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GENERALIZED CLOSE-TO-CONVEXITY RELATED WITH BOUNDED BOUNDARY ROTATION

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Abstract. The class $P_{\alpha,m}[A,B]$ consists of functions p, analytic in the open unit disc E with $p(0)=1$ and satisfy

$$p(z)=\left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z),\quad m\geq 2,$$

and p_1, p_2 are subordinate to strongly Janowski function $\left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, $\alpha\in(0,1]$ and $-1\leq B<A\leq 1$. The class $P_{\alpha,m}[A,B]$ is used to define $V_{\alpha,m}[A,B]$ and $T_{\alpha,m}[A,B;0;B_1]$, $B_1\in[-1,0)$. These classes generalize the concept of bounded boundary rotation and strongly close-to-convexity, respectively. In this paper, we study coefficient bounds, radius problem and several other interesting properties of these functions. Special cases and consequences of main results are also deduced.

1. Introduction

Let A denote the class of analytic functions defined in the open unit disc $E=\{z:|z|<1\}$ and be given by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad z\in E. \tag{1.1}$$

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Let $S \subset A$ be the class of univalent functions in E and let C, $S^\star$ and K be the subclasses of S consisting of convex, starlike and close-to-convex functions, respectively. For details, see [3].

For $f, g \in A$, we say f is subordinate to g in E, written as $f(z) \prec g(z)$, if there exists a Schwartz function $w(z)$ such that

$$f(z) = g(w(z)), \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Furthermore, if the function g is univalent in E, then we have the following equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

Convolution of f and g is defined as

$$(g * f)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The class $P_\alpha[A, B]$ of strongly Janowski functions is defined as follows.

Definition 1.1. *Let p be analytic in E with $p(0) = 1$. Then $p \in P_\alpha[A, B]$, if $p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^\alpha$, where $\alpha \in (0, 1]$ and $-1 \le B < A \le 1$.*

We denote $P_\alpha[0, B_1]$ as $P_\alpha[B_1]$, $-1 \le B_1 < 0$.

The class $P_\alpha[A, B]$ is generalized as:

Definition 1.2. *An analytic function $p : p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is in the class $P_{\alpha,m}[A, B]$, if and only if, there exist $p_1, p_2 \in P_\alpha[A, B]$ such that*

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), \quad m \ge 2. \tag{1.2}$$

It is obvious $P_{\alpha,2}[A, B] = P_\alpha[A, B]$. For the class $P_1[A, B] = P[A, B]$, we refer to [6].

About the class $P_\alpha[A, B]$, we observe the following.

(i) $p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^\alpha$ implies $p \in P_\alpha[A, B]$ and it can easily be shown that $\phi_\alpha(A, B; z) = \left(\frac{1+Az}{1+Bz}\right)^\alpha$ is convex univalent in E. In fact simple calculation yield that

$$Re\phi'_\alpha(A, B; z) \ge \alpha|A - B|\frac{(1 - |A|)^{\alpha-1}}{(1 - |B|)^{\alpha+1}} > 0, \quad z \in E.$$

This shows $\phi_\alpha(A, B; z)$ is univalent in E.

Also

$$Re\left\{\frac{(z\phi'_\alpha(A, B; z))'}{\phi'_\alpha(A, B; z)}\right\} \ge \frac{T(r)}{(1 + Ar)(1 + Br)},$$

where

$$T(r) = 1 - \alpha(A - B)r - ABr^2$$

is decreasing on $(0,1)$ and $T(0)=1$.

This implies $Re\left[\frac{(z\phi'_\alpha(A,B;z))'}{\phi'_\alpha(A,B;z)}\right] \geq 0$ in E.

(ii) For $A=1$, $B=-1$, $p \in \phi_\alpha(1,-1;z)$ implies

$$\left|\arg p(z)\right| \leq \frac{\alpha\pi}{2}, \quad z \in E.$$

Definition 1.3. *Let $f, g \in A$, $\frac{(g*f)'(z)}{z} \neq 0$, $z \in E$. Then $f \in V_{\alpha,m}[A,B;g]$, if and only if,*

$$\frac{(z(g*f)')'}{(g*f)'} \in P_{\alpha,m}[A,B], \quad z \in E,$$

with $F = zf'$, $F \in R_{\alpha,m}[A,B;g]$, if and only if, $f \in V_{\alpha,m}[A,B;g]$ in E.

Special Cases.

(i) $V_{1,m}[A,B;\frac{z}{1-z}] = V_m[A,B] \subset V_m[1,-1] = V_m$, where V_m is the well known class of functions of bounded boundary rotation. See, for example, [2,10,12].

(ii) $R_{1,m}[A,B;\frac{z}{1-z}] = R_m[A,B] \subset R_m$ and R_m is the class of functions with bounded radius rotation, see [9].

(iii) $V_{\alpha,m}[A,B;\frac{z}{(1-z)^2}] = R_{\alpha,m}[A,B;\frac{z}{1-z}] = R_{\alpha,m}[A,B]$.

Definition 1.4. *Let $f, g \in A$ with $(g*f)(z) \neq 0$. Then $f \in T_{\alpha,m}[A,B;0;B_1;g]$, if there exists $\psi \in V_{\alpha,m}[A,B;g]$ such that, for $B_1 \in [-1,0)$,*

$$\frac{(g*f)'}{(g*\psi)'} \in P_\alpha[B_1], \quad z \in E.$$

We note that $T_1[A,B;0;-1;\frac{z}{1-z}] = T_m[A,B]$. For certain special cases, see [8,11,12].

