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Controlled $K-g-$Fusion Frames in Hilbert C^*-Modules

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Abstract. Controlled frames have been the subject of interest because of its ability to improve the numerical efficiency of iterative algorithms for inverting the frame operator. In this paper, we introduce the concepts of controlled $g-$fusion frame and controlled $K-g-$fusion frame in Hilbert C^*-modules and we give some properties. Also, we study the perturbation problem of controlled $K-g-$fusion frame. Moreover, an illustrative example is presented to support the obtained results.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaefer [4] in 1952 to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [6] for signal processing.

Many generalizations of the concept of frame have been defined in Hilbert C^*-modules [5, 7, 9, 13–17].

Controlled frames in Hilbert spaces have been introduced by P. Balazs [3] to improve the numerical efficiency of iterative algorithms for inverting the frame operator.

Rashidi and Rahimi [10] are introduced the concept of Controlled frames in Hilbert C^*-modules.

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The paper is organized in the following manner. In section 3, we introduced the notion of $g-$fusion frames and controlled $g-$fusion frames in Hilbert C^*-modules and estabilish some properties. Section 4 is devoted to introduce the concept of controlled $K-g-$fusion frames in Hilbert C^*-modules and gives some results, finally in section 5 we study the perturbation of controlled $K-g-$fusion frames.

2. Preliminaires

Let $\mathcal{A}$ be a unital C^*-algebra, let J be countable index set. Throughout this paper H and L are countably generated Hilbert $\mathcal{A}-$modules and $\{H_j\}_{j\in J}$ is a sequence of submodules of L. For each $j\in J$, $End^*_{\mathcal{A}}(H,H_j)$ is the collection of all adjointable $\mathcal{A}-$linear maps from H to H_j, and $End^*_{\mathcal{A}}(H,H)$ is denoted by $End^*_{\mathcal{A}}(H)$. Also let $GL^+(H)$ be the set of all positive bounded linear invertible operators on H with bounded inverse.

Definition 2.1. *[8] Let $\mathcal{A}$ be a unital C^*-algebra and H be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and H are compatible. H is a pre-Hilbert $\mathcal{A}$-module if H is equipped with an $\mathcal{A}$-valued inner product $\langle .,.\rangle : H\times H\to\mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,*

(i) *$\langle f,f\rangle\geq 0$ for all $f\in H$ and $\langle f,f\rangle = 0$ if and only if $f=0$.*
(ii) *$\langle af+g,h\rangle = a\langle f,h\rangle+\langle g,h\rangle$ for all $a\in\mathcal{A}$ and $f,g,h\in H$.*
(iii) *$\langle f,g\rangle=\langle g,f\rangle^*$ for all $f,g\in H$.*

For $f\in H$, we define $||f|| = ||\langle f,f\rangle||^{\frac{1}{2}}$. If H is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in a C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on H is defined by $|f| = \langle f,f\rangle^{\frac{1}{2}}$ for $f\in H$.
Define $l^2(\{H_j\}_{j\in J})$ by

$$l^2(\{H_j\}_{j\in J}) = \{\{f_j\}_{j\in J} : f_j\in H_j, ||\sum_{j\in J}\langle f_j,f_j\rangle|| < \infty\}.$$

With $\mathcal{A}-$valued inner product is given by

$$\langle\{f_j\}_{j\in J},\{g_j\}_{j\in J}\rangle = \sum_{j\in J}\langle f_j,g_j\rangle,$$

$l^2(\{H_j\}_{j\in J})$ is a Hilbert $\mathcal{A}-$module.

The following lemmas was used to proof our results:

Lemma 2.1. *[1] If $\phi:\mathcal{A}\to\mathcal{B}$ is a $*-$homomorphism between C^*-algebras, then ϕ is increasing, that is, if $a\leq b$, then $\phi(a)\leq\phi(b)$.*

Lemma 2.2. *[2] Let $T\in End^*_{\mathcal{A}}(H,L)$ and H,L are Hilberts $\mathcal{A}-$modules.The following statemnts are multually equivalent:*

(i) *T is surjective.*

(ii) T^* *is bounded below with respect to the norm, i.e., there is* $m > 0$ *such that* $||T^*f|| \geq m||f||$ *for all* $f \in L$.

(iii) T^* *is bounded below with respect to the inner product, i.e, there is* $m^{'} > 0$ *such that* $\langle T^*f, T^*f\rangle \geq m^{'}\langle f, f\rangle$ *for all* $f \in L$.

Lemma 2.3. *[1] Let* H *and* L *are two Hilbert* $\mathcal{A}$*-modules and* $T \in End^*_{\mathcal{A}}(H, L)$. *Then:*

(i) *If* T *is injective and* T *has closed range, then the adjointable map* T^*T *is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If* T *is surjective, then the adjointable map* TT^* *is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 2.4. *[2] Let* H *be a Hilbert* $\mathcal{A}$*-module over a* C^**-algebra* $\mathcal{A}$, *and* $T \in End^*_{\mathcal{A}}(H)$ *such that* $T^* = T$. *The following statements are equivalent:*

(i) T *is surjective.*

(ii) *There are* $m, M > 0$ *such that* $m\|f\| \leq \|Tf\| \leq M\|f\|$, *for all* $f \in H$.

(iii) *There are* $m', M' > 0$ *such that* $m'\langle f, f\rangle \leq \langle Tf, Tf\rangle \leq M'\langle f, f\rangle$ *for all* $f \in H$.

Lemma 2.5. *[12] Let* H *be a Hilbert* $\mathcal{A}$*-module. If* $T \in End^*_{\mathcal{A}}(H)$, *then*

$$\langle Tf, Tf\rangle \leq \|T\|^2\langle f, f\rangle, \qquad \forall f \in H.$$

Lemma 2.6. *[18] Let* E, H *and* L *be Hilbert* $\mathcal{A}-$*modules,* $T \in End^*_{\mathcal{A}}(E, L)$ *and* $T^{'} \in End^*_{\mathcal{A}}(H, L)$. *Then the following two statements are equivalent:*

(1) $T^{'}(T^{'})^* \leq \lambda TT^*$ *for some* $\lambda > 0$;

(2) *There exists* $\mu > 0$ *such that* $\|(T^{'})^*z\| \leq \mu\|T^*z\|$ *for all* $z \in L$.

Lemma 2.7. *[11] Let* $\{W_j\}_{j\in J}$ *be a sequence of orthogonally complemented closed submodules of* H *and* $T \in End^*_{\mathcal{A}}(H)$ *invertible, if* $T^*TW_j \subset W_j$ *for each* $j \in J$, *then* $\{TW_j\}_{j\in J}$ *is a sequence of orthogonally complemented closed submodules and* $P_{W_j}T^* = P_{W_j}T^*P_{TW_j}$.

3. Controlled $g-$fusion frame in Hilbert C^*-modules

Firstly we give the definition of $g-$fusion frame in Hilbert C^*-modules.

Definition 3.1. *[11] Let* $\{W_j\}_{j\in J}$ *be a sequence of closed submodules orthogonally complemented of* H, $\{v_j\}_{j\in J}$ *be a family of weights in* $\mathcal{A}$, *ie., each* v_j *is positive invertible element frome the center of* $\mathcal{A}$ *and* $\Lambda_j \in End^*_{\mathcal{A}}(H, H_j)$ *for each* $j \in J$. *We say that* $\Lambda = \{W_j, \Lambda_j, v_j\}_{j\in J}$ *is a* $g-$*fusion frame for* H *if there exists* $0 < A \leq B < \infty$ *such that*

$$A\langle f, f\rangle \leq \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H. \tag{3.1}$$

The constants A and B are called the lower and upper bounds of the $g-$fusion frame, respectively. If $A = B$ then Λ is called tight $g-$fusion frame and if $A = B = 1$ then we say Λ is a Parseval $g-$fusion frame.

