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An Affirmative Result on Banach Space

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Abstract. The aim of this paper is to establish a common fixed point theorem on Banach space using occasionally weakly compatible (OWC) mappings.

1. Introduction

Fixed point theory is one of the most powerful topics of modern mathematics and might be taken as main subject of analysis.For the past many years, fixed point theory has been evolved as the area of research for many researchers. Banach contraction principle is one such result proposed by Banach to name a few.For the study of discontinuous and noncompatible mappings in fixed point theory we refer the literature like [4] and [5]. Pathak and others [1] proved a fixed point theorem on complete metric space using continuity and weakly compatible mappings.Thereafter Sushil Sharma, Bhavana Deshpande, and Alok Pandey [2] proved some more results on Banach space.Further several theorems [3], [6], [7], [8] , [9] and [10]are being generated on Banach space using various conditions.The focus of this work is now on proving a result in Banach space without a continuity constraint using OWC mappings to prove a common fixed point theorem.

Before we prove our theorem, we'll present some definitions and examples.

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2. Preliminaries

A pair (I, J) of a Banach space is said to be

Definition 2.1 Weakly commuting iff $\|IJ\alpha - JI\alpha\| \leq \|I\alpha - J\alpha\|$ for all $\alpha \in X$.

Definition 2.2 Cmpatible iff $\|IJ\alpha_j - JI\alpha_j\| = 0$ as $j \to \infty$ whenever $\{\alpha_j\}$ is a sequence in X such that $\|I\alpha_j - J\alpha_j\| = 0$ as $j \to \infty$ for some $\eta \in X$.

Definition 2.3 Weakly compatible if $I\eta = J\eta$ for some $\eta \in X$ such that $IJ\eta = JI\eta$

Definition 2.4 OWC if and only if there exists a point $\eta \in X$ such that $I\eta = J\eta$ implies $IJ\eta = JI\eta$.

We now discuss some examples to find the relation among the above definitions.

Example 2.5 Let $X = [0, 1]$ is a Banach space with $\|u - v\| = |u - v|$, $u, v \in X$.

Define T and S as

$$T(\alpha) = \begin{cases} \frac{1+2\alpha}{2} & \text{if } 0 \leq \alpha \leq \frac{2}{3}; \\ \frac{5\alpha+2}{8} & \text{if } \frac{2}{3} < \alpha \leq 1. \end{cases}$$

$$S(\alpha) = \begin{cases} 1 - 2\alpha & \text{if } 0 \leq \alpha \leq \frac{2}{3}; \\ \alpha & \text{if } \frac{2}{3} \leq \alpha \leq 1. \end{cases}$$

Take a sequence α_k as $\alpha_k = \frac{1}{6} - \frac{1}{k}$

$T(\alpha_k) = T(\frac{1}{6} - \frac{1}{k})=\frac{1+2(\frac{1}{6}-\frac{1}{k})}{2}=\frac{2}{3}$

$S(\alpha_k) = S(\frac{1}{6} - \frac{1}{k})=1 - 2(\frac{1}{6} - \frac{1}{k})=\frac{2}{3}$

$T\alpha_k=S\alpha_k=\frac{2}{3}$ as$k \to \infty$.

Now $TS(\alpha_k) = GJ(\frac{1}{6} - \frac{1}{k})=G(1 - 2(\frac{1}{6} - \frac{1}{k}))$

$=G(\frac{2}{3} + \frac{2}{k})=\frac{3(\frac{2}{3})+\frac{2}{k}+1}{5}=\frac{3}{5}$ as $k \to \infty$.

$ST\alpha_k = S[T(\frac{1}{6} - \frac{1}{k})]=S[1 - 2(\frac{1}{6} - \frac{1}{k})]$

$=S(\frac{2}{3} + \frac{2}{k})=\frac{2}{3} + \frac{2}{k}=\frac{2}{3}$as $k \to \infty$.

$\lim_k \to \infty\|(TS\alpha_k - ST\alpha_k)\| \neq 0$

Therefore the pair (T, S) is not compatible.

But $T(\frac{1}{6})=S(\frac{1}{6})=(\frac{2}{3})$.

