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n-Convexity via Delta-Integral Representation of Divided Difference on Time Scales

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Abstract. We introduce the delta-integral representation of divided difference on arbitrary time scales and utilize it to set criteria for n-convex functions involving delta-derivative on time scales. Consequences of the theory appear in terms of estimates which generalize and extend some important facts in mathematical analysis.

1. Introduction

Time scale calculus is a well known and rapidly growing theory in mathematical analysis which unifies two distinct well-known mathematical areas named as continuous and discrete analysis. For supplementary details and basics of time scale calculus, we invoke [1–3].

The notion of convexity with its various types have a noteworthy presence in literature, see [4–7] and the references therein. The notion is firstly generalized on an arbitrary time scale in 2008 by Cristian Dinu [8], subsequently a large number of estimation and inequalities for the functions that are convex on time scales are in the continuous state of development, some of them are present in [9, 10]. Here we consult with an exclusive variety of these functions, that is n-convex functions. The n-convexity or higher order convexity firstly investigated by Eberhard Hopf [11] in his scholarly thesis. Further it was discussed in different narrations by Popoviciu [12, 13]. A comprehensive review of this family of functions is elaborated in [5, 14]. In [15] M. Rozarija, and J. Pečarić discussed some "Jensen-Type Inequalities on Time Scales" involving real-valued n-convex functions. Higher order convex functions has been discussed on time scales with constant graininess function by H. A. Baig and N. Ahmad in [16], so there is a need to explore this class of functions on arbitrary time scales.

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This article is structured as follows. In section 2 we furnish few preliminaries, utilizing in the main results. Section 3 is dedicated to construct a relationship between nth delta derivatives and nth-order divided difference on arbitrary time scales. Afterward, we presented some mathematical inequalities as consequences of our main results in the last section.

2. Preliminaries

A time scale $\mathbb{T}$ is defined to be an arbitrary closed subset of the real numbers $\mathbb{R}$, with the standard inherited topology. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, where $\inf\phi = \sup\mathbb{T}$ and $\sup\phi = \inf\mathbb{T}$. Let $u : \mathbb{T} \to \mathbb{R}$, $u^{\Delta}(t)$ is representing the first delta derivative of function u at $t \in \mathbb{T}^{\kappa}$. The second-order delta derivative of u at t is defined as, provided it exists

$$u^{\Delta^2}(t) = u^{\Delta\Delta}(t) = (u^{\Delta}(t))^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$$

Similarly higher-order derivatives are defined as $u^{\Delta^n}(t) : \mathbb{T}^{\kappa^n} \to \mathbb{R}$. The definition for rd-continuous functions can be seen in [2]. The set of rd-continuous functions $u : \mathbb{T} \to \mathbb{R}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R}) = C_{rd}(\mathbb{T}).$$

The set consisting of first-order delta differentiable functions u and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}, \mathbb{R}) = C^1_{rd}(\mathbb{T}).$$

The substitution rule and first mean value theorem for delta-integrals in time scales are presented in [1–3].

Theorem 2.1. *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $u \in C_{rd}$ and $\nu \in C^1_{rd}$, then for $a, b \in \mathbb{T}$*

$$\int_a^b u(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} \left(u \circ \nu^{-1}\right)(s)\tilde{\Delta}s. \tag{2.1}$$

Theorem 2.2. *Let ν and u be bounded and integrable functions on $[a, b]$, and let ν be nonnegative (or nonpositive) on $[a, b]$. Let us set*

$$M = \sup\{u(t) : t \in [a, b)\} \qquad m = \inf\{u(t) : t \in [a, b)\}.$$

Then there exists a real number λ satisfying the inequalities $m < \lambda < M$ such that

$$\int_a^b u(t)\nu(t)\Delta t = \lambda \int_a^b \nu(t)\Delta t.$$

The time scale monomials have been defined in [1, 3, 17] recursively as

$$g_0(t, s) = h_0(t, s) = 1 \text{ for } s, t \in \mathbb{T},$$

$$g_{k+1}(t,s)=\int_s^t g_k(\sigma(\gamma),s)\,\Delta\gamma,\quad h_{k+1}(t,s)=\int_s^t h_k(\gamma,s)\,\Delta\gamma,\quad k\in\mathbf{N}_0. \tag{2.2}$$

These monomials satisfy the following relation for $t\in\mathbb{T}$ and $s\in\mathbb{T}^{\kappa}$:

$$g_n(t,s)=(-1)^n h_n(s,t)\,. \tag{2.3}$$

Remark 2.1. *[17] The functions h_n and g_n satisfy*

$$g_n(t,s)\geq 0 \text{ and } h_n(t,s)\geq 0 \text{ for all } t\geq s.$$

Let us recall the Taylor's formula defined on time scales from [17].