2. The class $V_{\alpha,m}[A,B;g]$

Theorem 2.1. *Let $f \in V_{\alpha,m}[A,B;g]$ and let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then, with f given by (1.1), $A_n = a_n b_n$,*

$$A_n = O(1)n^\sigma, \quad \sigma = \left\{\left(\frac{m}{2}+1\right)(1-\rho) - (\rho+2)\right\},$$

where $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$, $m \geq \frac{2(1+\rho)}{1-\rho}$ and $O(1)$ denotes a constant.

Proof. Let $F = f*g$. Then $F \in V_{\alpha,m}[A,B]$. Since $p \in P_\alpha[A,B]$ implies $Re\, p(z) > \rho$, $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$, it follows that $V_{\alpha,m}[A,B] \subset V_m(\rho)$.

Now, $F \in V_m(\rho)$, we can write

$$F'_1(z) = \left(F'_1(z)\right)^{1-\rho}, \quad F_1 \in V_m, \tag{2.1}$$

see [13].

Using a result due to Brannan [2], we can write

$$zF_1'(z) = \frac{(s_1(z))^{\left(\frac{m}{4}+\frac{1}{2}\right)(1-\rho)}}{(s_2(z))^{\left(\frac{m}{4}-\frac{1}{2}\right)(1-\rho)}}, \quad s_1, s_2 \in S^\star. \tag{2.2}$$

Therefore, from (2.1), (2.2) and Cauchy Theorem with $z = re^{\iota\theta}$, we have

$$\begin{aligned} n^2|A_n| &\leq \frac{1}{2\pi r^n}\int_0^{2\pi} \left|F_1'(z)h(z)\right|^{1-\rho} d\theta, \quad h \in P_{\alpha,m}[A,B] \subset P_m(\rho) \\ &= \frac{1}{2\pi r^{n+1}}\int_0^{2\pi} \frac{|s_1(z)|^{\left(\frac{m}{4}+\frac{1}{2}\right)(1-\rho)}}{|s_2(z)|^{\left(\frac{m}{4}-\frac{1}{2}\right)(1-\rho)}} \cdot |h(z)|^{1-\rho} d\theta. \end{aligned} \tag{2.3}$$

Applying distortion result for $s_2 \in S^\star$ and Holder's inequality in (2.3), we get

$$\begin{aligned} n^2|A_n| \leq \frac{1}{r^{n+1}}\left(\frac{4}{r}\right)^{\left(\frac{m}{4}-\frac{1}{2}\right)(1-\rho)} &\left(\frac{1}{2\pi}\int_0^{2\pi} \left|s_1(z)\right|^{\left\{\left(\frac{m}{4}+\frac{1}{2}\right)(1-\rho)\right\}\frac{2}{1+\rho}} d\theta\right)^{\frac{1+\rho}{2}} \\ &\cdot \left(\frac{1}{2\pi}\int_0^{2\pi} \left|h(z)\right|^2 d\theta\right)^{\frac{1-\rho}{2}} \end{aligned} \tag{2.4}$$

Now, for $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, we use Parsval identity to have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \left|h(z)\right|^2 d\theta &= \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \\ &\leq 1 + m^2(1-\rho)^2 \sum_{n=1}^{\infty} r^{2n} \\ &= \frac{1 + [m^2(1-\rho)^2 - 1]r^2}{1-r^2}, \end{aligned} \tag{2.5}$$

where we have used coefficient bounds $|c_n| \leq m(1-\rho)$, for $h \in P_m(\rho)$.

From (2.5) together with subordination for starlike functions, and a result due to Hayman [5] for $m \geq \frac{2(1+\rho)}{1-\rho}$, we have

$$n^2|A_n| \leq c_1(m,\rho)\left(\frac{1}{1-r}\right)^{\left\{\left(\frac{m}{2}+1\right)(1-\rho)\right\}-\rho}, \tag{2.6}$$

where $c_1(m,\rho)$ denotes a constant.

Taking $r = 1 - \frac{1}{n}$ in (2.6), we obtain the required result. □

Special Cases.

(i) Let $g(z) = \frac{z}{1-z}$, then $A_n = a_n$. Take $A = 0$, and in this case $f \in V_m$. This leads us to a known coefficient result that $a_n = O(1)n^{\left(\frac{m}{2}-1\right)}$.

(ii) Let $f \in V_{1,m}\left[0, -1, \frac{z}{(1-z)^2}\right] = R_m\left(\frac{1}{2}\right)$. Then $a_n = O(1)n^{\frac{m}{4}-2}$, $m \geq 6$.