The operator $S : H \to H$ defined by

$$Sf = \sum_{j\in J} v_j^2 P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} f, \qquad \forall f \in H.$$

Is called $g-$fusion frame operator.

Now we define the notion of $(C, C^{'})-$controlled $g-$fusion frame in Hilbert C^*-modules.

Definition 3.2. *Let C, $C^{'} \in GL^+(H)$, $\{W_j\}_{j\in J}$ be a sequence of closed submodules orthogonally complemented of H, $\{v_j\}_{j\in J}$ be a family of weights in $\mathcal{A}$, i.e., each v_j is a positive invertible element frome the center of $\mathcal{A}$ and $\Lambda_j \in End_{\mathcal{A}}^*(H, H_j)$ for each $j \in J$. We say that $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $g-$fusion frame for H if there exists $0 < A \leq B < \infty$ such that*

$$A\langle f, f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'} f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H. \tag{3.2}$$

The constants A and B are called the lower and upper bounds of the $(C, C^{'})-$controlled $g-$fusion frame, respectively. When $A = B$, the sequence $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is called $(C, C^{'})-$controlled tight $g-$fusion frame, and when $A = B = 1$, it is called a $(C, C^{'})-$controlled Parseval $g-$fusion frame. If only upper inequality of (3.2) hold, then $\Lambda_{CC'}$ is called an $(C, C^{'})-$controlled $g-$fusion bessel sequence for H.

Example 3.1. *Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}$, $v = \{v_j\}_{j\in\mathbb{N}} \in l^\infty$, we have*

$$uv = \{u_j v_j\}_{j\in\mathbb{N}}, u^* = \{\overline{u_j}\}_{j\in\mathbb{N}}, ||u|| = \sup_{j\in\mathbb{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, ||.||\}$ is a C^*-algebra.

Let $H = C_0$ be the set of all sequences converging to zero. For any u, $v \in H$ we define

$$\langle u, v\rangle = uv^* = \{u_j \overline{v_j}\}_{j\in\mathbb{N}}.$$

Then H is a Hilbert $\mathcal{A}-$module.

Now let $\{e_j\}_{j\in\mathbb{N}}$ be the standard orthonormal basis of H.

We construct $H_j = \overline{span}\{e_1, e_2, ..., e_j\}$ and $W_j = \overline{span}\{e_j\}$ for each $j \in \mathbb{N}$.
Define $\Lambda_j : H \to H_j$ by $\Lambda_j(f) = \sum_{k=1}^{j} \langle f, \frac{e_j}{\sqrt{j}}\rangle e_k$.
The adjoint operator $\Lambda_j^* : H_j \to H$ define by $\Lambda_j^*(g) = \sum_{k=1}^{j} \langle g, \frac{e_k}{\sqrt{j}}\rangle e_j$.
And the projection orthogonal P_{W_j} define by $P_{W_j}(f) = \langle f, e_j\rangle e_j$.

Let us define $Cf = 2f$ and $C^{'}f = \frac{1}{2}f$. Then for any $f \in H$, we have

$$\begin{aligned}\langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'} f\rangle &= \langle \frac{2}{\sqrt{j}}\langle f, e_j\rangle \sum_{k=1}^{j} e_k, \frac{1}{2\sqrt{j}}\langle f, e_j\rangle \sum_{k=1}^{j} e_k\rangle \\ &= \frac{1}{j}\langle f, e_j\rangle\langle e_j, f\rangle\langle \sum_{k=1}^{j} e_k, \sum_{k=1}^{j} e_k\rangle \\ &= \frac{1}{j}\langle f, e_j\rangle\langle e_j, f\rangle \sum_{k=1}^{j} \|e_k\|^2 \\ &= \frac{1}{j}\langle f, e_j\rangle\langle e_j, f\rangle j \\ &= \langle f, e_j\rangle\langle e_j, f\rangle.\end{aligned}$$

Therefore, for each $f \in H$,

$$\sum_{j\in\mathbb{N}} \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'} f\rangle = \sum_{j\in\mathbb{N}} \langle f, e_j\rangle\langle e_j, f\rangle = \langle f, f\rangle.$$

Hence $\{W_j, \Lambda_j, 1\}_{j\in\mathbb{N}}$ is a $(C, C^{'})-$controlled Parseval $g-$fusion frame for H.

Suppose that $\Lambda_{CC'}$ be a $(C, C^{'})-$controlled $g-$fusion bessel sequence for H. The bounded linear operator $T_{(C,C')} : l^2(\{H_j\}_{j\in J}) \to H$ define by

$$T_{(C,C')}(\{f_j\}_{j\in J}) = \sum_{j\in J} v_j (CC^{'})^{\frac{1}{2}} P_{W_j} \Lambda_j^* f_j, \qquad \forall \{f_j\}_{j\in J} \in l^2(\{H_j\}_{j\in J}). \tag{3.3}$$

is called the synthesis operator for the $(C, C^{'})-$controlled $g-$fusion frame $\Lambda_{CC'}$.
The adjoint operator $T^*_{(C,C')} : H \to l^2(\{H_j\}_{j\in J})$ given by

$$T^*_{(C,C')}(g) = \{v_j \Lambda_j P_{W_j} (C^{'}C)^{\frac{1}{2}} g\}_{j\in J} \tag{3.4}$$

is called the analysis operator for the $(C, C^{'})-$controlled $g-$fusion frame $\Lambda_{CC'}$.
When C and $C^{'}$ commute with each other, and commute with the operator $P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}$, for each $j \in J$, then the $(C, C^{'})-$controlled $g-$fusion frame operator $S_{(C,C')} : H \to H$ is defined as

$$S_{(C,C')}(f) = T_{(C,C')} T^*_{(C,C')}(f) = \sum_{j\in J} v_j^2 C^{'} P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} Cf, \qquad \forall f \in H. \tag{3.5}$$

And we have

$$\langle S_{(C,C')}(f), f\rangle = \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'} f\rangle, \qquad \forall f \in H. \tag{3.6}$$

From now we assume that C and $C^{'}$ commute with each other, and commute with the operator $P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}$, for each $j \in J$

Lemma 3.1. *Let $\Lambda_{CC'}$ be a $(C, C^{'})-$controlled $g-$fusion frame for H. Then the $(C, C^{'})-$controlled $g-$fusion frame operator $S_{(C,C')}$ is positive, self-adjoint and invertible.*

Proof. For each $f \in H$ we have

$$S_{(C,C')}(f) = \sum_{j\in J} v_j^2 C' P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} Cf$$

Then

$$\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C' f\rangle = \langle \sum_{j\in J} v_j^2 C' P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} Cf, f\rangle = \langle S_{(C,C')}(f), f\rangle.$$

Since $\Lambda_{CC'}$ is a $(C, C')-$controlled $g-$fusion frame for H, then

$$A\langle f, f\rangle \leq \langle S_{(C,C')}(f), f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H \tag{3.7}$$

It is clear that $S_{(C,C')}$ is positive, bounded and linear operator. On the other hand for each $f, g \in H$

$$\begin{aligned} \langle S_{(C,C')}(f), g\rangle &= \langle \sum_{j\in J} v_j^2 C' P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} Cf, g\rangle \\ &= \langle f, \sum_{j\in J} v_j^2 C P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} C' g\rangle \\ &= \langle f, S_{(C',C)}(g)\rangle. \end{aligned}$$

That implies $S^*_{(C,C')} = S_{(C',C)}$. Also as C and C' commute with each other, and commute with the operator $P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}$, for each $j \in J$, we have $S_{(C,C')} = S_{(C',C)}$. So the $(C, C')-$controlled $g-$fusion frame operator $S_{(C,C')}$ is self-adjoint. And from inequality (3.7) we have

$$AI_H \leq S_{(C,C')} \leq BI_H. \tag{3.8}$$

Therefore, the $(C, C')-$controlled $g-$fusion frame operator $S_{(C,C')}$ is invertible. □

We estabilish an equivalent definition of $(C, C')-$controlled $g-$fusion frame.