$TS(\frac{1}{6})=T[1 - 2(\frac{1}{6})]=G(\frac{2}{3})=\frac{2}{3}$ and $ST(\frac{1}{6}) = J(G(\frac{1}{6})) = \frac{2(\frac{2}{3})+1}{2} = \frac{2}{3}$.

Hence the pair (T, S) weakly compatiable.

Example 2.6 Let $X = [0, 1]$ is a Banach space with $\|u - v\| = |u - v|$, $\forall u, v \in X$.

Define four maps G,J,H and I as follows

$$G(\alpha) = I(\alpha) = \begin{cases} \frac{\alpha+2}{8} & \text{if } 0 \leq \alpha < \frac{1}{2}; \\ 1 - \alpha & \text{if } \alpha \geq \frac{1}{2}. \end{cases}$$

$$J(\alpha) = H(\alpha) = \begin{cases} \frac{\alpha+1}{4} & \text{if } 0 \leq \alpha < \frac{1}{2}; \\ \frac{4\alpha-1}{2} & \text{if } \alpha \geq \frac{1}{2}. \end{cases}$$

Here $G(0) = J(0) = \frac{1}{4}$ and $G(\frac{1}{2}) = J(\frac{1}{2}) = \frac{1}{2}$.

Clearly $\alpha = 0$ and $\alpha = \frac{1}{2}$ are two coincidence points.

If $\alpha = 0$ then $GJ(0) = G(\frac{1}{4}) = \frac{(\frac{1}{4}+2)}{8} = \frac{9}{32}$

and $JG(0) = J(\frac{1}{4}) = \frac{(\frac{1}{4})+1}{4} = \frac{5}{16}$.

Therefore $GJ(0) \neq JG(0)$.

Now $JG(\frac{1}{2}) = J(1-\frac{1}{2}) = J(\frac{1}{2}) = \frac{4(\frac{1}{2})-1}{2} = \frac{1}{2}$

$GJ(\frac{1}{2}) = G(\frac{1}{2}) = 1-\frac{1}{2} = \frac{1}{2}$

Therefore $GJ(\frac{1}{2}) = JG(\frac{1}{2})$.

Thus G and J are OWC, but not weakly compatible.

The following Theorem was proved in metric space [1].

3. Theorem

Let X be a comlete metic space and self mappings G,H,I and J are satisfying

$(b1)$ $G(X) \subseteq H(X)$ and $I(X) \subseteq J(X)$

$(b2)$ $d(Gu, Iv)^{2p} \leq [a\phi_0(d(Ju, Hv)^{2p}) + (1-a)max\{\phi_1(d(Ju, Hv)^{2p}),$

$\phi_2(d(Ju, Gv)^q d(Hv, Iv)^{q'}), \phi_3(d(Ju, Iv)^r d(Hv, Gv)^{r'}),$

$\phi_4(d(Ju, Gv)^s d(Hv, Gu)^{s'}), \phi_5(d(Ju, Iv)^l d(Hv, Iv)^{l'})\}]$

for all u,$v$$\in$$X$ where $\phi_j \in \phi$,$j = 0, 1, 2, 3, 4, 5$ and a takes values from 0 to 1 inclusive and $p, q, q', r, r'.s, s', l, l'$ takes from 0 exclusive to 1 inclusive such that $2p = q + q' = r + r' = s + s' = l + l'$.

$(b3)$ The mapping G or I is a continuous

$(b4)$ the pairs (G,J) and (I,H) are weakly compatible mappings.

Then G,H,I and J have a unique fixed point which is common.

Now we'll present a list of key lemmas that are useful to our main result.

Lemma 3.1 [10]

If $\phi_j \in \phi$ and $j \in \{0, 1, 2, 3, 4, 5\}$,ϕ is upper semicontinuous and also contractive modulus such that $\max\{\phi_j(t)\} \leq \phi(t)$ for all $t > 0$ and also $\phi(t) < t$ for $t > 0$.

Lemma 3.2 [1]

Let $\phi_j \in \phi$ and β_j be a non-negative real sequence.If $\beta_{j+1} \leq \phi(\beta_j)$ for $j \in N$, then the sequence converges to 0.

