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Exact and Sinusoidal Periodic Solutions of Lienard Equation Without Restoring Force

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Abstract. In this paper we study a Lienard equation without restoring force. Although this equation does not satisfy the classical existence theorems, we show, for the first time, that such an equation can exhibit harmonic periodic solutions. As such the usual existence theorems are not entirely adequate and satisfactory to predict the existence of periodic solutions.

1. Introduction

The Lienard equation

$$\ddot{x} + \vartheta(x)\dot{x} + h(x) = 0 \tag{1.1}$$

where overdot denotes derivative with respect to time, $\vartheta(x)$ and $h(x)$ are arbitrary functions of x, is a generalization of the conservative equation

$$\ddot{x} + h(x) = 0 \tag{1.2}$$

where $h(x)$ is a restoring force function, to take into account energy dissipation in real world dynamical systems. As such, the use of equation (1.2) constitutes a first approximation to model oscillations in dynamical systems so that the determination of periodic solutions of equation (1.1) has become a fundamental problem in mathematics and physics. In this way, a rich and various study has been carried out on the existence of periodic solutions of equation (1.1). One can find a vast literature on

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the theorems of existence and uniqueness ([1], [2], [3], [4], [5], [6]) of periodic solutions of equation (1.1). However, if the limit cycle of the Van der Pol equation ([1], [2], [3], [4])

$$\ddot{x}+\beta(x^2-1)\dot{x}+x=0 \tag{1.3}$$

of type (1.1), where β is an arbitrary constant, is well studied, one can then notice that exact and explicit periodic solutions of equation (1.1) for specific expressions of $\vartheta(x)$ and $h(x)$ are less obtained in the literature. A very limited number of authors have been able to find explicitly exact periodic solutions to equation (1.1) since its statement nearly a century ago, to model electrical circuits. In [7] Chandrasekar and his group investigated the case

$$\ddot{x}+\alpha x\dot{x}+\gamma x^3+\lambda x=0 \tag{1.4}$$

to extract for the first time, exact and explicit harmonic and isochronous periodic solutions, so that the authors qualified this equation of an unusual Lienard type nonlinear oscillator. Thus a lot of publications can be found on this equation from various mathematical points of view. In [8] the authors investigated a more general form of equation (1.4). In [9] the Lie point symmetries of the general class (1.1) are discussed. In a recent paper [10] Doutètien and coworkers studied analytically equation (1.4). Their study shows without ambiguity that the type of equation (1.4) investigated by Chandrasekar et al. [7] can exhibit unbounded periodic solutions. In [11] a Lienard nonlinear oscillator of type (1.1) is presented to represent periodic and isochronous oscillations in dynamical systems. The authors in [12] succeeded in exhibiting harmonic and isochronous periodic solution of a Lienard equation of type (1.1) using the first integral approach. In [13] the authors have successfully calculated a general sinusoidal function with isochronicity property for the exact and explicit general solution of an equation of type (1.1). The above shows the importance to investigate the Lienard type equation (1.1), as periodic solutions constitute an interesting and special class of solutions of differential equations. The theory of existence conditions of periodic solutions of equations of type (1.1) states ([1], [2], [3], [4], [5], [6]) often the requirement $xh(x)\succ 0$, for $x\neq 0$, that is to say equation (1.1) with $h(x)=0$, does not satisfy these conditions. In other words, $h(x)$ must be odd, so that the theorems of existence for periodic solutions of equation (1.1) eliminate the case where the restoring force $h(x)=0$, for all x. It seems that the question of existence of periodic solutions of such equations of the form

$$\ddot{x}+\vartheta(x)\dot{x}=0 \tag{1.5}$$

has not been formulated and solved in any study published to date explicitly. Nevertheless, it is reasonable to investigate such a question due to the results obtained previously in ([14], [15]). Indeed, the authors in ([14], [15]) succeeded in proving explicitly the existence of harmonic and isochronous periodic solutions of equations of type

$$\ddot{x} + \sigma(x, \dot{x})\dot{x} = 0 \tag{1.6}$$

where $\sigma(x, \dot{x})$ denotes the damping, as a function of x and $\dot{x}$, and the restoring force is null, whereas equation (1.6), as such, can be viewed as a more general form of equation (1.5). It was for the first time such a result is reached for the study of a Lienard type equation without restoring force in mathematics and physics. In this regard, the objective in the present paper is to show explicitly the existence of sinusoidal and isochronous periodic solutions of equation (1.5) under appropriate choice of function $\vartheta(x)$. For this purpose, the required theory is stated (section 2) so that we can present the equation of interest and solve it explicitly (section 3).The phase plane and existence theorem analysis is carried out (section 4) and a conclusion is given finally for the work.