Theorem 3.1. *$\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C')-$controlled $g-$fusion frame for H. If and only if there exists two constants $0 < A \leq B < \infty$ such that*

$$A||f||^2 \leq ||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C' f\rangle|| \leq B||f||^2, \qquad \forall f \in H. \tag{3.9}$$

Proof. If $\Lambda_{CC'}$ be a $(C, C')-$controlled $g-$fusion frame for H, then we have inequality (3.9).
Converselly, assume that (3.9) holds. From (3.4), the $(C, C')-$controlled $g-$fusion frame operator $S_{(C,C')}$ is positive, self-adjoint and invertible. Then we have for all $f \in H$

$$\langle (S_{(C,C')})^{\frac{1}{2}} f, (S_{(C,C')})^{\frac{1}{2}} f\rangle = \langle S_{(C,C')} f, f\rangle = \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C' f\rangle. \tag{3.10}$$

Using (3.9) and (3.10), we conclude that

$$\sqrt{A}||f|| \leq ||S^{\frac{1}{2}}_{(C,C')} f|| \leq \sqrt{B}||f||, \qquad \forall f \in H.$$

So by lemma 2.4, $\Lambda_{CC'}$ is a $(C, C')-$controlled $g-$fusion frame for H. □

Theorem 3.2. *Let $\{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $g-$fusion frame for H with frame operator S and let C, $C^{'} \in GL^{+}(H)$. Then $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $g-$fusion frame for H.*

Proof. Let $\{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $g-$fusion frame for H with frame bounds A and B. Then for each $f \in H$

$$A\langle f, f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle f, f\rangle \tag{3.11}$$

We have

$$||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'} f\rangle|| = ||\langle S_{(C,C^{'})} f, f\rangle|| = ||C||.||C^{'}||.||\langle Sf, f\rangle||, \tag{3.12}$$

Using (3.11) and (3.12), we conclude

$$A||C||.||C^{'}||||\langle f, f\rangle|| \leq ||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'} f\rangle|| \leq B||C||.||C^{'}||||\langle f, f\rangle||, \qquad \forall f \in H.$$

Therefore, $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $g-$fusion frame for H with bounds $A||C||.||C^{'}||$ and $B||C||.||C^{'}||$. □

Remark 3.1. *When $C = C^{'}$ we say that the sequence $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a C^2-controlled $g-$fusion frame for H.*

Theorem 3.3. *Let $C \in GL^{+}(H)$. The sequence $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $g-$fusion frame for H if and only if $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a C^2-controlled $g-$fusion frame for H.*

Proof. Suppose that $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $g-$fusion frame for H. with bounds A and B. Then

$$A\langle f, f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H.$$

We have for each $f \in H$,

$$\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} Cf\rangle \leq B\langle Cf, Cf\rangle \leq B||C||^2 \langle f, f\rangle. \tag{3.13}$$

On the other hand for each $f \in H$

$$\begin{aligned} A\langle f, f\rangle = A\langle C^{-1}Cf, C^{-1}Cf\rangle &\leq A||C^{-1}||^2 \langle Cf, Cf\rangle \\ &\leq ||C^{-1}||^2 \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} Cf\rangle. \end{aligned} \tag{3.14}$$

So from (3.13) and (4.1), we have

$$A||C^{-1}||^{-2} \langle f, f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} Cf\rangle \leq B||C||^2 \langle f, f\rangle, \qquad \forall f \in H.$$

We conclude that $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a C^2-controlled $g-$fusion frame for H.

Conversely, Let $\{W_j, \Lambda_j, v_j\}_{j\in J}$ be a C^2-controlled $g-$fusion frame for H with bounds A' and B'. Then for all $f \in H$,

$$A'\langle f, f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} Cf\rangle \leq B'\langle f, f\rangle$$

We have for each $f \in H$,

$$\begin{aligned}\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle &= \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} CC^{-1} f, \Lambda_j P_{W_j} CC^{-1} f\rangle \\ &\leq B'\langle C^{-1} f, C^{-1} f\rangle \\ &\leq B' \|C^{-1}\|^2 \langle f, f\rangle. \end{aligned} \tag{3.15}$$

Also for each $f \in H$,

$$A'\langle C^{-1} f, C^{-1} f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} CC^{-1} f, \Lambda_j P_{W_j} CC^{-1} f\rangle = \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle$$

And

$$A' \|(C^{-1}C^{-1})^{-1}\|^{-1} \langle f, f\rangle \leq A'\langle C^{-1} f, C^{-1} f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \tag{3.16}$$

From (3.15) and (3.16), we have

$$A' \|(C^{-2})^{-1}\|^{-1} \langle f, f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B' \|C^{-1}\|^2 \langle f, f\rangle, \qquad \forall f \in H.$$

Hence $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $g-$fusion frame for H. □

Theorem 3.4. *Let C, $C' \in GL^+(H)$, and C, C' commute with each other and commute with $P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}$ for all $j \in J$. Then $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C')-$controlled $g-$fusion bessel sequence for H with bound B if and only if the operator $T_{(C,C')} : l^2(\{H_j\}_{j\in J}) \to H$ given by*

$$T_{(C,C')}(\{g_j\}_{j\in J}) = \sum_{j\in J} v_j (CC')^{\frac{1}{2}} P_{W_j} \Lambda_j^* g_j, \qquad \forall \{g_j\}_{j\in J} \in l^2(\{H_j\}_{j\in J}).$$

is well defined and bounded operator with, $\|T_{(C,C')}\| \leq \sqrt{B}$.

Proof. Let $\Lambda_{CC'}$ is a $(C, C')-$controlled $g-$fusion bessel sequence with bound B for H. As a result of theorem 3.1,

$$\|\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C' f\rangle\| \leq B\|f\|^2, \qquad \forall f \in H. \tag{3.17}$$

For any $\{g_j\}_{j\in J} \in l^2(\{H_j\}_{j\in J})$,

$$\begin{aligned}
||T_{(C,C')}(\{g_j\}_{j\in J})|| &= \sup_{||f||=1} ||\langle T_{(C,C')}(\{g_j\}_{j\in J}), f\rangle|| \\
&= \sup_{||f||=1} ||\langle \sum_{j\in J} v_j (CC^{'})^{\frac{1}{2}} P_{W_j}\Lambda_j^* g_j, f\rangle|| \\
&= \sup_{||f||=1} ||\sum_{j\in J} \langle v_j (CC^{'})^{\frac{1}{2}} P_{W_j}\Lambda_j^* g_j, f\rangle|| \\
&= \sup_{||f||=1} ||\sum_{j\in J} \langle g_j, v_j \Lambda_j P_{W_j} (CC^{'})^{\frac{1}{2}} f\rangle|| \\
&\le \sup_{||f||=1} ||\sum_{j\in J} \langle g_j, g_j\rangle||^{\frac{1}{2}} ||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} (CC^{'})^{\frac{1}{2}} f, \Lambda_j P_{W_j} (CC^{'})^{\frac{1}{2}} f\rangle||^{\frac{1}{2}} \\
&= \sup_{||f||=1} ||\sum_{j\in J} \langle g_j, g_j\rangle||^{\frac{1}{2}} ||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C^{'} f\rangle||^{\frac{1}{2}} \\
&\le \sup_{||f||=1} ||\sum_{j\in J} \langle g_j, g_j\rangle||^{\frac{1}{2}} \sqrt{B}||f|| = \sqrt{B}||\{g_j\}_{j\in J}||.
\end{aligned}$$