Theorem 2.3. *Let u be n-times delta-differentiable on $\mathbb{T}^{\kappa^n}$, $t\in\mathbb{T}$ and $t_\alpha\in\mathbb{T}^{\kappa^{n-1}}$. We have*

$$u(t)-\sum_{k=0}^{n-1}h_k(t,t_\alpha)u^{\Delta^k}(t_\alpha)=\int_{t_\alpha}^{\rho^{n-1}(t)}h_{n-1}(t,\sigma(\gamma))u^{\Delta^n}(\gamma)\Delta\gamma, \tag{2.4}$$

similarly,

$$u(t)-\sum_{k=0}^{n-1}(-1)^n g_k(t_\alpha,t)u^{\Delta^k}(t_\alpha)=\int_{t_\alpha}^{\rho^{n-1}(t)}(-1)^n g_{n-1}(\sigma(\gamma),t)u^{\Delta^n}(\gamma)\Delta\gamma, \tag{2.5}$$

where $k\in\mathbb{N}_0$.

higher order convex functions defined on $\mathbb{R}$ as well as on $\mathbb{Z}$ through nth-order divided difference, in which we randomly select $n+1$ points $\{a_0,a_1,\ldots,a_n\}$ from $\mathbb{R}$ or from $\mathbb{Z}$, respectively and compute the nth-order divided difference by the formula

$$[a_0,a_1,\cdots,a_n;u]=\frac{[a_1,a_2,\cdots,a_n;u]-[a_0,a_1,\cdots,a_{n-1};u]}{a_n-a_0}. \tag{2.6}$$

If (2.6) is non-negative we say that u is an n-convex function. Here (2.6) remains same for every permutation of $n+1$ points.

To construct the criteria for n-convexity we need to introduce the forward operator σ in the definition of higher order convexity. So we adopt the same strategy as we did in [16]. Assume $n+1$ distinct points $t_0,\cdots,t_n\in\mathbb{T}$ and arrange them in an increasing order. Relabel these points in the time scale $\tilde{\mathbb{T}}$ in terms of forward operator, that is

$$\tilde{\mathbb{T}}=\{t_0,\sigma(t_0),\cdots,\sigma^n(t_0)\}.$$

Consequently we can define the nth-order divided difference for $n+1$ points as

$$[t_0,\sigma(t_0),\cdots,\sigma^n(t_0);u]=\frac{[\sigma(t_0),\sigma^2(t_0),\cdots,\sigma^n(t_0);u]-[t_0,\sigma(t_0),\cdots,\sigma^{n-1}(t_0);u]}{\sigma^n(t_0)-t_0}. \tag{2.7}$$

So a function $u:\mathbb{T}\to\mathbb{R}$, is said to be n-convex if

$$[t_0,\sigma(t_0),\cdots,\sigma^n(t_0);u]\geq 0, \tag{2.8}$$

where $\sigma:\mathbb{T}\bigcap\tilde{\mathbb{T}}\to\mathbb{T}\bigcap\tilde{\mathbb{T}}$.

3. Main Results

Here we want to establish a criteria for n-convex function on arbitrary time scales which is stated as $u \in C^n_{rd}$ is n-convex iff $u^{\Delta^n} \geq 0$. It is sufficient to prove this on $\tilde{\mathbb{T}}$. Firstly we introduce a new representation of divided difference in terms of delta-integral, that can be seen in the next Theorem.