Theorem 2.2. *Let $f \in V_{\alpha,m}[A,B;g]$. Then, for $F = f * g$, $z = re^{\iota\theta}$,*
$0 \le \theta_1 < \theta_2 \le 2\pi$, we have

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zF'(z))'}{F'(z)}\right\} d\theta > -\left(\frac{m}{2}-1\right)(1-\rho)\pi, \quad \rho = \left(\frac{1-A}{1-B}\right)^{\alpha}. \tag{2.7}$$

Proof. Proof is straight forward, since $V_{\alpha,m}[A,B] \subset V_m(\rho)$ and $F \in V_m(\rho)$ implies there exist $F_1 \in V_m$ with $F'(z) = (F_1'(z))^{(1-\rho)}$. Now, using essentially the same method given in [2], the required result follows. □

Remark 2.1. *Let $\beta\left(\frac{m}{2}-1\right)(1-\rho)$. Then, from a result of Goodman [4] and from (2.7), it follows that $F = f * g \in V_{\alpha,m}[A,B]$ is univalent for $\beta = \left(\frac{m}{2}-1\right)(1-\rho) \le 1$. That is $F \in S$ for $m \le \frac{2(2-\rho)}{1-\rho}$. As a special case, with $g(z) = \frac{z}{1-z}$, $A = 0$, $B = -1$ and $\alpha = 1$, we have $F = f$, $\rho = \frac{1}{2}$. Then $f \in V_{1,m}[0,-1]$ implies*

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zF'(z))'}{F'(z)}\right\} d\theta > -\left(\frac{m}{4}-\frac{1}{2}\right)\pi$$

For this, we can conclude that

$$V_{1,m}[0,-1] \subset S \quad for \quad 2 \le m \le 6.$$

Also, with $g(z) = \frac{z}{1-z}$, $A = 1$, $B = -1$, we have a well known result that $f \in V_m$ is univalent for $2 \le m \le 4$.

Theorem 2.3. *Let $f \in V_{\alpha,m}[A,B;g]$, $m \le \frac{2(2-\rho)}{1-\rho}$ and $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$. Then $F(E)$ with $F = f * g$, contains the disc d:*

$$d = \left\{w : |w| < \frac{4}{8 + \alpha m|A-B|}\right\}$$

Proof. From Theorem 2.2, F is univalent in E. Let w_0 $(w_0 \ne 0)$ be any complex number such that $F(z) \ne w_0$ for $z \in E$. Then the function

$$F_1(z) = \frac{w_0 F(z)}{w_0 - F(z)} = z + \left(A_2 + \frac{1}{w_0}\right) z^2 + \dots$$

is analytic and univalent in E. Using the well known Bieberbach Theorem for the best bound for second coefficient of univalent functions, see [3], we have

$$\frac{1}{|w_0|} - |A_2| \le \left|A_2 + \frac{1}{w_0}\right| \le 2.$$

This gives us

$$\begin{aligned} \frac{1}{|w_0|} &\le 2 + |A_2| \\ &\le 2 + \frac{\alpha_m|A-B|}{4} = \frac{8 + \alpha_m|A-B|}{4}. \end{aligned}$$

This completes the proof. □

Special Cases.

(i) Let $A = 1,\ B = -1,\ \alpha = 1$; $(\rho = 0)$ and so $F(E)$ contains the disc $|w| < \frac{2}{4+m},\ m \leq 4$.

(ii) With $A = 0,\ B = -1,\ \alpha = \frac{1}{2}$, we have $\rho = \frac{1}{4}$, and $F(E)$ contains the disc $|w| < \frac{8}{16+m},\ m \leq \frac{14}{3}$.

The following properties of the class $V_{\alpha,m}[A, B; g]$ can easily be proved with simple computations and well known results and therefore we omit the proof.

Theorem 2.4. (i) *The class $V_{\alpha,m}[A, B; g]$ is preserved under the integral operator $L : A \to A$ defined as*

$$L(z) = \int_0^z (L_1'(\xi))^{\beta} (L_2'(\xi))^{\gamma} d\xi,$$

where $L_i \in V_{\alpha,m}[A, B; g],\ i = 1, 2$ and β, γ are positively real with $\beta + \gamma = 1$.

(ii) *Let $f \in V_{\alpha,m}\left[A, B; \frac{z}{1-z}\right]$. Then, with $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$, $z \in E$ and $z = re^{\iota\theta}$, we have*

$$\frac{(1 - Br)^{(1-\rho)\left(\frac{m}{4}+\frac{1}{2}\right)}}{(1 + Br)^{(1-\rho)\left(\frac{m}{4}-\frac{1}{2}\right)}} \leq |f'(z)| \leq \frac{(1 + Br)^{(1-\rho)\left(\frac{m}{4}+\frac{1}{2}\right)}}{(1 - Br)^{(1-\rho)\left(\frac{m}{4}-\frac{1}{2}\right)}}.$$

For $\alpha = 1,\ f \in V_m[A, B]$ and $A = 1,\ B = -1$,the result reduces to $f \in V_m$ studied in [2].

(iii) *Let $f \in V_{\alpha,2}\left[A, B; \frac{z}{1-z}\right]$ and define $F \in A$ as*

$$F(z) = \frac{\beta + 1}{z^{\beta}} \int_0^z t^{\beta-1} f(t)dt, \beta > 0.$$

Then F is convex of order $\gamma(\rho)$, $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$, where

$$\gamma = \gamma(\rho) = \left\{ \frac{(\beta + 1)}{{}_2F_1\left(2(1 - \rho), 1; (\beta + 2); \frac{1}{2}\right)} - \beta \right\},$$

${}_2F_1$ represents Gauss hypergeometric function.

