal{P}_\mu(\mathcal{H}),$$

where $g \in S^*(\varphi, h)$.

From previous theorem, we can write

$$\frac{z\left(\varphi * f\right)'}{\left(\varphi * g\right)} = p_2(z) \in \mathcal{P}_\mu(\mathcal{H}).$$

Now,

$$\begin{aligned}\delta\frac{\left(z\left(\varphi*f\right)'\right)'}{\left(\varphi*g\right)'} &= \left[\left(1-\delta\right)\frac{z\left(\varphi*f\right)'}{\left(\varphi*g\right)}+\delta\frac{\left(z\left(\varphi*f\right)'\right)'}{\left(\varphi*g\right)'}\right]+\left(\delta-1\right)\frac{z\left(\varphi*f\right)'}{\left(\varphi*g\right)} \\ &= p_1(z)+\left(\delta-1\right)p_2(z).\end{aligned}$$

This implies

$$\frac{\left(z\left(\varphi*f\right)'\right)'}{\left(\varphi*g\right)'}=\left(1-\frac{1}{\delta}\right)p_2(z)+\frac{1}{\delta}p_1(z).$$

Since $p_1,\ p_2\in\mathcal{P}_\mu(\mathcal{H})$ and $\mathcal{P}_\mu(\mathcal{H})$ is convex set, then

$$\frac{\left(z\left(\varphi*f\right)'\right)'}{\left(\varphi*g\right)'}\in\mathcal{P}_\mu(\mathcal{H}).$$

Hence $f\in\mathcal{Q}^1_\mu\left(\varphi,h,\mathcal{H}\right)$. □

Theorem 3.3. *Let* $0\le\delta_1<\delta$. *Then*

$$\mathcal{Q}^\delta_\mu\left(\varphi,h,\mathcal{H}\right)\subset\mathcal{Q}^{\delta_1}_\mu\left(\varphi,h,\mathcal{H}\right).$$

Proof. If $\delta_1=0$, then it is obvious from Theorem 3.2.

For $\delta_1>0$. Let $f\in\mathcal{Q}^\delta_\mu\left(\varphi,h,H\right)$. Then, from Theorem 3.2

$$\frac{z\left(\varphi*f\right)'}{\left(\varphi*g\right)}=p_2(z)\in\mathcal{P}_\mu(\mathcal{H}). \tag{3.10}$$

As we can write

$$\begin{aligned}&\left(1-\delta_1\right)\frac{z\left(\varphi*f\right)'}{\left(\varphi*g\right)}+\delta_1\frac{\left(z\left(\varphi*f\right)'\right)'}{\left(\varphi*g\right)'} \\ =\ &\frac{\delta_1}{\delta}\left[\left(\frac{\delta}{\delta_1}-1\right)\frac{z\left(\varphi*f\right)'}{\left(\varphi*g\right)}+\left(1-\delta\right)\frac{z\left(\varphi*f\right)'}{\left(\varphi*g\right)}+\delta\frac{\left(z\left(\varphi*f\right)'\right)'}{\left(\varphi*g\right)'}\right].\end{aligned} \tag{3.11}$$

Since $f\in\mathcal{Q}^\delta_\mu\left(\varphi,h,\mathcal{H}\right)$, from definition of $\mathcal{Q}^\delta_\mu\left(\varphi,h,\mathcal{H}\right)$, we have

$$\left(1-\delta\right)\frac{z\left(\varphi*f\right)'}{\left(\varphi*g\right)}+\delta\frac{\left(z\left(\varphi*f\right)'\right)'}{\left(\varphi*g\right)'}=p_1(z)\in\mathcal{P}_\mu(\mathcal{H}). \tag{3.12}$$

From (3.10-3.12) and the convexity of $\mathcal{P}_\mu(\mathcal{H})$ implies

$$\left(1-\delta_1\right)\frac{z\left(\varphi*f\right)'}{\left(\varphi*g\right)}+\delta_1\frac{\left(z\left(\varphi*f\right)'\right)'}{\left(\varphi*g\right)'}\in\mathcal{P}_\mu(\mathcal{H}).$$

Hence $f\in\mathcal{Q}^{\delta_1}_\mu\left(\varphi,h,\mathcal{H}\right)$. □

Remark 3.4. *It is easy to see that the inclusion in Theorem 3.3 is true for all choices given in Remark 3.3.*

Theorem 3.4. *The class* $\mathcal{Q}^\delta_\mu\left(\varphi,h,\mathcal{H}\right)$ *is closed under the convex convolution.*

Proof. Let $f \in \mathcal{Q}_{\mu}^{\delta}(\varphi, h, \mathcal{H})$. Then, by definition,

$$(1-\delta)\frac{z(\varphi * f)'}{(\varphi * g)} + \delta\frac{\left(z(\varphi * f)'\right)'}{(\varphi * g)'} \in \mathcal{P}_{\mu}(\mathcal{H}). \tag{3.13}$$

First, we need to prove $\varsigma * f \in \mathcal{Q}_{\mu}^{0}(\varphi, h, \mathcal{H})$ for $\varsigma \in C$.