Theorem 3.1. *Suppose $u \in C^n_{rd}(\mathbb{T}, \mathbb{R})$. Let $t_0, t_1, \cdots, t_n$ be $n+1$ distinct points in $\mathbb{T}$, then*

$$[t_0, \sigma(t_0), \cdots, \sigma^n(t_0); u] = \int_0^1 \Delta s_1 \int_0^{s_1} \Delta s_2 \cdots \int_0^{s_{n-1}} \Delta s_n \times u^{\Delta^n}(s_n[\sigma^n(t_0) - \sigma^{n-1}(t_0)] + \cdots + s_1[\sigma(t_0) - t_0] + t_0), \tag{3.1}$$

where $n \geq 1$ and $s_i \in [0, 1]$.

Proof. Consider $t_0, t_1, \cdots, t_n$, $n+1$ distinct points and the corresponding time scale $\tilde{\mathbb{T}} = \{t_0, \sigma(t_0), \cdots, \sigma^n(t_0)\}$. We prove (4.3) by induction method. For this we first show that

$$[t_0, \sigma(t_0); u] = \int_0^1 u^{\Delta}(s_1[\sigma(t_0) - t_0] + t_0)\Delta s_1. \tag{3.2}$$

Let us use the time scales substitution rule for integration (2.1), let the new variable of integration β in the following manner (since $\sigma(t_0) \neq t_0$)

$$\beta = v^{-1}(s_1) = s_1[\sigma(t_0) - t_0] + t_0 \Rightarrow v(s_1) = \frac{s_1 - t_0}{\sigma(t_0) - t_0},$$

here $v^{-1} : [0, 1] \to \tilde{\mathbb{T}}$. By calculating delta derivative of $v(s_1)$ with respect to s_1 we get $v^{\Delta}(s_1) = \frac{1}{\sigma(t_0) - t_0}$ therefore, $s_1 \in [0, 1]$ and $v(s_1)$ is strictly increasing such that $v[t_0, \sigma(t_0)] = [0, 1]$. Hence the corresponding limits are

$$(s_1 = 0) \to (\beta = t_0); \qquad (s_1 = 1) \to (\beta = \sigma(t_0)).$$

Since $\sigma(t_0) \neq t_0$, thus (3.2) can be written as

$$\begin{aligned}
&\int_0^1 u^{\Delta}(s_1[\sigma(t_0) - t_0] + t_0)\Delta s_1 \\
&= \int_{v(t_0)}^{v(\sigma(t_0))} u^{\Delta}(v^{-1}(s_1))\Delta s_1 \\
&= \int_{t_0}^{\sigma(t_0)} \frac{u^{\Delta}(\beta)}{\sigma(t_0) - t_0}\Delta\beta \\
&= \frac{1}{\sigma(t_0) - t_0}\left(u(\beta)\Big|_{t_0}^{\sigma(t_0)} \right) \\
&= \frac{u(\sigma(t_0)) - u(t_0)}{\sigma(t_0) - t_0}.
\end{aligned}$$

Now we make the inductive hypothesis that

$$
\begin{aligned}
&[t_0, \sigma(t_0), \cdots, \sigma^{n-1}(t_0); u] \\
&= \int_0^1 \Delta s_1 \int_0^{s_1} \Delta s_2 \cdots \int_0^{s_{n-2}} \Delta s_{n-1} \\
&\quad \times u^{\Delta^{n-1}}(s_{n-1}[\sigma^{n-1}(t_0) - \sigma^{n-2}(t_0)] + \cdots + s_1[\sigma(t_0) - t_0] + t_0).
\end{aligned}
$$

In the integral in (3.1) we apply substitution rule of integration of time scales (2.1) by replacing the variable of integration s_n with β.

$$
\begin{aligned}
\beta =& v^{-1}(s_n) = s_n[\sigma^n(t_0) - \sigma^{n-1}(t_0)] + \cdots + s_1[\sigma(t_0) - t_0] + t_0 \\
\Rightarrow v(s_n) =& \frac{s_n - (s_{n-1}[\sigma^{n-1}(t_0) - \sigma^{n-2}(t_0)] + \cdots + s_1[\sigma(t_0) - t_0] + t_0)}{\sigma^n(t_0) - \sigma^{n-1}(t_0)}.
\end{aligned}
$$