Therefore, the sum $\sum_{j\in J} v_j (CC^{'})^{\frac{1}{2}} P_{W_j}\Lambda_j^* g_j$ is convergent, and we have

$$||T_{(C,C')}(\{g_j\}_{j\in J})|| \le \sqrt{B}||\{g_j\}_{j\in J}||$$

Hence the operator $T_{(C,C')}$ is well defined, bounded and $||T_{(C,C')}|| \le \sqrt{B}$.
For the converse, suppose that the operator $T_{(C,C')}$ is well defined, bounded and $||T_{(C,C')}|| \le \sqrt{B}$. For all $f \in H$, we have

$$\begin{aligned}
||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C^{'} f\rangle|| &= ||\sum_{j\in J} v_j^2 \langle C^{'} P_{W_j}\Lambda_j^* \Lambda_j P_{W_j} C f, f\rangle|| \\
&= ||\sum_{j\in J} v_j^2 \langle (CC^{'})^{\frac{1}{2}} P_{W_j}\Lambda_j^* \Lambda_j P_{W_j} (CC^{'})^{\frac{1}{2}} f, f\rangle|| \\
&= ||\langle T_{(C,C')}(\{g_j\}_{j\in J}), f\rangle|| \\
&\le ||T_{(C,C')}|| ||\{g_j\}_{j\in J}|| ||f|| \\
&= ||T_{(C,C')}|| ||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C^{'} f\rangle||^{\frac{1}{2}} ||f||
\end{aligned}$$

Where $g_j = v_j \Lambda_j P_{W_j} (CC^{'})^{\frac{1}{2}} f$.
Hence

$$||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C^{'} f\rangle||^{\frac{1}{2}} \le \sqrt{B}||f||$$

Then

$$||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C^{'} f\rangle|| \le B||f||^2 \tag{3.18}$$

The adjoint operator of $T_{(C,C')}$ is given by

$$T^*_{(C,C')}(g) = \{v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}}g\}_{j\in J}, \qquad \forall g \in H.$$

And we have for each $f \in H$

$$\begin{aligned}||\sum_{j\in J}\langle\Lambda_j P_{W_j}Cf, \Lambda_j P_{W_j}C^{'}f\rangle|| &= ||\sum_{j\in J} v_j^2\langle\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}}f, \Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}}f\rangle|| \\ &= ||\langle T^*_{(C,C')}(f), T^*_{(C,C')}(f)\rangle|| \\ &= ||T^*_{(C,C')}(f)||^2\end{aligned}$$

Frome (3.18), we have

$$||T^*_{(C,C')}(f)|| \leq \sqrt{B}||f||, \qquad \forall f \in H.$$

So, $T^*_{(C,C')}$ is bounded $\mathcal{A}-$linear operator, then there exist a constant $M > 0$ such that

$$\langle T^*_{(C,C')}f, T^*_{(C,C')}f\rangle \leq M\langle f, f\rangle, \qquad \forall f \in H.$$

Hence

$$\sum_{j\in J} v_j^2\langle\Lambda_j P_{W_j}Cf, \Lambda_j P_{W_j}C^{'}f\rangle \leq M\langle f, f\rangle. \qquad \forall f \in H.$$

This give that $\Lambda_{CC'}$ is a $(C, C^{'})-$controlled $g-$fusion bessel sequence for H. □

Theorem 3.5. *Let $\{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $(C, C^{'})-$controlled $g-$fusion frame for H with bounds A and B, with operator frame $S_{(C,C')}$. Let $\theta \in End^*_{\mathcal{A}}(H)$ be injective and has a closed range. Suppose that θ commute with C, $C^{'}$ and P_{W_j} for all $j \in J$. Then $\{W_j, \Lambda_j\theta, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $g-$fusion frame for H.*

Proof. Let $\{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $(C, C^{'})-$controlled $g-$fusion frame for H with bounds A and B, then

$$A\langle f, f\rangle \leq \sum_{j\in J} v_j^2\langle\Lambda_j P_{W_j}Cf, \Lambda_j P_{W_j}C^{'}f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H.$$

For each $f \in H$, we have

$$\begin{aligned}\sum_{j\in J} v_j^2\langle\Lambda_j\theta P_{W_j}Cf, \Lambda_j\theta P_{W_j}C^{'}f\rangle &= \sum_{j\in J} v_j^2\langle\Lambda_j P_{W_j}C\theta f, \Lambda_j P_{W_j}C^{'}\theta f\rangle \\ &\leq B\langle\theta f, \theta f\rangle \\ &\leq B||\theta||^2\langle f, f\rangle \qquad (3.19)\end{aligned}$$

And

$$A\langle\theta f, \theta f\rangle \leq \sum_{j\in J} v_j^2\langle\Lambda_j\theta P_{W_j}Cf, \Lambda_j\theta P_{W_j}C^{'}f\rangle,$$

By lemma 2.3, we have

$$A||(\theta^*\theta)^{-1}||^{-1}\langle f, f\rangle \leq A\langle\theta f, \theta f\rangle$$

So

$$A||(\theta^*\theta)^{-1}||^{-1}\langle f,f\rangle \leq \sum_{j\in J} v_j^2\langle \Lambda_j\theta P_{W_j}Cf, \Lambda_j\theta P_{W_j}C^{'}f\rangle \tag{3.20}$$

Using (3.19) and (3.20) we conclude that

$$A||(\theta^*\theta)^{-1}||^{-1}\langle f,f\rangle \leq \sum_{j\in J} v_j^2\langle \Lambda_j\theta P_{W_j}Cf, \Lambda_j\theta P_{W_j}C^{'}f\rangle \leq B||\theta||^2\langle f,f\rangle, \qquad \forall f\in H.$$

Therefore $\{W_j, \Lambda_j\theta, v_j\}_{j\in J}$ is a $(C,C^{'})-$controlled $g-$fusion frame for H. □

Theorem 3.6. *Let $\{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $(C,C^{'})-$controlled $g-$fusion frame for H. with bounds A and B. Let $\theta\in End^*_{\mathcal{A}}(L,H)$ be injective and has a closed range. Suppose that θ commute with $\Lambda_jP_{W_j}C$ and $\Lambda_jP_{W_j}C^{'}$ for all $j\in J$. Then $\{W_j, \theta\Lambda_j, v_j\}_{j\in J}$ be a $(C,C^{'})-$controlled $g-$fusion frame for H.*

Proof. Let $\{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $(C,C^{'})-$controlled $g-$fusion frame for H with bounds A and B, then

$$A\langle f,f\rangle \leq \sum_{j\in J} v_j^2\langle \Lambda_jP_{W_j}Cf, \Lambda_jP_{W_j}C^{'}f\rangle \leq B\langle f,f\rangle, \qquad \forall f\in H.$$