Now we genealize the existence of the above theorem by extending it on to Banach Space under the following modified conditions.

Theorem 3.3

Let $(X, \|.\|)$ be a Banach Space and self mappings G,H,I and J are satisfying

$(b1)$ $G(X) \subseteq H(X)$ and $I(X) \subseteq J(X)$

$(b2)$ $\|(Gu - Iv)\|^{2p} \leq [a\phi_0(\|(Ju - Hv)\|^{2p}) + (1-a)max\{\phi_1(\|Ju - Hv\|^{2p}),$
$\phi_2(\|Ju - Gu\|^{q}\|Hv - Iv\|^{q'}), \phi_3(\|Ju - Iv\|^{r}\|Hv - Gu\|^{r'}),$
$\phi_4(\frac{1}{2}\|Ju - Gu\|^{s}\|Hv - Iv\|^{s'}), \phi_5(\frac{1}{2}\|Ju - Iv\|^{l}\|Hv - Iv\|^{l'})\}]$

for all $u,v \in X$ where $\phi_j \in \phi, j = 0, 1, 2, 3, 4, 5$ and a takes values from 0 to 1 inclusive and $p, q, q', r, r', s, s', l, l'$ takes from 0 exclusive to 1 inclusive such that $2p = q + q' = r + r' = s + s' = l + l'$.

$(b3)$ Two pairs (G,J) and (I,H) have a coincidence point

$(b4)$ the pairs maps (G,J) and (I,H) are OWC.

Then G,H,I and J have a unique common fixed point.

Proof

Using the condition (b1), there is a point $u_0 \in X$ such that $Gu_0 = Hu_1$.For this point $u_1 \in X$ there exists a point u_2 in X such that $Iu_1 = Ju_2$ and so on.

Continuing this process it is possible to construct a sequence $\{v_j\}$ for j = 1,2,3......in X such that $v_{2j} = Gu_{2j} = Hu_{2j+1}, v_{2j+1} = Iu_{2j+1} = Ju_{2j+2}$ for $j \geq 0$.

We now prove $\{v_j\}$ is a cauchy sequence.

Putting $u = u_{2j}$ and $v = v_{2j+1}$ in (b2), we get

$\|v_{2j} - v_{2j+1}\|^{2p} \leq [a\phi_0(\|v_{2j-1} - v_{2j}\|^{2p}) + (1-a)max\{\phi_1(\|v_{2j-1} - v_{2j}\|^{2p}),$
$\phi_2(\|v_{2j-1} - v_{2j}\|^{q}\|v_{2j} - v_{2j+1}\|^{q'}), \phi_3(\|v_{2j-1} - v_{2j+1}\|^{r}\|v_{2j} - v_{2j}\|^{r'}),$
$\phi_4(\frac{1}{2}\|v_{2j-1} - v_{2j}\|^{s}\|v_{2j} - v_{2j}\|^{s'}), \phi_5(\frac{1}{2}\|v_{2j-1} - Iv_{2j+1}\|^{l}\|v_{2j} - v_{2j+1}\|^{l'})\}].$

Denote $\rho_j = \|v_j - v_{j+1}\|$

$(\rho_{2j})^{2p} \leq [a\phi_0(\rho_{2j-1})^{2p}) + (1-a)max\{\phi_1(\rho_{2j-1})^{2p}, \phi_2((\rho_{2j-1})^{q}(\rho_{2j})^{q'}), \phi_3(0),$
$\phi_4(0), \phi_5(\frac{1}{2}[(\rho_{2j-1})^{l} + (\rho_{2j})^{l'})](\rho_{2j})^{l})\}].$

$(\rho_{2j})^{2p} \leq [a\phi_0(\rho_{2j-1})^{2p}) + (1-a)max\{\phi_1(\rho_{2j-1})^{2p}, \phi_2((\rho_{2j-1})^{q}(\rho_{2j})^{q'}), \phi_3(0),$
$\phi_4(0), \phi_5(\frac{1}{2}[(\rho_{2j-1})^{l}(\rho_{2j})^{l} + (\rho_{2j}^{l'})(\rho_{2j})^{l})])\}].$