So that the delta derivative of $v(s_n)$ with respect to s_n gives us

$$
v^{\Delta}(s_n) = \frac{1}{\sigma^n(t_0) - \sigma^{n-1}(t_0)}.
$$

The corresponding limits are

$$
\begin{gathered}
(s_n = 0) \to \big(\beta = \beta_0 \equiv s_{n-1}[\sigma^{n-1}(t_0) - \sigma^{n-2}(t_0)] + \cdots + s_1[\sigma(t_0) - t_0] + t_0\big) \\
(s_n = s_{n-1}) \to (\beta = \beta_1 \equiv s_{n-1}[\sigma^n(t_0) - \sigma^{n-2}(t_0)] + s_{n-2}[\sigma^{n-2}(t_0) - \sigma^{n-3}(t_0)] + \\
\cdots + s_1[\sigma(t_0) - t_0] + t_0).
\end{gathered}
$$

Thus the innermost integral of (4.3) can transform in the following manner, since $\sigma^n(t_0) \neq \sigma^{n-1}(t_0)$

$$
\begin{aligned}
&\int_0^{s_{n-1}} u^{\Delta^n}(s_n[\sigma^n(t_0) - \sigma^{n-1}(t_0)]) + \cdots + s_1[\sigma(t_0) - t_0] + t_0)\Delta s_n \\
&= \int_{\beta_0}^{\beta_1} \frac{u^{\Delta^n}(\beta)}{\sigma^n(t_0) - \sigma^{n-1}(t_0)} \Delta\beta \\
&= \frac{1}{\sigma^n(t_0) - \sigma^{n-1}(t_0)} \left(u^{\Delta^{n-1}}(\beta)\Big|_{\beta_0}^{\beta_1} \right) \\
&= \frac{u^{\Delta^{n-1}}(\beta_1) - u^{\Delta^{n-1}}(\beta_0)}{\sigma^n(t_0) - \sigma^{n-1}(t_0)}.
\end{aligned}
$$

However, by applying the inductive hypothesis we have

$$
\begin{aligned}
&\int_0^1 \Delta s_1 \int_0^{s_1} \Delta s_2 \cdots \int_0^{s_{n-2}} \Delta s_{n-1} \left(\frac{u^{\Delta^{n-1}}(\beta_1) - u^{\Delta^{n-1}}(\beta_0)}{\sigma^n(t_0) - \sigma^{n-1}(t_0)} \right) \\
&= \frac{u[t_0, \sigma(t_0), \cdots, \sigma^{n-2}(t_0), \sigma^n(t_0)] - u[t_0, \sigma(t_0), \cdots, \sigma^{n-2}(t_0), \sigma^{n-1}(t_0)]}{\sigma^n(t_0) - \sigma^{n-1}(t_0)} \\
&= [t_0, \sigma(t_0), \cdots, \sigma^n(t_0); u].
\end{aligned}
$$

□

In the next Theorem we establish a relation between nth-order divided difference and nth-delta derivative on arbitrary time scales, since in this result the points $t_i \in \mathbb{T}$ need not to be distinct.

Theorem 3.2. *Let $u \in C^n_{rd}(\mathbb{T}, \mathbb{R})$, then for $n+1$ points form $\mathbb{T}$ we have*

$$[t_0, \sigma(t_0), \cdots, \sigma^n(t_0); u] = u^{\Delta^n}(\xi)\left(h_i(s_{n-i}, 0)\right), \tag{3.3}$$

where $s_0 = 1$, $0 \le i \le n$, and $\xi \in [t_0, \sigma^n(t_0)]_{\mathbb{T}}$.

Proof. By using the time scale monomials (2.2) we can write a general notation for the integral $\int_0^1 \Delta s_1 \int_0^{s_1} \Delta s_2 \cdots \int_0^{s_{n-1}} \Delta s_n$, that is

$$h_i(s_{n-i}, 0) = \int_0^{s_{n-i}} h_{i-1}(s_{n-i+1}, 0)\Delta s_{n-i+1}. \tag{3.4}$$

By the Remark 2.1 we can conclude that $h_n(s_i, 0) > 0$ in (3.4) because all $s_i > 0$. Now by applying Theorem 2.2, (3.1) yields

$$x\left(h_i(s_{n-i}, 0)\right) \le [t_0, \sigma(t_0), \cdots, \sigma^n(t_0); u] \le X\left(h_i(s_{n-i}, 0)\right),$$

or

$$x \le \frac{[t_0, \sigma(t_0), \cdots, \sigma^n(t_0); u]}{(h_i(s_{n-i}, 0))} \le X,$$

where $x \equiv \min u^{\Delta^n}(t)$ and $X \equiv \max u^{\Delta^n}(t)$ for $t \in [t_0, \sigma^n(t_0)]_{\mathbb{T}}$. Then by the rd-continuity of u^{Δ^n} there exists a λ in this interval that is $u^{\Delta^n}(\xi) = \lambda$, such that

$$\frac{[t_0, \sigma(t_0), \cdots, \sigma^n(t_0); u]}{(h_i(s_{n-i}, 0))} = u^{\Delta^n}(\xi).$$

□

Here, we can directly achieve the next result.