(iv) *The set of all points $\log f'(z)$ for a fixed $z \in E$ and f ranging over the class $V_{\alpha,m}[A, B; g]$ is convex.*

(v) *Let $f \in V_{\alpha,m}\left[A, B; \frac{z}{1-z}\right]$, $B \neq 0$. Then f is close-to-convex for $|z| < r_1$, where*

$$r_1 = \left\{ \sin\left(\frac{\pi}{B(\gamma - 2)}\right), \quad B \neq 0, \quad m > \frac{2}{\gamma}, \quad \gamma = 1 - \left(\frac{1 - A}{1 - B}\right)^{\alpha} \right.$$

(vi) *Let $f \in V_{\alpha,m}[A, B; g]$, and let $F = f * g$. Then F is convex of order $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$ for $|z| < r_m$, where*

$$r(m) = \frac{m - \sqrt{m^2 - 4}}{2}, \quad m \geq 2.$$

Theorem 2.5. *Let $f_1, f_2 \in V_{\alpha,m}[A, B; g]$, $\beta,\ \delta,\ c$ and ν be positively real, $c \geq \beta \geq 1$, $(\nu + \delta) = \beta$. Let $F = F_1 * g$, $G_i = f_i * g$, $i = 1, 2$ and define*

$$[F(z)]^{\beta} = cz^{(\beta-c)} \int_0^z t^{c-1} (G_1'(t))^{\delta} (G_2'(t))^{\nu} dt. \tag{2.8}$$

Then, for $z = re^{\iota\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$, $\frac{zF'}{F} = p$, *we have*

$$\int_{\theta_1}^{\theta_2} Re\left\{p(z) + \frac{\frac{1}{\beta}zp'(z)}{p(z)+\frac{1}{\beta}(c-\beta)}\right\}d\theta > -(1-\rho)\left(\frac{m}{2}-1\right)\pi, \quad \rho = \left(\frac{1-A}{1-B}\right)^{\alpha}.$$

Proof. First we show that there exists a function $F \in A$ satisfying (2.8). We assume $F_1 * g \neq 0$, $f_i * g \neq 0$, $z \in E$. Let

$$Q(z) = (G_1'(z))^{\delta}(G_2'(z))^{\nu} = 1 + d_1 z + d_2 z^2 + \ldots$$

and choose the branches which equal 1, when $z = 0$.

For $K(z) = z^{c-1}(G_1'(z))^{\delta}(G_2'(z))^{\nu} = z^{c-1}Q(z)$, we have

$$N(z) = \frac{c}{z^c}\int_0^z K(t)dt = 1 + \frac{c}{c+1}d_1 z + \ldots$$

Hence N is well defined and analytic.

Now let

$$F(z) = \left[z^{\beta}N(z)\right]^{\frac{1}{\beta}} = z\left[N(z)\right]^{\frac{1}{\beta}},$$

where we choose the branch of $[N(z)]^{\frac{1}{\beta}}$ which equal 1 when $z = 0$. Thus $F \in A$ and satisfies (2.8). We write

$$\frac{zF'(z)}{F(z)} = p(z), \quad F = F_1 * g. \tag{2.9}$$

From (2.8) and (2.9) with some calculations

$$\beta p(z) + \frac{\beta zp'(z)}{(c-\beta)+\beta p(z)} = \delta\left[\frac{(zG_1'(z))'}{G_1'(z)}\right] + \nu\left[\frac{(zG_1'(z))'}{G_2'(z)}\right].$$

That is

$$p(z) + \frac{\frac{1}{\beta}zp'(z)}{p(z)+\frac{1}{\beta}(c-\beta)} = \frac{]}{\delta}\beta\left[\frac{(zG_1'(z))'}{G_1'(z)}\right] + \frac{\nu}{\beta}\left[\frac{(zG_1'(z))'}{G_2'(z)}\right].$$

We now apply Theorem 2.2 and obtain the required result.

For $m \le \frac{2(2-\rho)}{1-\rho}$ and applying a result proved in [14], it can easily be deduced that

$$\int_{\theta_1}^{\theta_2} Re\left\{p(z)\right\}d\theta > -\pi, \quad p(z) = \frac{z(F_1 * g)'}{F_1 * g}.$$

Taking $g(z) = \frac{z}{(1-z)^2}$, it follows that $F_1 \in S$ in E, see [4]. □

3. The class $T_{\alpha,m}[A,B;0;B_1;g]$

Theorem 3.1. *Let* $f \in T_{\alpha,m}\left[A,B;0;B_1;\frac{z}{1-z}\right] = T_{\alpha,m}[A,B;0;B_1]$. *Then, for* $z = re^{\iota\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$, $\rho_1 = \left(\frac{1}{2}\right)^{\alpha}$, $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$,

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zf'(z))'}{f'(z)}\right\}d\theta > -\beta\pi, \quad \beta = \left[(1-\rho_1) + \left(\frac{m}{2}-1\right)(1-\rho)\right].$$

Proof. For $f \in T_{\alpha,m}[A,B;0;B_1]$, we can write

$$\frac{f'(z)}{\psi'(z)} = h(z), \quad \psi \in V_{\alpha,m}[A,B], \quad h \in P_\alpha[0,B_1].$$