2. Theory

Let us consider the Lienard type equation ([14], [15], [16])

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x} + ax^{\ell}\frac{f'(x)}{g(x)}\dot{x} = 0 \tag{2.1}$$

where prime means differentiation with respect to x, a and ℓ are arbitrary parameters, and $f(x)$ and $g(x) \neq 0$ are arbitrary functions of x. Equation (2.1) can be written

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + \frac{ax^{\ell}}{g(x)}(\ell\frac{f(x)}{x} + f'(x))\dot{x} = 0 \tag{2.2}$$

when $x \neq 0$.

For $\ell = 0$, equation (2.2) becomes

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\frac{f'(x)}{g(x)}\dot{x} = 0 \tag{2.3}$$

Under the requirement $g(x) = 1$, equation (2.3) reduces to

$$\ddot{x} + af'(x)\dot{x} = 0 \tag{2.4}$$

Equation (2.4) is equivalent to the system

$$\dot{x} = y, \quad \dot{y} = -af'(x)y \tag{2.5}$$

which can lead to

$\frac{dy}{dx} = -af'(x)$

so that one can get

$$dy = -af'(x)dx \tag{2.6}$$

that is, the integral curves

$$y = -af(x) + C \tag{2.7}$$

where C is a constant of integration. The problem to solve becomes now the choice of function $f(x)$ that reduces equation (2.7) to that of an ellipse centered at the origin if we expect to have harmonic periodic solution.

3. Equation of interest

Let $f(x) = \sqrt{\mu^2 - x^2}$, where μ is an arbitrary constant. Then, equation (2.4) turns into the equation of interest

$$\ddot{x} - \frac{ax}{\sqrt{\mu^2 - x^2}}\dot{x} = 0 \tag{3.1}$$

where $h(x) = 0$. Using equation (2.7), the solution of equation (3.1) is given by the quadrature, as expected ([14], [15], [16]), defined in the form

$$-a(t + K) = \int \frac{dx}{f(x)} \tag{3.2}$$

where K is an arbitrary constant and $C = 0$. From the expression of $f(x)$, equation (3.2) takes the form

$$-a(t + K) = \int \frac{dx}{\sqrt{\mu^2 - x^2}} \tag{3.3}$$

so that, after integration, one can obtain

$$\sin^{-1}(\frac{x}{\mu}) = -a(t + K) \tag{3.4}$$

In this context, by inverting, the exact harmonic and isochronous periodic solution of equation (3.1) is secured as

$$x(t) = \mu \sin(-a(t + K)) \tag{3.5}$$

where $\mu \succ 0$ and $a \prec 0$. Solution (3.5) shows that equation (3.1) and the linear harmonic oscillator

$$\ddot{x} + (-a)^2 x = 0 \tag{3.6}$$

with amplitude of oscillations $\mu \succ 0$, and angular frequency $\omega = -a$, have identical solutions. From physical point of view, the Hamiltonian of oscillator (3.1) can, using equation (2.7), reads

$$H = \frac{1}{2}y^2 + \frac{1}{2}a^2x^2 \tag{3.7}$$

As H is a constant, the damped equation (3.1) is then, a conservative nonlinear system.

4. Phase plane and existence theorem analysis

According to equation (2.5), equation (3.1) is equivalent to the dynamical system

$$\dot{x} = y, \quad \dot{y} = \frac{axy}{\sqrt{\mu^2 - x^2}} \tag{4.1}$$

The equilibrium points in the (x, y) phase plane are given by $y = 0$, and $\frac{axy}{\sqrt{\mu^2 - x^2}} = 0$. This means that if $y = 0$, then $x \neq 0$, and inversely, if $x = 0$, then $y \neq 0$. This is in opposition to results (3.5) and (3.7) showing that the origin is a center, that is a single equilibrium point. According, for example, to Theorem 11.3 of the book [[2], p.390] equation (3.1) has a center at the origin when $h(x) \succ 0$, for $x \succ 0$, and is odd. Since $h(x) = 0$, then equation (3.1) cannot have a center at the origin. As previously, this prediction of the preceding existence theorem is in contradiction with the analytical results (3.5) and (3.7). As seen, the classical existence theorems exclude some cases of Lienard nonlinear differential equations while the exact and explicit results show the existence of harmonic and isochronous periodic solutions. Now, we can address a conclusion for the work.

5. Conclusion

In this paper a Lienard equation without restoring force is studied. Although the equation does not satisfy the usual existence theorems, we have successfully shown that it can exhibit sinusoidal and isochronous periodic solution as the linear harmonic oscillator.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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