Corollary 3.1. *Let $u : \mathbb{T} \to \mathbb{R}$ is n-convex function iff $u^{\Delta^n} \ge 0$, given that u^{Δ^n} exists.*

Another useful property of n-convex function is represented in the next result.

Theorem 3.3. *Let $u(t) \in C^n_{rd}(\mathbb{T}, \mathbb{R})$ is n-convex function, then for every $r \in \mathbb{N}$, $1 \le r \le n-1$, u^{Δ^r} is $(n-r)$-convex.*

Proof. By Corollary 3.1 $u^{\Delta^n} \ge 0$. Since u^{Δ^r} exists for every $1 \le r \le n-1$. Let us choose $(n-r+1)$ points from $[t_a, t_b]_{\mathbb{T}}$ such that $\tilde{\mathbb{T}} = \{t_0, \sigma(t_0), \cdots \sigma^{n-r}(t_0)\}$, then by using (3.3) we can write

$$\begin{aligned}[t_0, \sigma(t_0), \cdots, \sigma^{n-r}(t_0); u^{\Delta^r}] &= \left(u^{\Delta^r}(\xi)\right)^{\Delta^{n-r}}\left(h_{n-r}(s_r, 0)\right) \\ &= \left(u(\xi)\right)^{\Delta^n}\left(h_{n-r}(s_r, 0)\right) \ge 0, \end{aligned} \tag{3.5}$$

where $\xi \in [t_0, \sigma^{n-r}(t_0)]_{\mathbb{T}}$. Thus (3.5) shows that u^{Δ^r} is $(n-r)$-convex for every $1 \le r \le n-1$.

□

4. Applications: Inequalities for n-convex functions

Let us present Levinson's type inequality for higher-order convex functions on time scales for this we require the next result. Let $t_i \in [t_a, t_b]_{\mathbb{T}}$, for $i = 1, \cdots, z$. Let $b_i > 0$ such that $\sum_{i=1}^{z} b_i = 1$ therefore $t \in [t_a, t_b]_{\mathbb{T}}$ denoted by $\sum_{i=1}^{z} b_i t_i$.

Theorem 4.1. *Let u is $(n+2)$-convex on $\mathbb{T}$. Then for every $t \in \mathbb{T}$ the function*

$$U(t) = [t, \sigma(t), \cdots, \sigma^n(t); u], \tag{4.1}$$

is a convex function.

Proof. By using (3.1), (4.1) can be expressed as

$$\begin{aligned} U(t) &= [t, \sigma(t), \cdots, \sigma^n(t); u] \\ &= \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} u^{\Delta^n}(s_n[\sigma^n(t) - \sigma^{n-1}(t)] + \cdots + s_1[\sigma(t) - t] + t)\Delta s_n \cdots \Delta s_1. \end{aligned}$$

Therefore u^{Δ^n} is convex by Theorem 3.3, thus for fixed s_j, $\sigma^j(t)$ for $j = 1, \cdots, n$ we can write

$$u^{\Delta^n}\left(\sum_{j=1}^{n} s_j[\sigma^j(t) - \sigma^{j-1}(t)] + \sum_{i=1}^{z} b_i t_i\right) \leq \sum_{i=1}^{z} b_i u^{\Delta^n}\left(\sum_{j=1}^{n} s_j[\sigma^j(t) - \sigma^{j-1}(t)] + t_i\right),$$

which concludes the proof. □

Theorem 4.2. *If u is $(n+2)$-convex on $\mathbb{T}$, then the given inequality is true*

$$u[t, \sigma(t), \cdots, \sigma^n(t)] \leq \sum_{i=1}^{z} b_i[t_i, \sigma(t_i), \cdots, \sigma^n(t_i); u]. \tag{4.2}$$

Proof. The proof is the direct consequence of Theorem 4.1. □

Remark 4.1. *Let $\mathbb{T} = \mathbb{R}$ in Theorem 4.2, inequality (4.2) coincides with inequality (4) in [18], this Levinson's type inequality itself having a great importance in literature which is used to develop further divided difference estimates for n-convex functions in [19].*

Further, we present certain useful inequalities involving n-convex functions on time scales by using the criteria for n-convexity, that is $u^{\Delta^n} \geq 0$.