To prove this result, we shall essentially follow the method due to Kaplan [4].
For $\psi \in V_{\alpha,m}[A,B]$, it implies that $\psi \in V_m(\rho)$, where $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$.
Also $h \in P_\alpha[0,B_1]$, $B_1 \in [-1,0)$ is equivalent to $h \prec \left(\frac{1}{1+B_1 z}\right)^\alpha$. That is,
$h \in P(\alpha_1) \subset P$, $\alpha_1 = \left(\frac{1}{2}\right)^\alpha$.
Now, with $z = re^{\iota\theta}$, write $p(z) = \arg f'(z)$ and $q(z) = \arg \psi'(z)$. Then

$$|p(z) - q(z)| < \left(1 - \left(\frac{1}{2}\right)^\alpha\right)\frac{\pi}{2} \tag{3.1}$$

Let $P(r,\theta) = p(re^{\iota\theta}) + \theta$, $\quad Q(r,\theta) = q(re^{\iota\theta}) + \theta$ be defined for $0 \le r < 1$ and for all θ. This gives us

$$|P(r,\theta) - Q(r,\theta)| < \left(1 - \left(\frac{1}{2}\right)^\alpha\right)\frac{\pi}{2}. \tag{3.2}$$

From Theorem 2.2, for $\psi \in V_{\alpha,m}[A,B] \subset V_m(\rho)$, we have

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(z\psi'(z))'}{\psi'(z)}\right\} d\theta > -\left(\frac{m}{2} - 1\right)\left(1 - \left(\frac{1-A}{1-B}\right)^\alpha\right)\pi, \quad (z = re^{\iota\theta}).$$

Thus

$$|Q(r,\theta_1) - Q(r,\theta_2)| < \left(1 - \left(\frac{1-A}{1-B}\right)^\alpha\right)\left(\frac{m}{2} - 1\right)\pi. \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\begin{aligned}
&|P(r,\theta_1) - P(r,\theta_2)| \\
&= |\{P(r,\theta_1) - Q(r,\theta_1)\} - \{P(r,\theta_2) - Q(r,\theta_2)\} + \{Q(r,\theta_1) - Q(r,\theta_2)\}| \\
&< \left(1 - \left(\frac{1}{2}\right)^\alpha\right)\frac{\pi}{2} + \left(1 - \left(\frac{1}{2}\right)^\alpha\right)\frac{\pi}{2} + \left(1 - \left(\frac{1-A}{1-B}\right)^\alpha\right)\left(\frac{m}{2} - 1\right)\pi \\
&= \left[\left(1 - \left(\frac{1}{2}\right)^\alpha\right) + \left(1 - \left(\frac{1-A}{1-B}\right)^\alpha\right)\left(\frac{m}{2} - 1\right)\right]\pi \\
&= \left[(1 - \rho_1) + \left(\frac{m}{2} - 1\right)(1 - \rho)\right]\pi = \beta\pi,
\end{aligned}$$

and this proves our result. □

Special Cases.

(i) Let $\alpha = 1$, $A = 1$ and $B = -1$. Then $\beta = \frac{m-1}{2} = 1$ for $m = 3$. This implies $f \in T_{1,m}[1,-1;0;-1]$ is univalent for $2 \le m \le 3$.

(ii) For $A = 0$, $B = -1$, $\alpha = 1$ we have $\beta = \frac{m}{4}$ and, in this case, f is univalent for $2 \le m \le 4$.

Remark 3.1. *For $F \in A$, Goodman [4] introduced a class $K(\beta)$ as*

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zF'(z))'}{F'(z)}\right\} d\theta > -\beta\pi, \quad z = re^{\iota\theta}, \quad 0 \le \theta_1 < \theta_2 \le 2\pi,$$

and $\beta \ge 0$. When $0 \le \beta \le 1$, $K(\beta)$ consists of univalent functions (close-to-convex), whilest for $\beta > 1$, F need not even be finitely-valent, see [4].

We note that, for $\rho_1 = \left(\frac{1}{2}\right)^{\alpha}$, $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$.

$$T_{\alpha,m}[A, B, 0, B_1] \subset K\left(\frac{m}{2}(1-\rho) + (\rho - \rho_1)\right).$$

This implies $F \in T_{\alpha,m}[A, B; 0; -1]$ is univalent for $m \le 2\left[1 + \frac{\rho_1}{1-\rho}\right]$.

Theorem 3.2. *For $g(z) = \frac{z}{1-z}$, let $f \in T_{\alpha,2}[A, B, 0, B_1]$ and for $\gamma, \beta > 0$, let F_1 be defined by*

$$F_1(z) = \left[(1+\beta)z^{-\beta}\int_0^z t^{\beta-1} f^{\gamma}(t)dt\right]^{\frac{1}{\gamma}}. \tag{3.4}$$

Then $F_1 \in T_{1,2}[A, B; 0; B_1]$ in E.

Proof. We can write (3.4) as

$$F_1(z) = \left[\left(\frac{f(z)}{z}\right)^{\gamma} * \left(\frac{\phi_{\gamma,\beta}(z)}{z}\right)\right]^{\frac{1}{\gamma}}, \tag{3.5}$$

where

$$\phi_{\gamma,\beta}(z) = \sum_{n=1}^{\infty}\left(\frac{z^n}{n+\gamma+\beta}\right) \tag{3.6}$$

is convex in E.

Since $f \in T_{\alpha,2}[A, B; 0; B_1]$, there exists $\psi_1 = z\psi' \in R_{\alpha,2}[A, B]$ such that $\frac{f'}{\psi'} \in P_{\alpha}[0, B_1]$, $\psi = V_{\alpha,2}[A, B]$ in E. Let

$$G_1(z) = \left[(\beta+1)z^{-\beta}\int_0^z t^{\beta-1}\psi_1^{\gamma}(t)dt\right]^{\frac{1}{\gamma}}, \quad G_1 = zG'. \tag{3.7}$$

We first show that $G \in V_{\alpha,2}[A, B]$.