We have for each $f\in H$

$$\begin{aligned}\sum_{j\in J} v_j^2\langle \theta\Lambda_jP_{W_j}Cf, \theta\Lambda_jP_{W_j}C^{'}f\rangle &\leq ||\theta||^2\sum_{j\in J} v_j^2\langle \Lambda_jP_{W_j}Cf, \Lambda_jP_{W_j}C^{'}f\rangle \\ &\leq B||\theta||^2\langle f,f\rangle \end{aligned}\tag{3.21}$$

On the other hand,

$$A\langle\theta f,\theta f\rangle \leq \sum_{j\in J} v_j^2\langle \theta\Lambda_jP_{W_j}Cf, \theta\Lambda_jP_{W_j}C^{'}f\rangle = \sum_{j\in J} v_j^2\langle \Lambda_jP_{W_j}C\theta f, \Lambda_jP_{W_j}C^{'}\theta f\rangle$$

By lemma 2.3, we have

$$A||(\theta^*\theta)^{-1}||^{-1}\langle f,f\rangle \leq \sum_{j\in J} v_j^2\langle \theta\Lambda_jP_{W_j}Cf, \theta\Lambda_jP_{W_j}C^{'}f\rangle \tag{3.22}$$

Using (3.21) and (3.22) , we conclude that

$$A||(\theta^*\theta)^{-1}||^{-1}\langle f,f\rangle \leq \sum_{j\in J} v_j^2\langle \theta\Lambda_jP_{W_j}Cf, \theta\Lambda_jP_{W_j}C^{'}f\rangle \leq B||\theta||^2\langle f,f\rangle, \qquad \forall f\in H.$$

Hence, $\{W_j, \theta\Lambda_j, v_j\}_{j\in J}$ is a $(C,C^{'})-$controlled $g-$fusion frame for H. □

Under wich conditions a $(C,C^{'})-$controlled $g-$fusion frame for H with H a C^*-module over a unital C^*-algebras $\mathcal{A}$ is also a $(C,C^{'})-$controlled $g-$fusion frame for H with H a C^*-module over a unital C^*-algebras $\mathcal{B}$. the following theorem answer this questions. We teak in next theorem $H_j\subset H$, $\forall j\in J$.

Theorem 3.7. *Let $(H, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ and $(H, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ be two Hilbert C^*-modules and let $\phi : \mathcal{A} \to \mathcal{B}$ be a $*-$homomorphisme and θ be a map on H such that $\langle \theta f, \theta g \rangle_{\mathcal{B}} = \phi(\langle f, g \rangle_{\mathcal{A}})$ for all $f,\ g \in H$. Suppose that $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C')-$controlled $g-$fusion frame for $(H, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ with frame operator $S_{\mathcal{A}}$ and lower and upper bounds A and B respectively. If θ is surjective such that $\theta \Lambda_j P_{W_j} = \Lambda_j P_{W_j} \theta$ for each $j \in J$ and $\theta C = C\theta$ and $\theta C' = C'\theta$.*
Then $\{W_j, \Lambda_j, \phi(v_j)\}_{j\in J}$ is a $(C, C')-$controlled $g-$fusion frame for $(H, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ with frame operator $S_{\mathcal{B}}$ and lower and upper bounds A and B respectively and $\langle S_{\mathcal{B}} \theta f, \theta g \rangle_{\mathcal{B}} = \phi(\langle S_{\mathcal{A}} f, g \rangle_{\mathcal{A}})$.

Proof. Since θ is surjective, then for every $g \in H$ there exists $f \in H$ such that $\theta f = g$. Using the definition of $(C, C')-$controlled $g-$fusion frame for $(H, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ we have

$$A\langle f, f \rangle_{\mathcal{A}} \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C' f \rangle_{\mathcal{A}} \leq B\langle f, f \rangle_{\mathcal{A}}$$

By lemma 2.1 we have

$$\phi\big(A\langle f, f \rangle_{\mathcal{A}}\big) \leq \phi\bigg(\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C' f \rangle_{\mathcal{A}}\bigg) \leq \phi\big(B\langle f, f \rangle_{\mathcal{A}}\big)$$

Frome the definition of $*-$homomorphisme we have

$$A\phi\big(\langle f, f \rangle_{\mathcal{A}}\big) \leq \sum_{j\in J} \phi(v_j^2)\phi\bigg(\langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C' f \rangle_{\mathcal{A}}\bigg) \leq B\phi\big(\langle f, f \rangle_{\mathcal{A}}\big)$$

Using the relation betwen θ and ϕ we get

$$A\langle \theta f, \theta f \rangle_{\mathcal{B}} \leq \sum_{j\in J} \phi(v_j)^2 \langle \theta \Lambda_j P_{W_j} C f, \theta \Lambda_j P_{W_j} C' f \rangle_{\mathcal{B}} \leq B\langle \theta f, \theta f \rangle_{\mathcal{B}}$$

Since $\theta \Lambda_j P_{W_j} = \Lambda_j P_{W_j} \theta$ for each $j \in J$ and $\theta C = C\theta$ and $\theta C' = C'\theta$ we have

$$A\langle \theta f, \theta f \rangle_{\mathcal{B}} \leq \sum_{j\in J} \phi(v_j)^2 \langle \Lambda_j P_{W_j} C \theta f, \Lambda_j P_{W_j} C' \theta f \rangle_{\mathcal{B}} \leq B\langle \theta f, \theta f \rangle_{\mathcal{B}}$$

Therefore,

$$A\langle g, g \rangle_{\mathcal{B}} \leq \sum_{j\in J} \phi(v_j)^2 \langle \Lambda_j P_{W_j} C g, \Lambda_j P_{W_j} C' g \rangle_{\mathcal{B}} \leq B\langle g, g \rangle_{\mathcal{B}}, \qquad \forall g \in H.$$

This implies that $\{W_j, \Lambda_j, \phi(v_j)\}_{j\in J}$ is a $(C, C')-$controlled $g-$fusion frame for $(H, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ with bounds A and B. Moreover we have

$$\begin{aligned}\phi\big(\langle S_{\mathcal{A}} f, g \rangle_{\mathcal{A}}\big) &= \phi\bigg(\langle \sum_{j\in J} v_j^2 C' P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} C f, g \rangle_{\mathcal{A}}\bigg) \\ &= \sum_{j\in J} \phi(v_j)^2 \phi\bigg(\langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C' g \rangle_{\mathcal{A}}\bigg)\end{aligned}$$

$$= \sum_{j\in J} \phi(v_j)^2 \langle \theta\Lambda_j P_{W_j} Cf, \theta\Lambda_j P_{W_j} C^{'} g\rangle_{\mathcal{B}}$$
$$= \langle \sum_{j\in J} \phi(v_j)^2 C^{'} P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} C\theta f, \theta g\rangle_{\mathcal{B}}$$
$$= \langle S_{\mathcal{B}}\theta f, \theta g\rangle_{\mathcal{B}}.$$

□

4. $(C, C^{'})-$controlled $K-g-$fusion frames in Hilbert C^*-modules

Firstly we give the definition of $K-g-$fusion frame in Hilbert C^*-modules.