Theorem 4.3. *Let $t_\alpha, t_\beta \in \mathbb{T}^{\kappa^n}$, suppose $u \in C_{rd}^{n+1}(\mathbb{T}, \mathbb{R})$ be $(n+1)$-convex function on $[t_\alpha, t_\beta]$. Then for each $t \in (t_\alpha, t_\beta)$, the following inequalities hold*

$$\begin{aligned} \sum_{k=0}^{n-1} h_k(t, t_\alpha) u^{\Delta^k}(t_\alpha) + u^{\Delta^n}(t_\alpha) \int_{t_\alpha}^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\gamma))\Delta\gamma \leq u(t) \\ \leq \sum_{k=0}^{n-1} h_k(t, t_\alpha) u^{\Delta^k}(t_\alpha) + u^{\Delta^n}(t_\beta) \int_{t_\alpha}^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\gamma))\Delta\gamma, \end{aligned} \tag{4.3}$$

where $t_\alpha < \rho^{n-1}(t_\beta)$. *If n is odd, then*

$$\sum_{k=0}^{n-1} h_k(t,t_\beta)u^{\Delta^k}(t_\beta) + u^{\Delta^n}(t_\beta)\int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))\Delta\gamma \le u(t)$$
$$\le \sum_{k=0}^{n-1} h_k(t,t_\beta)u^{\Delta^k}(t_\beta) + u^{\Delta^n}(t_\alpha)\int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))\Delta\gamma, \tag{4.4}$$

and if n is even, the given inequality holds

$$\sum_{k=0}^{n-1} h_k(t,t_\beta)u^{\Delta^k}(t_\beta) + u^{\Delta^n}(t_\alpha)\int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))\Delta\gamma \le u(t)$$
$$\le \sum_{k=0}^{n-1} h_k(t,t_\beta)u^{\Delta^k}(t_\beta) + u^{\Delta^n}(t_\beta)\int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))\Delta\gamma. \tag{4.5}$$

Proof. If u is $(n+1)-$convex on $\mathbb{T}^{\kappa^n}$ which implies that $u^{\Delta^{n+1}} \ge 0$, then u^{Δ^n} is increasing on $\mathbb{T}^{\kappa^n}$, i.e $u^{\Delta^n}(t_\alpha) \le u^{\Delta^n}(\gamma) \le u^{\Delta^n}(t_\beta)$ for each $\gamma \in [t_\alpha, t_\beta]$, let $\sigma(\gamma) \le t$ so that $h_{n-1}(t,\sigma(\gamma))$ is non-negative, then from (2.4) we get

$$\int_{t_\alpha}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))u(t_\alpha)\Delta\gamma \le u(t) - \sum_{k=0}^{n-1} h_k(t,t_\alpha)u^{\Delta^k}(t_\alpha)$$
$$\le \int_{t_\alpha}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))u^{\Delta^n}(t_\beta)\Delta\gamma,$$

which executes the proof for (4.3).

Let n is odd and $t \le \sigma(\gamma)$ so that $g_{n-1}(\sigma(\gamma),t) \ge 0$, thus we can write

$$\int_{\rho^{n-1}(t)}^{t_\beta} (-1)^{n-1} g_{n-1}(\sigma(\gamma),t)u^{\Delta^n}(t_\alpha)\Delta\gamma$$
$$\le \int_{\rho^{n-1}(t)}^{t_\beta} (-1)^{n-1} g_{n-1}(\sigma(\gamma,t))u^{\Delta^n}(\gamma)\Delta\gamma$$
$$\le \int_{\rho^{n-1}(t)}^{t_\beta} (-1)^{n-1} g_{n-1}(\sigma(\gamma),t)u^{\Delta^n}(t_\beta)\Delta\gamma,$$
$$\Rightarrow u^{\Delta^n}(t_\beta)\int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))\Delta\gamma$$
$$\le \int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))u^{\Delta^n}(\gamma)\Delta\gamma$$
$$\le u^{\Delta^n}(t_\alpha)\int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(\sigma(\gamma),t)\Delta\gamma,$$