We take $\delta = 0$, then (3.13) implies

$$\frac{z(\varphi * f)'}{(\varphi * g)} \in \mathcal{P}_{\mu}(\mathcal{H}). \tag{3.14}$$

Let

$$\begin{aligned} \frac{z(\varphi * (\varsigma * f))'(z)}{(\varphi * (\varsigma * g))(z)} &= \frac{\varsigma * \frac{z(\varphi * f)'}{(\varphi * g)}((\varphi * g))(z)}{\varsigma * (\varphi * g)(z)} \\ &= \frac{\varsigma * h_0(z)((\varphi * g))(z)}{\varsigma * (\varphi * g)(z)}, \end{aligned}$$

where $h_0(z) = \frac{z(\varphi * f)'}{(\varphi * g)} \in \mathcal{P}_{\mu}(\mathcal{H})$. Since $g \in S^*(\varphi, h)$ implies $\varphi * g \in S^*(h) \subset S^*$; $h \in \mathcal{P}$. Thus, by Lemma 2.3, we conclude

$$\frac{z(\varphi * (\varsigma * f))'(z)}{(\varphi * (\varsigma * g))(z)} \in \mathcal{P}_{\mu}(\mathcal{H}). \tag{3.15}$$

Similarly, for $\delta = 1$, we can easily prove

$$\frac{z\left(\varphi * (\varsigma * f)'\right)'(z)}{(\varphi * (\varsigma * g))'(z)} \in \mathcal{P}_{\mu}(\mathcal{H}). \tag{3.16}$$

Our required result follows from (3.15) and (3.16). □

Corollary 3.3. *The class $\mathcal{Q}_{\mu}^{\delta}(\varphi, h, \mathcal{H})$ is closed under the following operators.*

(i) $f_1(z) = \int_0^z \frac{f(t)}{t}dt$.
(ii) $f_2(z) = \frac{2}{z}\int_0^z f(t)dt$, (Libera's operator [7]).
(iii) $f_3(z) = \int_0^z \frac{f(t)-f(xt)}{t-xt}dt$, $|x| \le 1$, $x \neq 1$.
(iv) $f_4(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)$, $Re(c) \ge 0$, (Generalized Bernardi operator [1]).

Proof. We may write, $f_i(z) = f(z) * \phi_i(z)$, where $\phi_i(z)$, $i = 1, 2, 3, 4$, are convex and given by

$\phi_1(z) = -\log(1-z) = \sum\limits_{n=1}^{\infty}\frac{1}{n}z^n$,

$\phi_2(z) = \frac{-2[z-\log(1-z)]}{z} = \sum\limits_{n=1}^{\infty}\frac{2}{n+1}z^n$,

$\phi_3(z) = \frac{1}{1-x}\log\left(\frac{1-xz}{1-z}\right) = \sum\limits_{n=1}^{\infty}\frac{1-x^n}{(1-x)^n}z^n$, $|x| \le 1$, $x \neq 1$,

$\phi_4(z) = \sum\limits_{n=1}^{\infty}\frac{1+c}{n+c}z^n$, $Re(c) \ge 0$.

The proof follows easily by using Theorem 3.4. □

3.2. Radius Problem.

Theorem 3.5. *Let* $f \in M_1^0\left(\varphi, \frac{1+Az}{1+Bz}\right)$*. Then,* $f \in M_1^\delta\left(\varphi, \frac{1+z}{1-z}\right)$ *for* $|z| < r_\delta$*, where*

$$r_\delta = \frac{2A^2}{\{\delta(A-B)+2A\} + \sqrt{\delta^2(A-B)^2 + 4A\delta(A-B)}}.$$

Proof. Let $f \in M_1^0\left(\varphi, \frac{1+Az}{1+Bz}\right)$. Then, by definition,

$$\frac{z(\varphi * f)'}{(\varphi * f)} = p(z) \prec \frac{1+Az}{1+Bz}. \tag{3.17}$$

On logrithmic differentiation of (3.17), we get

$$\frac{\left(z(\varphi * f)'\right)'}{(\varphi * f)'} = \frac{z(\varphi * f)'}{(\varphi * f)} + \frac{zp'(z)}{p(z)}. \tag{3.18}$$

By (3.17) and (3.18), we obtain

$$\frac{\left(z(\varphi * f)'\right)'}{(\varphi * f)'} = p(z) + \frac{zp'(z)}{p(z)}. \tag{3.19}$$

Now,

$$(1-\delta)\frac{z(\varphi * f)'}{(\varphi * g)} + \delta\frac{\left(z(\varphi * f)'\right)'}{(\varphi * g)'} = p(z) + \delta\frac{zp'(z)}{p(z)}.$$

$$\Re\left(J_\delta(f(z))\right) \geq \frac{A^2r^2 - \{\delta(A-B)+2A\}r + 1}{(1-Ar)(1-Br)}.$$

For $\Re\left(J_\delta(f(z))\right) > 0$ in $\mathcal{U}$, we get

$$r_\delta = \frac{2A^2}{\{\delta(A-B)+2A\} + \sqrt{\delta^2(A-B)^2 + 4A\delta(A-B)}}.$$

□

Corollary 3.4. *Let* $f \in M_1^0\left(\frac{z}{1-z}, \frac{1+z}{1-z}\right) = S^*$*. Then*

$$f \in M_1^\delta\left(\frac{z}{1-z}, \frac{1+z}{1-z}\right) = M(\delta),$$

for $|z| < r_\delta = \frac{1}{(1+\delta)+\sqrt{\delta^2+2\delta}}$*. Moreover, for* $\delta = 1$*, we have well known result*

$$S^* \subset C, \textit{ for } |z| < r_1 = \frac{1}{2+\sqrt{3}}.$$

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