From (3.7), it follows that

$$\left\{z^{\beta}G_1^{\gamma}(z)\right\}' = z^{\beta-1}\left(\psi_1^{\gamma}(z)\right) \tag{3.8}$$

That is

$$\left(G_1^{\gamma}(z)\right)\left[\beta + \gamma H_1(z)\right] = \psi_1^{\gamma}(z), \quad H_1(z) = \frac{zG_1'(z)}{G_1(z)} \tag{3.9}$$

Logarithmic differentiation of (3.9) and simple computations give us

$$H_1(z) + \frac{zH_1'(z)}{\gamma H_1(z) + \beta} = \frac{z\psi_1'}{\psi_1(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \prec \left(\frac{1+Az}{1+Bz}\right). \tag{3.10}$$

Now, using Theorem 3.3 of [7, p: 109], It follows from (3.10) that $H_1 \in P[A,B]$ and $G_1 = zG'$ belongs to $R_{1,2}[A,B] = S^\star[A,B]$. Therefore $G \in V_{1,2}[A,B] = C[A,B]$.

From (3.4), we have

$$\frac{zF_1'(z)F_1^{\gamma-1}(z)}{G_1^\gamma(z)} = \frac{\phi_{\gamma,\beta}(z) * z\left(\frac{\psi_1(z)}{z}\right)^\gamma \left(zf'(z)\cdot\frac{f^{\gamma-1}(z)}{\psi_1^\gamma(z)}\right)}{\phi_{\gamma,\beta}(z) * z\left(\frac{\psi_1(z)}{z}\right)^\gamma}$$
$$= \frac{\phi_{\gamma,\beta}(z) * z\left(\frac{\psi_1(z)}{z}\right)^\gamma h(z)}{\phi_{\gamma,\beta}(z) * z\left(\frac{\psi_1(z)}{z}\right)^\gamma}, \quad h \in P_\alpha(B_1).$$

Since $h(z)$ is analytic in E, $h(0)=1$, and $\phi_{\gamma,\beta}(z)$ is convex, $\psi_1 \in S^\star$, we use a result due to Ruscheweyh and Sheil-Small [17] to conclude that $\left(\frac{zF_1'F^{\gamma-1}}{G_1^\gamma}\right)(E) \subset \bar{C}oh(E)$, where $\bar{C}oh(E)$ denotes convex hull of $h(E)$. This implies $F_1 \in T_{1,2}[A,B;0;B_1]$ in E. or $\gamma = 1$ in (3.4), we obtain the well known Bernardi integral operator, see [7]. □

Theorem 3.3. *Let $F = f * g$, $f \in T_{\alpha,m}[A,B;0;B]$, $B \neq 0$. Then with $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$ and $\gamma = \frac{A-B}{3B}$,*

(i)

$$\left(\frac{1}{1+Br}\right)^\alpha \frac{(1-Br)^{\gamma(1-\rho)(\frac{m}{4}+\frac{1}{2})}}{(1+Br)^{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})}} \le |F'(z)| \le \frac{(1+Br)^{\gamma(1-\rho)(\frac{m}{4}+\frac{1}{2})}}{(1-Br)^{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})}}\cdot\left(\frac{1}{1-Br}\right)^\alpha \tag{3.11}$$

(ii)

$$\frac{2^{\gamma(1-\rho)}}{a|B|}\left[G_{12}(a,b;c;-1) - r_1^{-a}G_{12}(a,b;c;-r_1)\right]$$
$$\le |F(z)|$$
$$\le \frac{2^{\gamma(1-\rho)}}{a|B|}\cdot\left[G_{12}(a,b;c;-1) - r_2^{-a}G_{12}(a,b;c;-r_2)\right],$$

where $r_1 = -r_2^{-1} = \frac{1+Br}{1-Br}$, $m \le \left[\frac{4(1-\alpha)}{\gamma(1-\rho)} + 2\right]$ and a is given in (3.16).

Proof. We can write for $F \in T_{\alpha,m}[A,B;0;B]$,

$$F'(z) = G'(z)h(z), \quad h \in P_\alpha[B_1], \quad G = \psi * g \in V_{\alpha,m}[A,B].$$

Since $h \in P_\alpha[B]$, it easily follows that

$$\left(\frac{1}{1+Br}\right)^\alpha \le |h(z)| \le \left(\frac{1}{1-Br}\right)^\alpha \tag{3.12}$$

From Theorem 2.4 (ii) and (3.12), the proof of (i) is established.

We know proceed to prove (ii).

Let d_r denote the radius of the largest schlicht disc centered at the origin contained in the image of $|z| < r$ under $F(z)$. Then there is a point z_0, $|z_0| = r$, such that $|F(z_0)| = d_r$. The ray from 0 to $F(z_0)$ lies entirely in the image and the inverse image of this ray is a curve in $|z| < r$.