which gets the form

$$\begin{aligned} &u^{\Delta^n}(t_\beta)\int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\gamma))\Delta\gamma \\ &\le u(t)-\sum_{k=0}^{n-1} h_k(t,t_\beta)u^{\Delta^k}(t_\beta) \\ &\le u^{\Delta^n}(t_\alpha)\int_{t_\beta}^{\rho^{n-1}(t)} h_{n-1}(\sigma(\gamma),t)\Delta\gamma, \end{aligned}$$

which executes the proof for (4.4).

Let n is even then we have

$$(-1)^{n-1}u^{\Delta^n}(t_\beta)\le(-1)^{n-1}u^{\Delta^n}(\gamma)\le(-1)^{n-1}u^{\Delta^n}(t_\alpha),$$

then by adopting the same steps we can prove (4.5). □

Therefore, we can extract the particular cases of Theorem 4.3 by considering different time scales. First by taking $\mathbb{T}=\mathbb{R}$ we obtained the following result which agrees Theorem 1 in [20].

Theorem 4.4. *Let $u(t)$ be $(n+1)-$convex on $[t_\alpha,t_\beta]$. Then for all $t\in(t_\alpha,t_\beta)$, the following inequality holds*

$$\sum_{k=0}^{n}\frac{u^{(k)}(t_\alpha)}{k!}(t-t_\alpha)^k\le u(t)\le\sum_{k=0}^{n-1}\frac{u^{(k)}(t_\alpha)}{k!}(t-t_\alpha)^k+\frac{u^{(n)}(t_\beta)}{n!}(t-t_\alpha)^n. \tag{4.6}$$

For odd n the following inequality is true

$$\sum_{k=0}^{n}\frac{u^{(k)}(t_\beta)}{k!}(t-t_\beta)^k\le u(t)\le\sum_{k=0}^{n-1}\frac{u^{(k)}(t_\beta)}{k!}(t-t_\beta)^k+\frac{u^{(n)}(t_\alpha)}{n!}(t-t_\beta)^n, \tag{4.7}$$

and for even n the following inequality holds

$$\sum_{k=0}^{n-1}\frac{u^{(k)}(t_\beta)}{k!}(t-t_\beta)^k+\frac{u^{(n)}(t_\alpha)}{n!}(t-t_\beta)^n\le u(t)\le\sum_{k=0}^{n}\frac{u^{(k)}(t_\beta)}{k!}(t-t_\beta)^k. \tag{4.8}$$

Now by considering $\mathbb{T}=\mathbb{Z}$ in Theorem 4.3 we get the discrete analogues of the inequalities (4.6), (4.7) and (4.8). Therefore, $\sigma(t)=t+1$, $\sigma^n(t)=t+n$, $\rho(t)=t-1$ and $\rho^n(t)=t-n$.

Theorem 4.5. *Let $u_t:[t_\alpha,t_\beta]\to\mathbb{R}$ be an $(n+1)-$convex sequence. Then for all $t\in(t_\alpha,t_\beta)$, the following inequality holds*

$$\sum_{k=0}^{n-1}\frac{\Delta^k u_{t_\alpha}}{k!}(t-t_\alpha)^k+\Delta^n u_{t_\alpha}\sum_{\gamma=t_\alpha}^{t-n}\frac{(t-\gamma-1)^{(n-1)}}{(n-1)!}\le u_t \tag{4.9}$$

$$\le\sum_{k=0}^{n-1}\frac{\Delta^k u_{t_\alpha}}{k!}(t-t_\alpha)^k+\Delta^n u_{t_\beta}\sum_{\gamma=t_\alpha}^{t-n}\frac{(t-\gamma-1)^{(n-1)}}{(n-1)!}. \tag{4.10}$$

