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Weighted Ostrowski's Type Integral Inequalities for Mapping Whose First Derivative Is Bounded

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Abstract. The aim of paper is to develop the inequalities for L_∞, L_p and L_1 norms. Applications for some special weight functions and Perturbed expressions are also determined via Chebychev functional. We recaptured the previous results for different weights.

1. Introduction

In 1938, Ostrowski established the interesting integral inequality for differentiable mappings with bounded derivative [10]. Cerone [3] also worked on this inequality. Different authors worked on the generalization of Ostrowski's type inequalities that is [1]- [2] and [9]. Further work done by Iftikhar et al. [6], Mustafa et al. [7] and Qayyum et al. [12]- [14].

Let the functional $S\left(f;\varpi;\hat{J},\check{k}\right)$ be defined as:

$$S\left(f;\varpi;\hat{J},\check{k}\right)=f\left(\ddot{z}\right)-\ddot{M}\left(f;\varpi;\hat{J},\check{k}\right), \tag{1.1}$$

where $f\left(\ddot{z}\right):\left[\hat{J},\check{k}\right]\rightarrow\mathbb{R}$ be a continuous mapping, $\ddot{M}\left(f;\varpi;\hat{J},\check{k}\right)$ is weighted integral mean and is defined as:

$$\ddot{M}\left(f;\varpi;\hat{J},\check{k}\right)=\frac{1}{\check{k}-\hat{J}}\int_{\hat{J}}^{\check{k}}f\left(\check{r}\right)\varpi\left(\check{r}\right)d\check{r}. \tag{1.2}$$

The functional $S\left(f;\varpi;\hat{J},\check{k}\right)$ represents the deviation of $f\left(\ddot{z}\right)$ from its integral mean over $\left[\hat{J},\check{k}\right]$.

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We assume non-negative weight function $\varpi : (\hat{J}, \check{k}) \to [0, \infty)$ is integrable

$$\int_{\hat{J}}^{\check{k}} \varpi(\check{r}) d\check{r} < \infty. \tag{1.3}$$

We define m, m_1 and μ as

$$m\left(\hat{J}, \check{k}\right) = \int_{\hat{J}}^{\check{k}} \varpi(\check{r}) d\check{r}, \; m_1\left(\hat{J}, \check{k}\right) = \int_{\hat{J}}^{\check{k}} \check{r}\varpi(\check{r}) d\check{r}$$

$$\text{and } \mu\left(\hat{J}, \check{k}\right) = \frac{m_1\left(\hat{J}, \check{k}\right)}{m\left(\hat{J}, \check{k}\right)}. \tag{1.4}$$

2. Main Result

Theorem 2.1. *Let $f : \left[\hat{J}, \check{k}\right] \to R$ be continuous on $\left[\hat{J}, \check{k}\right]$ and differentiable mapping on $\left(\hat{J}, \check{k}\right)$, then the following weighted peano kernel, define $\dot{G}\left(.,.\right) : \left[\hat{J}, \check{k}\right] \to \mathbb{R}$ as:*

$$\dot{G}\left(\ddot{z}, \check{r}\right) = \begin{cases} \frac{\epsilon}{(\epsilon+\delta)\left(\ddot{z}-\hat{J}\right)} \int_{\hat{J}}^{\check{r}} \varpi\left(u\right) du, & \text{if } \check{r} \in \left[\hat{J}, \ddot{z}\right] \\ \\ \frac{\delta}{(\epsilon+\delta)\left(\check{k}-\ddot{z}\right)} \int_{\check{k}}^{\check{r}} \varpi\left(u\right) du, & \text{if } \check{r} \in \left(\ddot{z}, \check{k}\right] \end{cases} \tag{2.1}$$

$\forall\ \check{r} \in \left[\hat{J}, \check{k}\right]$, $\ddot{z} \in \left[\hat{J}, \check{k}\right]$, ϖ is weight function as stated in (1.3) and $\epsilon, \delta \in \mathbb{R}$ non-negative and both are not zero at a time. Then the following weighted integral identity

$$\begin{aligned} &\tau\left(\varpi; \ddot{z}; \epsilon, \delta\right) \\ &= \int_{\hat{J}}^{\check{k}} \dot{G}\left(\ddot{z}, \check{r}\right) f'\left(\check{r}\right) d\check{r} \\ &= Bf\left(\ddot{z}\right) - \frac{1}{\epsilon+\delta}\left[\epsilon\ddot{M}\left(f; \varpi; \hat{J}, \ddot{z}\right) + \delta\ddot{M}\left(f; \varpi; \ddot{z}, \check{k}\right)\right], \end{aligned} \tag{2.2}$$

holds, where

$$B = \frac{1}{\epsilon+\delta}\left[\frac{\epsilon}{\ddot{z}-\hat{J}} m\left(\hat{J}, \ddot{z}\right) + \frac{\delta}{\check{k}-\ddot{z}} m\left(\ddot{z}, \check{k}\right)\right],$$

$\ddot{M}\left(f; \varpi; \hat{J}, \check{k}\right)$ is weighted integral mean as defined in (1.2).