If $\rho_{2j} > \rho_{2j-1}$ then we have

$(\rho_{2j})^{2p} \leq [a\phi_0(\rho_{2j})^{2p}) + (1-a)max\{\phi_1(\rho_{2j})^{2p}, \phi_2(\rho_{2j})^{q+q'}, \phi_3(0), \phi_4(0),$
$\phi_5(\frac{1}{2}[(\rho_{2j})^{l+l'}(\rho_{2j})^{l+l'} + (\rho_{2j}^{l+l'})])\}(\rho_{2j})^{2p})].$
$\leq [a\phi_0(\rho_{2j})^{2p}) + (1-a)max\{\phi_1(\rho_{2j})^{2p}, \phi_2(\rho_{2j})^{2p}, \phi_3(0), \phi_4(0), \phi_5(\rho_{2j})^{2p})].$

Using Lemma (3.2)

$(\rho_{2j})^{2p} \leq \phi(\rho_{2j})^{2p} < (\rho_{2j})^{2p}.$

a contradiction.

This implies $\rho_{2j} \leq \rho_{2j-1}$
then using this inequality the condition (b2) yields

$$\rho_{2j} \leq \phi(\rho_{2j-1}). \tag{3.1}$$

Similarly taking $u = u_{2j+2}$ and $v = u_{2j+1}$ in (b2), we get

$\|v_{2j+1} - v_{2j+2}\|^{2p} \leq [a\phi_0(\|v_{2j} - v_{2j+1}\|^{2p}) + (1-a)max\{\phi_1(\|v_{2j} - v_{2j+1}\|^{2p}),$
$\phi_2(\|v_{2j+1} - v_{2j+2}\|^{q}\|v_{2j} - v_{2j+1}\|^{q'}), \phi_3(\|v_{2j+1} - v_{2j+1}\|^{r}\|v_{2j} - v_{2j+1}\|^{r'}),$
$\phi_4(\frac{1}{2}\|v_{2j+11} - v_{2j+2}\|^{s}\|v_{2j} - v_{2j+2}\|^{s'}), \phi_5(\frac{1}{2}\|v_{2j+2} - Iv_{2j+1}\|^{l}\|v_{2j} - v_{2j+1}\|^{l'})\}].$

$(\rho_{2j+1})^{2p} \leq [a\phi_0(\rho_{2j})^{2p}) + (1-a)max\{\phi_1(\rho_{2j})^{2p}, \phi_2((\rho_{2j+1})^{q}(\rho_{2j})^{q'}), \phi_3(0),$
$\phi_4(\frac{1}{2}[(\rho_{2j+1})^{s}(\rho_{2j})^{s'} + (\rho_{2j+1})^{s'}], \phi_5(0)\}].$

$(\rho_{2j+1})^{2p} \leq [a\phi_0(\rho_{2j})^{2p}) + (1-a)max\{\phi_1(\rho_{2j})^{2p}, \phi_2((\rho_{2j+1})^{q}(\rho_{2j})^{q'}), \phi_3(0),$
$\phi_4(\frac{1}{2}[(\rho_{2j+1})^{s}(\rho_{2j})^{s'} + (\rho_{2j+1})^{s'}(\rho_{2j+1})^{s'}]), \phi_5(0)\}].$
If $\rho_{2j+1} > \rho_{2j}$,then we have
$(\rho_{2j+1})^{2p} \leq [a\phi_0(\rho_{2j+1})^{2p}) + (1-a)max\{\phi_1(\rho_{2j+1})^{2p}, \phi_2((\rho_{2j+1})^{q+q'}), \phi_3(0),$
$\phi_4(\rho_{2j+1}), \phi_5(0)\}].$
Using Lemma(3.2)
$(\rho_{2j+1})^{2p} \leq \phi(\rho_{2j+1})^{2p} < (\rho_{2j+1})^{2p}$
which is a contradiction.
Thus we must have $\rho_{2j+1} \leq \rho_{2j}$.
Again applying (b2) to the above inequality,we obtain