Definition 4.1. *[11] Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a countably generated Hilbert $\mathcal{A}-$module. let $\left(v_j\right)_{j\in \mathrm{J}}$ be a family of weights in $\mathcal{A}$,i.e.,each v_j is a positive invertible element frome the center of $\mathcal{A}$, let $\left(W_j\right)_{j\in \mathrm{J}}$ be a collection of orthogonally complemented closed submodules of $\mathcal{H}$. Let $\left(\mathcal{K}_j\right)_{j\in \mathrm{J}}$ a sequence of closed submodules of $\mathcal{K}$ and $\Lambda_j \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}_j)$ for each $j \in \mathrm{J}$ and $K \in End_{\mathcal{A}}^*(\mathcal{H})$. We say $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ is $K-g-$fusion frame for $\mathcal{H}$ with respect to $(\mathcal{K}_j)_{j\in \mathrm{J}}$ if there exist real constants $0 < A \leq B < \infty$ such that*

$$A\langle K^* f, K^* f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in \mathcal{H}. \tag{4.1}$$

The constants A and B are called a lower and upper bounds of $K-g-$fusion frame, respectively. If the left-hand inequality of (4.1) is an equality, we say that Λ is a tight $K-g-$fusion frame. If $K = I_{\mathcal{H}}$ then Λ is a $g-$fusion frame and if $K = I_{\mathcal{H}}$ and $\Lambda_j = P_{W_j}$ for any $j \in \mathrm{J}$, then Λ is a fusion frame for $\mathcal{H}$

Definition 4.2. *Let $C, C^{'} \in GL^+(H)$ and $K \in End_{\mathcal{A}}^*(H)$. $\{W_j\}_{j\in J}$ be a sequence of closed submodules orthogonally complemented of H, $\{v_j\}_{j\in J}$ be a family of weights in $\mathcal{A}$, i.e., each v_j is a positive invertible element frome the center of $\mathcal{A}$ and $\Lambda_j \in End_{\mathcal{A}}^*(H, H_j)$ for each $j \in J$. We say $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $K-g-$fusion frame for H if there exists $0 < A \leq B < \infty$ such that*

$$A\langle K^* f, K^* f\rangle \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'} f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H. \tag{4.2}$$

The constants A and B are called a lower and upper bounds of $(C, C^{'})-$controlled $K-g-$fusion frame, respectively. If the left-hand inequality of (4.2) is an equality, we say that $\Lambda_{CC'}$ is a tight $(C, C^{'})-$controlled $K-g-$fusion frame for H.

Remark 4.1. *If $\Lambda_{CC'}$ is a $(C, C^{'})-$controlled $K-g-$fusion frame for H with bounds A and B we have*

$$AKK^* \leq S_{(C,C')} \leq BI_{H.} \tag{4.3}$$

From equality (3.6) and inequality (4.3) we have

Proposition 4.1. *Let $K \in End_{\mathcal{A}}^{*}(H)$, and $\Lambda_{CC'}$ be a $(C, C^{'})-$controlled $g-$fusion bessel sequence for H. Then $\Lambda_{CC'}$ is a $(C, C^{'})-$controlled $K-g-$fusion frame for H if and only if there exist a constant $A > 0$ such that $AKK^{*} \leq S_{(C,C')}$ where $S_{(C,C')}$ is the frame operator for $\Lambda_{CC'}$.*

Theorem 4.1. *Let $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ and $\Gamma_{CC'} = \{V_j, \Gamma_j, u_j\}_{j\in J}$ be two $(C, C^{'})-$controlled $g-$fusion bessel sequences for H with bounds B_1 and B_2, respectively. Suppose that $T_{\Lambda_{CC'}}$ and $T_{\Gamma_{CC'}}$ be their synthesis operators such that $T_{\Gamma_{CC'}}T^{*}_{\Lambda_{CC'}} = K^{*}$ for some $K \in End_{\mathcal{A}}^{*}(H)$. Then, both $\Lambda_{CC'}$ and $\Gamma_{CC'}$ are $(C, C^{'})-$controlled K and $K^{*}-g-$fusion frames for H, respectively.*

Proof. For each $f \in H$, we have

$$\begin{aligned}\langle K^{*}f, K^{*}f\rangle = \langle T_{\Gamma_{CC'}}T^{*}_{\Lambda_{CC'}}f, T_{\Gamma_{CC'}}T^{*}_{\Lambda_{CC'}}f\rangle &\leq \|T_{\Gamma_{CC'}}\|^{2}\langle T^{*}_{\Lambda_{CC'}}, T^{*}_{\Lambda_{CC'}}f\rangle\\ &\leq B_2\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}Cf, \Lambda_j P_{W_j}C^{'}f\rangle,\end{aligned}$$

Hence

$$B_2^{-1}\langle K^{*}f, K^{*}f\rangle \leq \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}Cf, \Lambda_j P_{W_j}C^{'}f\rangle.$$

This means that $\Lambda_{CC'}$ is a $(C, C^{'})-$controlled $K-g-$fusion frame for H. Similarly, $\Gamma_{CC'}$ is a $(C, C^{'})-$controlled $K^{*}-g-$fusion frame for H with the lower bound B_1^{-1}. □

Theorem 4.2. *Let $U \in End_{\mathcal{A}}^{*}(H)$ be an invertible operator on H and $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $(C, C^{'})-$controlled $K-g-$fusion frame for H for some $K \in End_{\mathcal{A}}^{*}(H)$. Suppose that $U^{*}UW_j \subset W_j$, $\forall j \in J$ and C, $C^{'}$ commute with U. Then $\Gamma_{CC'} = \{UW_j, \Lambda_j P_{W_j}U^{*}, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $UKU^{*}-g-$fusion frame for H.*

Proof. Since $\Lambda_{CC'}$ is a $(C, C^{'})-$controlled $K-g-$fusion frame for H, $\exists\ A, B > 0$ such that

$$A\langle K^{*}f, K^{*}f\rangle \leq \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}Cf, \Lambda_j P_{W_j}C^{'}f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H.$$

Also, U is an invertible linear operator on H, so for any $j \in J$, UW_j is closed in H. Now, for each $f \in H$, using lemma 2.7, we obtain

$$\begin{aligned}\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}U^{*}P_{UW_j}Cf, \Lambda_j P_{W_j}U^{*}P_{UW_j}C^{'}f\rangle &= \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}U^{*}Cf, \Lambda_j P_{W_j}U^{*}C^{'}f\rangle\\ &= \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}CU^{*}f, \Lambda_j P_{W_j}C^{'}U^{*}f\rangle\\ &\leq B\langle U^{*}f, U^{*}f\rangle\\ &\leq B\|U\|^{2}\langle f, f\rangle.\end{aligned}$$

On the other hand, for each $f \in H$

$$\begin{aligned}
A\langle (UKU^*)^*f, (UKU^*)^*f\rangle &= A\langle UK^*U^*f, UK^*U^*f\rangle \\
&\leq A||U||^2\langle K^*U^*f, K^*U^*f\rangle \\
&\leq ||U||^2\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} C(U^*f), \Lambda_j P_{W_j} C^{'}(U^*f)\rangle \\
&= ||U||^2\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} U^*Cf, \Lambda_j P_{W_j} U^*C^{'}f\rangle \\
&\leq ||U||^2\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} U^*P_{UW_j}Cf, \Lambda_j P_{W_j} U^*P_{UW_j}C^{'}f\rangle,
\end{aligned}$$

Then

$$\frac{A}{||U||^2}\langle (UKU^*)^*f, (UKU^*)^*f\rangle \leq \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} U^*P_{UW_j}Cf, \Lambda_j P_{W_j} U^*P_{UW_j}C^{'}f\rangle$$

Therefore, $\Gamma_{CC'}$ is a $(C, C^{'})-$controlled $UKU^* - g-$fusion frame for H. □