Proof. From (2.1), we have

$$\begin{aligned} &\int_{\hat{J}}^{\check{k}} \dot{G}\left(\ddot{z}, \check{r}\right) f'\left(\check{r}\right) d\check{r} \\ &= \frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}} \int_{\hat{J}}^{\ddot{z}} \varpi\left(\check{r}\right) d\check{r} + \frac{\delta}{\check{k}-\ddot{z}} \int_{\ddot{z}}^{\check{k}} \varpi\left(\check{r}\right) d\check{r}\right\} f\left(\ddot{z}\right) \\ &- \frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}} \int_{\hat{J}}^{\ddot{z}} f\ \left(\check{r}\right)\varpi\left(\check{r}\right) d\check{r} + \frac{\delta}{\check{k}-\ddot{z}} \int_{\ddot{z}}^{\check{k}} f\ \left(\check{r}\right)\varpi\left(\check{r}\right) d\check{r}\right\}, \end{aligned}$$

where the integration by parts formula has been utilized on the separate interval $\left[\hat{J}, \ddot{z}\right]$ and $\left(\ddot{z}, \check{k}\right]$. Simplification of the expressions readily produces the identity as stated in (2.2). □

Theorem 2.2. *Let $f : [\hat{J}, \check{k}] \to R$ be continuous on $[\hat{J}, \check{k}]$ and differentiable mapping on $(\hat{J}, \check{k})$, whose first derivative $f' : [\hat{J}, \check{k}] \to R$ is bounded on $(\hat{J}, \check{k})$, then following weighted integral inequalities*

$$|\tau(\varpi;\ddot{z};\epsilon,\delta)| \leq \begin{cases} \left(\frac{\epsilon m(\hat{J},\ddot{z})}{\ddot{z}-\hat{J}}\left\{\ddot{z}-\mu(\hat{J},\ddot{z})\right\} + \frac{\delta m(\ddot{z},\check{k})}{\check{k}-\ddot{z}}\left\{\ddot{z}-\mu(\ddot{z},\check{k})\right\}\right)\frac{\|f'\|_\infty}{\epsilon+\delta} & \text{for } f' \in L_\infty[\hat{J},\check{k}] \\ \frac{\|f'\|_p \varpi(\ddot{z})}{(\epsilon+\delta)(\ddot{q}+1)^{\frac{1}{\ddot{q}}}}\left[\epsilon^{\ddot{q}}(\ddot{z}-\hat{J}) + \delta^{\ddot{q}}(\check{k}-\ddot{z})\right]^{\frac{1}{\ddot{q}}} & \text{for } f' \in L_p[\hat{J},\check{k}] \\ \frac{\vartheta}{\epsilon+\delta}\left[1+\frac{|\rho|}{\vartheta}\right]\frac{\|f'\|_1}{2} & \text{for } f' \in L_1[\hat{J},\check{k}] \end{cases} \tag{2.3}$$

are hold for all $\check{r} \in [\hat{J}, \check{k}]$, $\ddot{z} \in [\hat{J}, \check{k}]$, ϖ is weight function as stated in (1.3) and $\epsilon, \delta \in \mathbb{R}$ non-negative and both are not zero at a time, where

$$\vartheta = \frac{1}{(\ddot{z}-\hat{J})(\check{k}-\ddot{z})}\left(\epsilon m(\hat{J},\ddot{z})(\check{k}-\ddot{z}) + \delta m(\ddot{z},\check{k})(\ddot{z}-\hat{J})\right)$$

and

$$\rho = \frac{1}{(\ddot{z}-\hat{J})(\check{k}-\ddot{z})}\left(\epsilon m(\hat{J},\ddot{z})(\check{k}-\ddot{z}) - \delta m(\ddot{z},\check{k})(\ddot{z}-\hat{J})\right).$$