$$\rho_{2j+1} \leq \phi(\rho_{2j}). \tag{3.2}$$

From (2.1) and (2.2), in general $\rho_{j+1} \leq \phi(\rho_j)$, for j=0,1,2,3....
by Lemma 3.3 we get $\rho_j \to 0$ as $j \to \infty$.
This shows that $\rho_j = \|v_j - v_{j+1}\| \to 0$ as $j \to \infty$.
Hence $\{v_j\}$ is a cauchy sequence.
Since X Banach space, $\exists$ a point $\eta \in X$ such that $v_j \to \eta$ as $j \to \infty$. Consequently, the subsequences $G\alpha_{2j}, J\alpha_{2j}, I\alpha_{2j+1}$ and $H\alpha_{2j}$ of $\{v_j\}$ also converge to the same point $\eta \in X$. Since the pair (G, J) is OWC ,there exists $u \in C(G, J)$ such that $Gu = Ju = \eta$(say) and $GJu = JGu = \eta'$(say).
Hence we have

$$G\eta = J\eta = \eta'(say) \tag{3.3}$$

Since the pair (I, H) is OWC ,there exists $v \in C(I, H)$
such that $Iv = Hv = \delta$(say) and $IHv = HIv = \delta'$(say).

Hence we have

$$I\delta = H\delta = \delta^{'}(say) \tag{3.4}$$

Now we claim that $\eta^{'} = \delta'$

substitute $u = \eta, v = \delta$ in (b2)

$\|(G\eta - I\delta)\|^{2p} \leq [\phi_0(\|(J\eta - H\delta)\|^{2p}) + (1-a)max\{\phi_1(\|J\eta - H\delta\|^{2p}),$

$\phi_2(\|J\eta - G\eta\|^{q}\|H\delta - I\delta\|^{q'}), \phi_3(\|J\eta - I\delta\|^{r}\|H\delta - G\eta\|^{r'}),$

$\phi_4(\frac{1}{2}\|J\eta - G\eta\|^{s}\|H\delta - I\delta\|^{s'}), \phi_5(\frac{1}{2}\|J\eta - I\delta\|^{l}\|H\delta - I\delta\|^{l'})\}].$

Using (3) and (4), we get $\|\eta' - \delta'\|^{2p} \leq [\phi_0(\|\eta' - \delta'\|^{2p}) + (1-a)max\{\phi_1(\|\eta' - \delta'\|^{2p}),$

$\phi_2(\|\eta' - \eta'\|^{q}\|\delta' - \delta\|^{q'}), \phi_3(\|\eta' - \delta'\|^{r}\|\delta' - \eta'\|^{r'}),$

$\phi_4(\frac{1}{2}\|\eta' - \eta'\|^{s}\|\delta' - \eta'\|^{s'}), \phi_5(\frac{1}{2}\|\eta' - \delta'\|^{l}\|\delta' - \delta'\|^{l'})\}].$

$\|\eta' - \delta'\|^{2p} \leq [\phi_0(\|\eta' - \delta'\|^{2p}) + (1-a)max\{\phi_1(\|\eta' - \delta'\|^{2p}), \phi_2(0), \phi_3(\|\eta' - \delta'\|^{2p}), \phi_4(0), \phi_5(0)\}].$

Since by Lemma (3.2)

$\|\eta' - \delta'\|^{2p} \leq \phi(\|\eta' - \delta'\|)^{2p} < \|\eta' - \delta'\|^{2p}$

a contradiction.

Therefore $\eta' = \delta'$.

Hence from (2.3) we get

$$G\eta = J\eta = \delta^{'}. \tag{3.5}$$

Next claim that $\eta = \delta'$

substitute $u = u$ and $v = \delta$ in (b2)

$\|(Gu - I\delta)\|^{2p} \leq [\phi_0(\|(Ju - H\delta)\|^{2p}) + (1-a)max\{\phi_1(\|Ju - H\delta\|^{2p}),$

$\phi_2(\|Ju - Gu\|^{q}\|H\delta - I\delta\|^{q'}), \phi_3(\|Ju - I\delta\|^{r}\|H\delta - Gu\|^{r'}),$

$\phi_4(\frac{1}{2}\|Ju - Gu\|^{s}\|H\delta - I\delta\|^{s'}), \phi_5(\frac{1}{2}\|Ju - I\delta\|^{l}\|H\delta - I\delta\|^{l'})\}].$