Theorem 4.3. *Let $U \in End_{\mathcal{A}}^*(H)$ be an invertible operator on H and $\Gamma_{CC'} = \{UW_j, \Lambda_j P_{W_j} U^*, v_j\}_{j\in J}$ be a $(C, C^{'})-$controlled $K - g-$fusion frame for H for some $K \in End_{\mathcal{A}}^*(H)$. Suppose that $U^*UW_j \subset W_j$, $\forall j \in J$ and C, $C^{'}$ commute with U. Then $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $U^{-1}KU - g-$fusion frame for H.*

Proof. Since $\Gamma_{CC'} = \{UW_j, \Lambda_j P_{W_j} U^*, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $K - g-$fusion frame for H, $\exists$ A, $B > 0$ such that

$$A\langle K^*f, K^*f\rangle \leq \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} U^*P_{UW_j}Cf, \Lambda_j P_{W_j} U^*P_{UW_j}C^{'}f\rangle \leq B\langle f, f\rangle. \qquad \forall f \in H.$$

Let $f \in H$, we have

$$\begin{aligned}
A\langle (U^{-1}KU)^*f, (U^{-1}KU)^*f\rangle &= A\langle U^*K^*(U^{-1})^*f, U^*K^*(U^{-1})^*f\rangle \\
&\leq A||U||^2\langle K^*(U^{-1})^*f, K^*(U^{-1})^*f\rangle \\
&\leq ||U||^2\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} U^*P_{UW_j}C(U^{-1})^*f, \Lambda_j P_{W_j} U^*P_{UW_j}C^{'}(U^{-1})^*f\rangle \\
&\leq ||U||^2\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} U^*C(U^{-1})^*f, \Lambda_j P_{W_j} U^*C^{'}(U^{-1})^*f\rangle \\
&= ||U||^2\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} U^*(U^{-1})^*Cf, \Lambda_j P_{W_j} U^*(U^{-1})^*C^{'}f\rangle \\
&= ||U||^2\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'}f\rangle.
\end{aligned}$$

Then, for each $f \in H$, we have

$$\frac{A}{||U||^2}\langle (U^{-1}KU)^*f, (U^{-1}KU)^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j} Cf, \Lambda_j P_{W_j} C^{'}f\rangle.$$

Also, for each $f \in H$, we have

$$\begin{aligned}\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C^{'} f\rangle &= \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C U^*(U^{-1})^* f, \Lambda_j P_{W_j} C^{'} U^*(U^{-1})^* f\rangle \\ &= \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} U^* C(U^{-1})^* f, \Lambda_j P_{W_j} U^* C^{'}(U^{-1})^* f\rangle \\ &= \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} U^* P_{UW_j} C(U^{-1})^* f, \Lambda_j P_{W_j} U^* P_{UW_j} C^{'}(U^{-1})^* f\rangle \\ &\leq B\langle (U^{-1})^* f, (U^{-1})^* f\rangle \\ &\leq B\|U^{-1}\|^2 \langle f, f\rangle.\end{aligned}$$

Thus, $\Lambda_{CC'}$ is a $(C, C^{'})-$controlled $U^{-1}KU - g-$fusion frame for H. $\square$

Theorem 4.4. *Let $K \in End_{\mathcal{A}}^*(H)$ be an invertible operator on H and $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $(C, C^{'})-$controlled $g-$fusion frame for H with frame bounds A, B and $S_{(C,C')}$ be the associated $(C, C^{'})-$controlled $g-$fusion frame operator. Suppose that for all $j \in J$, $T^*TW_j \subset W_j$, where $T = KS_{(C,C')}^{-1}$ and C, $C^{'}$ commute with T. Then $\{KS_{(C,C')}^{-1}W_j, \Lambda_j P_{W_j} S_{(C,C')}^{-1} K^*, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $K - g-$fusion frame for H with the corresponding $(C, C^{'})-$controlled $g-$fusion frame operator $KS_{(C,C')}^{-1}K^*$.*

Proof. We now $T = KS_{(C,C')}^{-1}$ is invertible on H and $T^* = (KS_{(C,C')}^{-1})^* = S_{(C,C')}^{-1}K^*$. For each $f \in H$, we have

$$\begin{aligned}\langle K^* f, K^* f\rangle &= \langle S_{(C,C')} S_{(C,C')}^{-1} K^* f, S_{(C,C')} S_{(C,C')}^{-1} K^* f\rangle \\ &\leq \|S_{(C,C')}\|^2 \langle S_{(C,C')}^{-1} K^* f, S_{(C,C')}^{-1} K^* f\rangle \\ &\leq B^2 \langle S_{(C,C')}^{-1} K^* f, S_{(C,C')}^{-1} K^* f\rangle.\end{aligned}$$

Now for each $f \in H$, we get

$$\begin{aligned}\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} T^* P_{TW_j} C(f), \Lambda_j P_{W_j} T^* P_{TW_j} C^{'}(f)\rangle &= \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} T^* C(f), \Lambda_j P_{W_j} T^* C^{'}(f)\rangle \\ &= \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C T^*(f), \Lambda_j P_{W_j} C^{'} T^*(f)\rangle \\ &\leq B\langle T^* f, T^* f\rangle \\ &\leq B\|T\|^2 \langle f, f\rangle \\ &\leq B\|S_{(C,C')}^{-1}\|^2 \|K\|^2 \langle f, f\rangle \\ &\leq \frac{B}{A^2}\|K\|^2 \langle f, f\rangle.\end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned}
\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} T^* P_{TW_j} C(f), \Lambda_j P_{W_j} T^* P_{TW_j} C^{'}(f)\rangle &= \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} T^* C(f), \Lambda_j P_{W_j} T^* C^{'}(f)\rangle \\
&= \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C T^*(f), \Lambda_j P_{W_j} C^{'} T^*(f)\rangle \\
&\geq A\langle T^* f, T^* f\rangle \\
&= A\langle S_{(C,C')}^{-1} K^* f, S_{(C,C')}^{-1} K^* f\rangle \\
&\geq \frac{A}{B^2}\langle K^* f, K^* f\rangle.
\end{aligned}$$

Thus $\{KS_{(C,C')}^{-1} W_j, \Lambda_j P_{W_j} S_{(C,C')}^{-1} K^*, v_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $K - g-$fusion frame for H.
For each $f \in H$, we have

$$\begin{aligned}
\sum_{j\in J} v_j^2 C^{'} P_{TW_j} (\Lambda_j P_{W_j} T^*)^* (\Lambda_j P_{W_j} T^*) P_{TW_j} C f &= \sum_{j\in J} v_j^2 C^{'} P_{TW_j} T P_{W_j} \Lambda_j^* (\Lambda_j P_{W_j} T^*) P_{TW_j} C f \\
&= \sum_{j\in J} v_j^2 C^{'} (P_{W_j} T^* P_{TW_j})^* \Lambda_j^* \Lambda_j (P_{W_j} T^* P_{TWj}) C f \\
&= \sum_{j\in J} v_j^2 C^{'} T P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} T^* C f \\
&= \sum_{j\in J} v_j^2 T C^{'} P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} C T^* f \\
&= T(\sum_{j\in J} v_j^2 C^{'} P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} C T^* f) \\
&= T S_{(C,C')} T^*(f) = K S_{(C,C')}^{-1} K^*(f).
\end{aligned}$$

This implies that $KS_{(C,C')}^{-1} K^*$ is the associated $(C, C^{'})-$controlled $g-$fusion frame operator. □

The next theorem we give an equivqlent definition of $(C, C^{'})-$controlled $K - g-$fusion frame.