Proof. Taking the modulus of (2.2) and using (1.2)

$$\begin{aligned} &|\tau(\varpi;\ddot{z};\epsilon,\delta)| \\ &= \left|\int_{\hat{J}}^{\check{k}} \dot{G}(\ddot{z},\check{r}) f'(\check{r})\, d\check{r}\right| \leq \int_{\hat{J}}^{\check{k}} \left|\dot{G}(\ddot{z},\check{r})\right| \left|f'(\check{r})\right| d\check{r}, \end{aligned} \tag{2.4}$$

where we use properties of the integral and modulus. Thus for $f' \in L_\infty[\hat{J}, \check{k}]$ from (2.4)

$$|\tau(\varpi;\ddot{z};\epsilon,\delta)| \leq \|f'\|_\infty \int_{\hat{J}}^{\check{k}} \left|\dot{G}(\ddot{z},\check{r})\right| d\check{r}$$

from which a simple calculation using (2.1), gives

$$\begin{aligned} &\int_{\hat{J}}^{\check{k}} \dot{G}(\ddot{z},\check{r})\, d\check{r} \\ &= \frac{1}{\epsilon+\delta}\left[\frac{\epsilon}{\ddot{z}-\hat{J}}\left\{\ddot{z}m(\hat{J},\ddot{z}) - m_1(\hat{J},\ddot{z})\right\} \right. \\ &\left. + \frac{\delta}{\check{k}-\ddot{z}}\left\{\ddot{z}m(\ddot{z},\check{k}) - m_1(\ddot{z},\check{k})\right\}\right]. \end{aligned}$$

From above, first inequality given in (2.3) is obtained.

Further, using Hölder's Inequality, we have for $f^{'} \in L_p\left[\hat{J}, \check{k}\right]$ from (2.4)

$$\left|\tau\left(\varpi; \ddot{z}, \epsilon, \delta\right)\right| \leq \left\|f^{'}\right\|_p \left(\int_{\hat{J}}^{\check{k}} \left|\dot{G}\left(\ddot{z}, \check{r}\right)\right|^{\ddot{q}} d\check{r}\right)^{\frac{1}{\ddot{q}}},$$

where $\frac{1}{p} + \frac{1}{\ddot{q}} = 1, \;\; p > 1.$

With the help of mean value theorem and by using the technique Qayyum et al. [11], we get

$$\left(\int_{\hat{J}}^{\check{k}} \dot{G}\left|\left(\ddot{z}, \check{r}\right)\right|^{\ddot{q}} d\check{r}\right)^{\frac{1}{\ddot{q}}}$$
$$= \frac{\varpi\left(\ddot{z}\right)}{\left(\epsilon+\delta\right)\left(\ddot{q}+1\right)^{\frac{1}{\ddot{q}}}} \left[\epsilon^{\ddot{q}}\left(\ddot{z}-\hat{J}\right) - (-1)^{\ddot{q}+1}\delta^{\ddot{q}}\left(\check{k}-\ddot{z}\right)\right]^{\frac{1}{\ddot{q}}}.$$

So the second inequality given in (2.3) is obtained.

Finally, for $f^{'} \in L_1\left[\hat{J}, \check{k}\right]$ we have from (2.4) and using (2.1)

$$\left|\tau\left(\varpi; \ddot{z}; \epsilon, \delta\right)\right| \leq \sup_{\check{r}\in\left[\hat{J}, \check{k}\right]} \left|\dot{G}\left(\ddot{z}, \check{r}\right)\right| \left\|f^{'}\right\|_1,$$

where

$$\sup_{\check{r}\in\left[\hat{J}, \check{k}\right]} \left|\dot{G}\left(\ddot{z}, \check{r}\right)\right|$$
$$= \frac{1}{\epsilon+\delta} \max\left(\frac{\epsilon}{\ddot{z}-\hat{J}} m\left(\hat{J}, \ddot{z}\right), \frac{\delta}{\check{k}-\ddot{z}} m\left(\ddot{z}, \check{k}\right)\right)$$
$$= \frac{1}{2\left(\epsilon+\delta\right)\left(\ddot{z}-\hat{J}\right)\left(\check{k}-\ddot{z}\right)} \left[\epsilon m\left(\hat{J}, \ddot{z}\right)\left(\check{k}-\ddot{z}\right)\right.$$
$$\left. + \delta m\left(\ddot{z}, \check{k}\right)\left(\ddot{z}-\hat{J}\right)\right]$$
$$\times \left[1 + \frac{\left|\frac{1}{\left(\ddot{z}-\hat{J}\right)\left(\check{k}-\ddot{z}\right)}\left[\epsilon m\left(\hat{J}, \ddot{z}\right)\left(\check{k}-\ddot{z}\right) - \delta m\left(\ddot{z}, \check{k}\right)\left(\ddot{z}-\hat{J}\right)\right]\right|}{\frac{1}{\left(\ddot{z}-\hat{J}\right)\left(\check{k}-\ddot{z}\right)}\left[\epsilon m\left(\hat{J}, \ddot{z}\right)\left(\check{k}-\ddot{z}\right) + \delta m\left(\ddot{z}, \check{k}\right)\left(\ddot{z}-\hat{J}\right)\right]}\right].$$