Using $Gu = Ju = \eta$ and $I\delta = H\delta = \delta^{'}$, we get

$\|\eta - \delta'\|^{2p} \leq [\phi_0(\|\eta - \delta'\|^{2p}) + (1-a)max\{\phi_1(\|\eta - \delta'\|^{2p}), \phi_2(\|\eta - \eta\|^{q}\|\delta - \delta\|^{q'}), \phi_3(\|\eta - \delta'\|^{r}\|\delta' - \eta\|^{r'}),$

$\phi_4(\frac{1}{2}\|\eta - \eta\|^{s}\|\delta' - \eta\|^{s'}), \phi_5(\frac{1}{2}\|\eta - \delta'\|^{l}\|\delta' - \delta'\|^{l'})\}].$

$\|\eta - \delta'\|^{2p} \leq [\phi_0(\|\eta - \delta'\|^{2p}) + (1-a)max\{\phi_1(\|\eta - \delta'\|^{2p}), \phi_2(0), \phi_3(\|\eta - \delta'\|^{2p}), \phi_4(0), \phi_5(0)\}].$

By Lemma(3.2)

$\|\eta - \delta'\|^{2p} \leq \phi(\|\eta - \delta'\|)^{2p} < \|\eta - \delta'\|^{2p}$

which is a contradiction, and hence $\eta = \delta'$.

From (2.5),we get

$$G\eta = J\eta = \eta \tag{3.6}$$

and

$$I\delta = H\delta = \eta. \tag{3.7}$$

Now we claim that $\eta = \delta$.

Again in (b2) putting $u = \eta$ and $v = v$

$\|G\eta - Iv\|^{2p} \leq [\phi_0(\|(J\eta - Hv)\|^{2p}) + (1-a)max\{\phi_1(\|J\eta - Hv\|^{2p}),$

$\phi_2(\|J\eta - G\eta\|^q\|Hv - Iv\|^{q'}), \phi_3(\|J\eta - Iv\|^r\|Hv - G\eta\|^{r'}),$
$\phi_4(\frac{1}{2}\|J\eta - G\eta\|^s\|Hv - Iv\|^{s'}), \phi_5(\frac{1}{2}\|Ju - Iv\|^l\|Hv - Iv\|^{l'})\}].$
Using $G\eta = J\eta = \eta$ and $Iv = Hv = \delta$
$\|\eta-\delta\|^{2p} \leq [\phi_0(\|\eta-\delta\|^{2p})+(1-a)max\{\phi_1(\|\eta-\delta\|^{2p}), \phi_2(\|\eta-\eta\|^q\|\delta-\delta\|^{q'}), \phi_3(\|\eta-\delta\|^r\|\eta-\delta\|^{r'}),$
$\phi_4(\frac{1}{2}\|\eta - \eta\|^s\|\delta - \eta\|^{s'}), \phi_5(\frac{1}{2}\|\eta - \delta\|^l\|\delta - \delta\|^{l'})\}].$
$\|\eta - \delta\|^{2p} \leq [\phi_0(\|\eta - \delta\|^{2p}) + (1 - a)max\{\phi_1(\|\eta - \delta\|^{2p}), \phi_2(0), \phi_3(\|\eta - \delta\|^{2p}), \phi_4(0), \phi_5(0)\}].$
By Lemma(3.2)
$\|\eta - \delta\|^{2p} \leq \phi(\|\eta - \delta\|)^{2p} < \|\eta - \delta\|^{2p}$
which is a contradiction.

Therefore $\eta = \delta$.

From (2.7), we get

$$I\eta = H\eta = \eta. \tag{3.8}$$

From (2.6 and (2.8), we get
$G\eta = J\eta = I\eta = H\eta = \eta.$
Hence this gives that η is a common fixed point for G, H, I and J.