Using (3.11), we have

$$d_r = |F(z_0)| = \int_C |F'(z)||dz|, \quad r = \frac{A-B}{2B}$$
$$\geq \int_0^{|z|} \left[\frac{(1-Bs)^{\gamma\{(1-\rho)(\frac{m}{4}+\frac{1}{2})\}}}{(1+Bs)^{\{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})+\alpha\}}} \right] ds$$
$$(3.13) \qquad = \int_0^{|z|} \left[\left(\frac{1-Bs}{1+Bs} \right)^{\{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})+\alpha\}} \cdot (1-Bs)^{\gamma(1-\rho-\alpha)} \right] ds,$$

Let $\frac{1+Bs}{1-Bs} = t$. Then $\frac{2B}{(1-Bs)^2} = dt$, and $1 - Bs = \frac{2}{1+t}$. This implies $ds = \frac{2}{B}\left(\frac{1}{1+t}\right)^2 dt$. Therefore, from (3.13), we have

$$|F(z_0)| \geq \int_1^{\frac{1+Br}{1-Br}} t^{-\{(1-\rho)(\frac{m}{4}-\frac{1}{2})-\alpha\}} \cdot \left(\frac{2}{1+t}\right)^{1-\rho-\alpha} \cdot \frac{2}{B}\left(\frac{1}{1+t}\right)^2 dt$$
$$= \frac{-2^{(1-\rho)}}{|B|} \left[\int_0^{\frac{1+Br}{1-Br}} t^{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})-\alpha} \cdot (1+t)^{\gamma(1-\rho-\alpha)} dt \right.$$
$$\left. - \int_0^1 t^{r(1-\rho)(\frac{m}{4}-\frac{1}{2})+\alpha} \cdot (1+t)^{r(1-\rho-\alpha)} dt \right]$$
$$(3.14) \qquad = \frac{2^{\gamma(1-\rho)}}{|B|}[I_1 + I_2].$$

Now put $t = r_1 u$ with $r_1 = \frac{1+Br}{1-Br}$. Then $dt = r_1 du$ and

$$I_1 = \int_0^1 (r_1 u)^{-\left[\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})-\alpha\right]} \cdot (1+r_1 u)^{1-\rho-\alpha} du$$
$$= r_1^{-\{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})+\alpha-1\}} \int_0^1 u^{-\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})-\alpha} \cdot (1+r_1 u)^{-\{\gamma(1-\rho)+\alpha\}} du$$
$$(3.15) \qquad = r_1^{-\{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})+\alpha-1\}} \cdot \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} G_{12}(a, b; c; -r_1),$$

where Γ and G_{12}, respectively denote gamma and Gauss hypergeometric functions. Also, here, b, c are positively real for $m \leq 2\left\{1 + \frac{2(1-\alpha)}{1-\rho}\right\}$ and are given as

$$a = -\gamma(1-\rho)\left(\frac{m}{4} - \frac{1}{2}\right) - \alpha + 1, \quad \gamma = \frac{A-B}{2B}, \quad B \neq 0$$
$$b = -\gamma(1-\rho) + \alpha,$$
$$(3.16) \qquad c = -\gamma(1-\rho)\left(\frac{m}{4} - \frac{1}{2}\right) - \alpha + 2, \quad (c-a) > 0.$$

Similarly, we calculate I_2 and have

$$(3.17) \qquad I_2 = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} G_{12}(a, b; c; -1).$$

Using (3.15), (3.16) and (3.17) in (3.14), we obtain the lower bound of $|F(z)|$. For the upper bound, we proceed in similar way and have

$$\begin{aligned}|F(z)| &\leq \int_0^{|z|} \frac{(1+Bs)^{\gamma(1-\rho)(\frac{m}{4}+\frac{1}{2})}}{(1-Bs)^{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})}} \cdot \left(\frac{1}{1-Bs}\right)^{\alpha} ds \\ &= \int_0^{|z|} \left(\frac{1+Bs}{1-Bs}\right)^{\gamma(1-\rho)(\frac{m}{4}-\frac{1}{2})+\alpha} \cdot (1+Bs)^{(1-\rho-\alpha)} ds.\end{aligned}$$

Now similar computations yield the required bound and the proof is complete. □

By choosing suitable and permissible values of involved parameters, we obtain several new and also known results.

Remark 3.2. (i) *We use a result of Pommerenke [16] together with Theorem 3.1 and easily deduce that the class* $T_{\alpha,m}[A,B;0;-1]$, $m \leq 2\left\{1+\frac{\rho_1}{1-\rho}\right\}$, $\rho_1 = \left(\frac{1}{2}\right)^{\alpha}$, $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$, *is a linearly invariant family of order* $B_2 = \left\{\frac{m}{2}(1-\rho)+(\rho-\rho_1)+1\right\}$. *With similar argument given in [16], we have the covering result for* $T_{\alpha,m}[A,B;0;-1]$ *as:*

The image of E *under* $F = f*g \in T_{\alpha,m}[A,B;0;-1]$ *contains the Schlicht disc* $|z| = \frac{1}{2B_2}$, *where* $B_2 = \left\{\frac{m}{2}(1-\rho)+1+\rho-\rho_1\right\}$, *and* $F(z) = z + \sum\limits_{n=2}^{\infty} A_n z^n$.