For odd n the following inequality is true

$$\sum_{k=0}^{n-1}\frac{\Delta^k u_{t_\beta}}{k!}(t-t_\beta)^k+\Delta^n u_{t_\beta}\sum_{\gamma=t_\beta}^{t-n}\frac{(t-\gamma-1)^{(n-1)}}{(n-1)!}\le u_t \tag{4.11}$$

$$\le\sum_{k=0}^{n-1}\frac{\Delta^k u_{t_\beta}}{k!}(t-t_\beta)^k+\Delta^n u_{t_\alpha}\sum_{\gamma=t_\beta}^{t-n}\frac{(t-\gamma-1)^{(n-1)}}{(n-1)!}, \tag{4.12}$$

and for even n the following inequality holds

$$\sum_{k=0}^{n-1}\frac{\Delta^k u_{t_\beta}}{k!}(t-t_\beta)^k+\Delta^n u_{t_\alpha}\sum_{\gamma=t_\beta}^{t-n}\frac{(t-\gamma-1)^{(n-1)}}{(n-1)!}\le u_t \tag{4.13}$$

$$\le\sum_{k=0}^{n-1}\frac{\Delta^k u_{t_\beta}}{k!}(t-t_\beta)^k+\Delta^n u_{t_\beta}\sum_{\gamma=t_\beta}^{t-n}\frac{(t-\gamma-1)^{(n-1)}}{(n-1)!}. \tag{4.14}$$

The next result is obtained by considering $n=1$ in (4.3) and (4.4).

Corollary 4.1. *Let $t_\alpha, t_\beta \in \mathbb{T}^\kappa$, if u is convex on $[t_\alpha, t_\beta]$, then the given inequalities hold for all $t\in[t_\alpha,t_\beta]$*

$$\max\{u(t_\alpha)+u^\Delta(t_\alpha)(t-t_\alpha), u(t_\beta)+u^\Delta(t_\beta)(t-t_\beta)\}\le u(t)$$
$$\le \min\{u(t_\alpha)+u^\Delta(t_\beta)(t-t_\alpha), u(t_\beta)+u^\Delta(t_\alpha)(t-t_\beta)\}. \tag{4.15}$$

The next result is obtained by considering $n=2$ in (4.3) and (4.5).

Corollary 4.2. *Let $t_\alpha, t_\beta \in \mathbb{T}^{\kappa^2}$, if u is $3-$convex on $[t_\alpha, t_\beta]$, then the given inequalities hold for all $t\in[t_\alpha,t_\beta]_\mathbb{T}$*

$$\max\left\{u(t_\alpha)+u^\Delta(t_\alpha)(t-t_\alpha)+u^{\Delta^2}(t_\alpha)\int_{t_\alpha}^{\rho(t)}(\gamma-t_\alpha)\Delta\gamma, u(t_\beta)+u^\Delta(t_\beta)(t-t_\beta)\right.$$
$$\left.+u^{\Delta^2}(t_\alpha)\int_{t_\alpha}^{\rho(t)}(\gamma-t_\beta)\Delta\gamma\right\}\le u(t)\le \min\left\{u(t_\alpha)+u^\Delta(t_\beta)(t-t_\alpha)\right.$$
$$\left.+u^{\Delta^2}(t_\alpha)\int_{t_\alpha}^{\rho(t)}(\gamma-t_\beta)\Delta\gamma, u(t_\beta)+u^\Delta(t_\alpha)(t-t_\beta)+u^{\Delta^2}(t_\beta)\int_{t_\alpha}^{\rho(t)}(\gamma-t_\beta)\Delta\gamma\right\}.$$

Remark 4.2. *When we take $\mathbb{T}=\mathbb{R}$ in Corollaries 4.1 and 4.2 we get the results which coincide with Corollary 1 and Corollary 2 in [20] respectively. Moreover Corollary 4.1 for $\mathbb{T}=\mathbb{R}$ is used to derive more useful result in [21].*

5. Conclusion

The notion of n-convexity has been discussed in [16], on specific time scales that are $\mathbb{R}$ or $h\mathbb{Z}$. Here we extend the theory on arbitrary time scale and developed the relationship between the delta derivatives of order n and the nth-order divided difference using integral representation of nth-order divided difference on time scales, see [5, 22]. Further we utilized this relationship to derive some dynamic inequalities from which we are able to extract some difference inequalities that are equally important in the study of difference equations and their applications.

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