Theorem 4.5. *Let $K \in End_{\mathcal{A}}^*(H)$. Then $\Lambda_{CC'}$ is a $(C, C^{'})-$controlled $K - g-$fusion frame for H if and only if there exist constants A, $B > 0$ such that*

$$A||K^* f||^2 \leq ||\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} C^{'} f\rangle|| \leq B||f||^2, \qquad \forall f \in H. \tag{4.4}$$

Proof. Evidently, every $(C, C^{'})-$controlled $K - g-$fusion frame for H satisfies (4.4).
For the converse, we suppose that (4.4) holds. For any $\{f_j\}_{j\in J} \in l^2(\{H_j\}_{j\in J})$,

$$\begin{aligned}
||\sum_{j\in J} v_j (CC^{'})^{\frac{1}{2}} P_{W_j} \Lambda_j^* f_j|| &= \sup_{||g||=1} ||\langle \sum_{j\in J} v_j (CC^{'})^{\frac{1}{2}} P_{W_j} \Lambda_j^* f_j, g\rangle|| \\
&= \sup_{||g||=1} ||\sum_{j\in J} \langle v_j (CC^{'})^{\frac{1}{2}} P_{W_j} \Lambda_j^* f_j, g\rangle||
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{||g||=1} ||\sum_{j\in J}\langle f_j, v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} g\rangle||\\
&\leq \sup_{||g||=1} ||\sum_{j\in J}\langle f_j, f_j\rangle||^{\frac{1}{2}} ||\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} g, \Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} g\rangle||^{\frac{1}{2}}\\
&= \sup_{||g||=1} ||\sum_{j\in J}\langle f_j, f_j\rangle||^{\frac{1}{2}} ||\sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}Cg, \Lambda_j P_{W_j}C^{'}g\rangle||^{\frac{1}{2}}\\
&\leq \sup_{||g||=1} ||\sum_{j\in J}\langle f_j, f_j\rangle||^{\frac{1}{2}}\sqrt{B}||g|| = \sqrt{B}||\{f_j\}_{j\in J}||.
\end{aligned}
$$

Thus the series $\sum_{j\in J} v_j(CC^{'})^{\frac{1}{2}} P_{W_j}\Lambda_j^* f_j$ converges in H unconditionally. Since

$$\langle Tf, \{f_j\}_{j\in J}\rangle = \sum_{j\in J}\langle v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f, f_j\rangle = \langle f, \sum_{j\in J} v_j(CC^{'})^{\frac{1}{2}} P_{W_j}\Lambda_j^* f_j\rangle.$$

T is adjointable. Now for each $f\in H$ we have

$$\langle Tf, Tf\rangle = \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}Cf, \Lambda_j P_{W_j}C^{'}f\rangle \leq ||T||^2\langle f, f\rangle.$$

On the other hand the left-hand inequality of (4.4) gives

$$||K^*f||^2 \leq \frac{1}{A}||Tf||^2, \qquad \forall f\in H.$$

Then the lemma 2.6 implies that there exist a constant $\mu > 0$ such that

$$KK^* \leq \mu T^*T,$$

And hence

$$\frac{1}{\mu}\langle K^*f, K^*f\rangle \leq \langle Tf, Tf\rangle = \sum_{j\in J} v_j^2\langle \Lambda_j P_{W_j}Cf, \Lambda_j P_{W_j}C^{'}f\rangle, \qquad \forall f\in H.$$

Consequently, $\Lambda_{CC'}$ is a $(C, C^{'})-$controlled $K-g-$fusion frame for H. □

5. perturbation of $(C, C^{'})-$controlled $K-g-$fusion frame in Hilbert C^*-modules

Theorem 5.1. *Let $\Lambda_{CC'} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ be a $(C, C^{'})-$controlled $K-g-$fusion frame for H with frame bounds A, B and $\Gamma_j \in End_{\mathcal{A}}^*(H, H_j)$. Suppose that for each $f\in H$,*

$$
\begin{aligned}
||((v_j\Lambda_j P_{W_j} - u_j\Gamma_j P_{V_j})(CC^{'})^{\frac{1}{2}} f)_{j\in J}|| \leq &\lambda_1||(v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}|| +\\
&\lambda_2||(u_j\Gamma_j P_{V_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}|| + \epsilon||K^*f||.
\end{aligned}
$$

where $0<\lambda_1, \lambda_2<1$ and $\epsilon>0$ such that $\epsilon < (1-\lambda_1)\sqrt{A}$.

Then $\{W_j, \Gamma_j, u_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $K-g-$fusion frame for H.

Proof. We have for each $f \in \mathcal{H}$

$$\begin{aligned}
\|\sum_{j\in J} u_j^2 \langle \Gamma_j P_{V_j} Cf, \Gamma_j P_{V_j} C^{'} f\rangle\|^{\frac{1}{2}} &= \|(u_j\Gamma_j P_{V_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&= \|(u_j\Gamma_j P_{V_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J} + (v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J} \\
&\qquad - (v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&\leq \|((u_j\Gamma_j P_{V_j} - v_j\Lambda_j P_{W_j})(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&\qquad + \|(v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&\leq (\lambda_1 + 1)\|(v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&\qquad + \lambda_2\|(u_j\Gamma_j P_{V_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| + \epsilon\|K^* f\|.
\end{aligned}$$

So

$$(1-\lambda_2)\|(u_j\Gamma_j P_{V_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \leq (\lambda_1+1)\sqrt{B}\|f\| + \epsilon\|K^* f\|.$$

Then

$$\begin{aligned}
\|(u_j\Gamma_j P_{V_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| &\leq \frac{(\lambda_1+1)\sqrt{B}\|f\| + \epsilon\|K^* f\|}{1-\lambda_2} \\
&\leq (\frac{(\lambda_1+1)\sqrt{B} + \epsilon\|K\|}{1-\lambda_2})\|f\|.
\end{aligned}$$

Hence

$$\|\sum_{j\in J} u_j^2 \langle \Gamma_j P_{V_j} Cf, \Gamma_j P_{V_j} C^{'} f\rangle\| \leq (\frac{(\lambda_1+1)\sqrt{B} + \epsilon\|K\|}{1-\lambda_2})^2\|f\|^2.$$

On the other hand for each $f \in \mathcal{H}$

$$\begin{aligned}
\|\sum_{j\in J} u_j^2 \langle \Gamma_j P_{V_j} Cf, \Gamma_j P_{V_j} C^{'} f\rangle\|^{\frac{1}{2}} &= \|(u_j\Gamma_j P_{V_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&= \|((u_j\Gamma_j P_{V_j} - v_j\Lambda_j P_{W_j})(CC^{'})^{\frac{1}{2}} f)_{j\in J} \\
&\qquad + (v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&\geq \|(v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&\qquad - \|((u_j\Gamma_j P_{V_j} - v_j\Lambda_j P_{W_j})(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&\geq (1-\lambda_1)\|(v_j\Lambda_j P_{W_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| \\
&\qquad - \lambda_2\|(u_j\Gamma_j P_{V_j}(CC^{'})^{\frac{1}{2}} f)_{j\in J}\| - \epsilon\|K^* f\|.
\end{aligned}$$

Hence

$$\|\sum_{j\in J} u_j^2 \langle \Gamma_j P_{V_j} Cf, \Gamma_j P_{V_j} C^{'} f\rangle\| \geq (\frac{(1-\lambda_1)\sqrt{A} - \epsilon}{1+\lambda_2})^2\|K^* f\|^2.$$

By theorem 4.5, we conclude that $\{V_j, \Gamma_j, u_j\}_{j\in J}$ is a $(C, C^{'})-$controlled $K-g-$fusion frame for H. $\square$

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