For Uniqueness:

Suppose η and η^* $(\eta \neq \eta^*)$ are two common fixed points.
Then put $u = \eta$ and $v = \eta^*$ in the inequality (b2)
$\|G\eta - I\eta^*\|^{2p} \leq [a\phi_0(\|J\eta - H\eta^*\|^{2p}) + (1 - a)max\{\phi_1(\|J\eta - H\eta^*\|^{2p}),$
$\phi_2(\|J\eta - G\eta\|^q\|H\eta^* - I\eta^*\|^{q'}), \phi_3(\|J\eta - I\eta^*\|^r\|H\eta^* - G\eta\|^{r'}),$
$\phi_4(\frac{1}{2}\|J\eta - G\eta\|^s\|H\eta^* - G\eta\|^{s'}), \phi_5(\frac{1}{2}\|J\eta - I\eta^*\|^l\|H\eta^* - I\eta^*\|^{l'})\}]$
$\|\eta - \eta^*\|^{2p} \leq [a\phi_0(\|\eta - \eta^*\|^{2p}) + (1 - a)max\{\phi_1(\|\eta - \eta^*\|^{2p}), \phi_2(0), \phi_3(\|\eta - \eta^*\|^{2p}), \phi_4(0), \phi_5(0)\}]$
$\|\eta - \eta^*\|^{2p} \leq [\phi(\|\eta - \eta^*\|^{2p}) < \|\eta - \eta^*\|^{2p}$
a contradiction.
Which gives $\eta = \eta^*$, this proves the uniqueness.

Example

Now we continue to discuss the Example (2.6) to justyfy our Theorem(3.3).

Now G(X)=I(X)=$[\frac{1}{4}, \frac{5}{16}) \cup (\frac{1}{2})$ while J(X)=H(X)= $[\frac{1}{4}, \frac{3}{8}) \cup (\frac{1}{2})$
clearly $G(X) \subseteq H(X), I(X) \subseteq J(X)$
therefore **(**$b1$) is satisfied.

Now we verify the condition ($b2$)

$Case(i)$.
If $u, v \in [0, \frac{1}{2})$,then we have $\|(Gu - Iv)\| = |G\alpha - Iv|$
put $u = \frac{1}{3}, v = \frac{1}{5}$.
Then the inequality ($b2$) implies
$\|G(\frac{1}{3}) - I(\frac{1}{5})\|^{2p} \leq [a\phi_0(\|J(\frac{1}{3}) - H(\frac{1}{5}))\|^{2p}) + (1 - a)max\{\phi_1(\|J(\frac{1}{3}) - H(\frac{1}{5})\|^{2p}),$

$\phi_2(\|J(\frac{1}{3}) - G(\frac{1}{3})\|^q \|H(\frac{1}{5}) - I(\frac{1}{5})\|^{q'}), \phi_3(\|J(\frac{1}{3}) - I(\frac{1}{3})\|^r \|H(\frac{1}{5}) - G(\frac{1}{3})\|^{r'}),$
$\phi_4(\frac{1}{2}\|J(\frac{1}{3}) - G(\frac{1}{3})\|^s \|H(\frac{1}{5}) - I(\frac{1}{5})\|^{s'}), \phi_5(\frac{1}{2}\|J(\frac{1}{3}) - I(\frac{1}{5})\|^l \|H(\frac{1}{5}) - I(\frac{1}{5})\|^{l'})\}]$
for $a = \frac{1}{2}$ and $p = q = q' = r = r' = s = s' = l = l' = \frac{1}{2}$

$\|0.0166\| \leq [\frac{1}{2}\phi_0\|(0.033)\| + (1 - \frac{1}{2})max\{\phi_1\|(0.01665)\|,$
$\phi_2\|(0.0322)\|, \phi_3\|(0.01870)\|, \phi_4\|(0.0161)\|, \phi_5\|(0.019)\}]$
$|0.0166| < |0.0327|.$

Hence the inequality $(b2)$ holds.

Also the verification in the remaining intervals is also simple. Here it is evident that $\frac{1}{2}$ is the unique common fixed point for the four self mappings.

4. Conclusion

In this paper we proved a common fixed point theorem on Banach space using OWC mappings.It is also clear from the example proved that the mappings are neither compatible nor weakly compatible but OWC mappings.Moreover the continuity condition is dropped.Hence we conclude that our Theorem stands as an improvement of Theorem (3).

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