(ii) *Let* F_* *be defined as*

$$F_*(z) = \frac{1}{B_1}\left[\left(\frac{1+z}{1-z}\right)^{B_2} - 1\right] = z + \sum_{n=2}^{\infty} A_n^* z^n,$$

where

$$\begin{aligned}B_1 &= \left\{\frac{m}{2}(1-\rho)+(\rho-\rho_1)+2\right\}, \\ B_2 &= \left\{\frac{m}{2}(1-\rho)+(\rho-\rho_1)+1\right\}.\end{aligned}$$

It can be shown, with some computations, that F_* *belongs to the linearly invariant family of* $T_{\alpha,m}[A,B;0;-1]$. *Using this concept, together with the same argument of Pommerenke [16], we have* $|A_n| \leq |A_n^*|$, $n \geq 1$ *and* $L_r(F) \leq L_r(F_*)$, $F \in T_{\alpha,m}[A,B;0;-1]$ *when* $L_r(F)$ *is the length of the image of the circle* $|z| = r$ *under* F, $0 \leq r < 1$.

Theorem 3.4. *Let* $f \in T_{\alpha,m}[A,B;0;-1;g]$ *and let* $F = f*g \neq 0$ *in* E *with*

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n.$$

Then, for $m > 2$,

$$A_n = O(1)\cdot n^{\gamma_1}, \quad \gamma_1 = \left\{\frac{m}{2}(1-\rho)+[\rho_1-(1+\rho)]\right\},$$

where $O(1)$ *is a constant depending on* m, α, A *and* B *only.*

Proof. For $F \in T_{\alpha,m}[A,B;0;-1]$, we can write

$$F' = G'h, \quad G \in V_{\alpha,m}[A,B], \quad G = \psi * g, \quad \psi \in V_{\alpha,m}[A,B;g].$$

Since $V_{\alpha,m}[A,B] \subset V_m(\rho)$, $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$, and it is well known that there exists $G_i \in V_m$ such that $G'(z) = (G'(z))^{1-\rho}$ for $z \in E$.
Also $h \prec \left(\frac{1}{1+z}\right)^{\alpha}$, which implies $|\arg h(z)| < \frac{\rho_1 \pi}{2}$, $\rho_1 = \left(\frac{1}{2}\right)^{\alpha}$.
Therefore we have

$$F' = \left(G_1'\right)^{1-\rho}(h_1)^{\rho_1}, \quad Reh_1 > 0$$

in E.

For $G_1 \in V_m$, there exists $s \in S^{\star}$ such that $G_1' = sh_2^{(\frac{m}{2}-1)}$, $m > 2$ and

$Reh_2 > 0$ in E, see [1].

Thus, for $F \in T_{\alpha,m}[A,B;0;-1]$, it follows that

$$F' = (s)^{1-\rho}(h_2)^{(1-\rho)(\frac{m}{2}-1)}(h_1)^{\rho_1}, \quad h_i \in P, \quad i = 1,2 \tag{3.18}$$

So, by Cauchy Theorem and (3.18), we have for $z = re^{\iota\theta}$.

$$\begin{aligned} n|A_n| &\leq \frac{1}{2\pi r^n}\int_0^{2\pi} |s|^{1-\rho}|h_1|^{\rho_1}|h_2|^{(1-\rho)(\frac{m}{2}-1)}d\theta \\ &\leq \frac{1}{r^n}\left(\frac{r}{(1-r)^2}\right)^{(1-\rho)}\left[\left(\frac{1}{2\pi}\int_0^{2\pi}|h_1|^2 d\theta\right)^{\frac{\rho_1}{2}} \cdot \left(\frac{1}{2\pi}\int_0^{2\pi}|h_2|^{\frac{2\delta}{2-\rho_1}}d\theta\right)^{\frac{2-\rho_1}{2}}\right], \end{aligned}$$

where $\delta = (1-\rho)\left(\frac{m}{2}-1\right)$ and we have used distortion result for $s \in S^{\star}$ and Holder inequality.
Now, for $m > \left\{2 + \frac{2-\rho_1}{1-\rho}\right\}$, we apply a result due to Hayman [5] for $h_i \in P$ and obtain

$$n|A_n| \leq c(\rho,\rho_1,m) \cdot \left(\frac{1}{1-r}\right)^{1+\delta+\rho_1-2\rho} \tag{3.19}$$

where $c(\rho,\rho_1,\delta)$ is a constant.
Setting $r = 1 - \frac{1}{n}$, $n \to \infty$ in (3.19), the required result follows as

$$A_n = O(1) \cdot n^{\left\{\frac{m}{2}(1-\rho)+[\rho_1-(\rho+1)]\right\}}, \quad \rho_1 = \left(\frac{1}{2}\right)^{\alpha}, \quad \rho = \left(\frac{1-A}{1-B}\right)^{\alpha},$$

and $m > \left\{2 + \frac{2-\rho_1}{1-\rho}\right\}, \quad n \geq 2.$ □

Special Cases.

(i) $A = 1$ implies that $\rho = 0$ and for $\alpha = 1$, $\rho_1 = \frac{1}{2}$. Then

$$A_n = O(1) \cdot n^{\frac{m}{2}-\frac{1}{2}}, \quad m > \frac{7}{2}$$

Taking $m = 4$, we have $A_n = O(1) \cdot n^{\frac{3}{2}}$.

(ii) $A = \frac{1}{2}$, $B = -1$, $\alpha = 1 \Rightarrow \rho = \frac{1}{4}$. Also $\rho_1 = \frac{1}{2}$. Then $m = 5 > 4$ implies $A_n = O(1) \cdot n^{\frac{9}{8}}$.

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