Hence proved. □

Remark 2.1. *Triangular Inequality from* (2.2) *and* (1.1) *is*

$$\left(\epsilon+\delta\right)\tau\left(\varpi; \ddot{z}; \epsilon, \delta\right) = \epsilon S\left(f; \varpi; \hat{J}, \ddot{z}\right) + \delta S\left(f; \varpi; \ddot{z}, \check{k}\right)$$

then using triangular inequality in (2.3), we get

$$
\begin{aligned}
&|(\epsilon+\delta)\,\tau\,(\varpi;\ddot{z};\epsilon,\delta)| \\
\leq &\begin{cases}
\begin{array}{l}\frac{\epsilon}{2}\left(\frac{m(\hat{J},\ddot{z})}{\ddot{z}-\hat{J}}\left\{\ddot{z}-\mu\left(\hat{J},\ddot{z}\right)\right\}\right)\left\|f'\right\|_{\infty,[\hat{J},\ddot{z}]} \\ +\frac{\delta}{2}\left(\frac{m(\ddot{z},\check{k})}{\check{k}-\ddot{z}}\left\{\ddot{z}-\mu\left(\ddot{z},\check{k}\right)\right\}\right)\left\|f'\right\|_{\infty,[\ddot{z},\check{k}]}\end{array} & \text{for } f' \in L_{\infty}\left[\hat{J},\check{k}\right] \\
\\
\begin{array}{l}\epsilon\varpi(\ddot{z})\left(\frac{\ddot{z}-\hat{J}}{\ddot{q}+1}\right)^{\frac{1}{\ddot{q}}}\left\|f'\right\|_{p,[\hat{J},\ddot{z}]} \\ +\delta\varpi(\ddot{z})\left(\frac{\check{k}-\ddot{z}}{\ddot{q}+1}\right)^{\frac{1}{\ddot{q}}}\left\|f'\right\|_{p,[\ddot{z},\check{k}]}\end{array} & \text{for } f' \in L_{p}\left[\hat{J},\check{k}\right] \\
\\
\frac{\vartheta}{2}\left\|f'\right\|_{1,[\hat{J},\ddot{z}]}+\frac{|\rho|}{2}\left\|f'\right\|_{1,[\ddot{z},\check{k}]} & \text{for } f' \in L_{1}\left[\hat{J},\check{k}\right].
\end{cases}
\end{aligned}
\tag{2.5}
$$

Remark 2.2. *Since we may write (2.2) as*

$$
\begin{aligned}
&\epsilon\ddot{M}\left(f;\varpi;\hat{J},\ddot{z}\right)+\delta\ddot{M}\left(f;\varpi;\ddot{z},\check{k}\right) \\
&=\epsilon\ddot{M}\left(f;\varpi;\hat{J},\ddot{z}\right)+\frac{\delta}{\check{k}-\ddot{z}}\left(\int_{\hat{J}}^{\check{k}}\varpi\left(\check{r}\right)f\left(\check{r}\right)d\check{r}-\int_{\hat{J}}^{\ddot{z}}\varpi\left(\check{r}\right)f\left(\check{r}\right)d\check{r}\right) \\
&=\left[\epsilon-\delta\left(\frac{\ddot{z}-\hat{J}}{\check{k}-\ddot{z}}\right)\right]\ddot{M}\left(f;\varpi;\hat{J},\ddot{z}\right)+\delta\left(\frac{\check{k}-\hat{J}}{\check{k}-\ddot{z}}\right)\ddot{M}\left(f;\varpi;\ddot{z},\check{k}\right),
\end{aligned}
$$

thus $\tau\left(\varpi;\ddot{z};\epsilon,\delta\right)$ *is*

$$
\begin{aligned}
&\frac{1}{B}\tau\left(\varpi;\ddot{z};\epsilon,\delta\right) \\
&=f\left(\ddot{z}\right)-\frac{1}{B}\left[\left(1-\frac{\delta}{\epsilon+\delta}\lambda\right)\ddot{M}\left(f;\varpi;\hat{J},\ddot{z}\right)\right. \\
&\left.+\frac{\delta}{\epsilon+\delta}\lambda\ddot{M}\left(f;\varpi;\ddot{z},\check{k}\right)\right],
\end{aligned}
$$

where

$$
\lambda=\frac{\check{k}-\hat{J}}{\check{k}-\ddot{z}},
$$

same as $\left[\hat{J},\check{k}\right]$, $\ddot{M}\left(f;\varpi;\hat{J},\check{k}\right)$ *is also fixed.*

Corollary 2.1. *Let the conditions of Theorem 2.2 holds. Then the results for $\delta = \epsilon$*

$$\left|\tau\left(\varpi;\ddot{z};\epsilon,\epsilon\right)\right| \leq \begin{cases} \left(\frac{m\left(\hat{J},\ddot{z}\right)}{\ddot{z}-\hat{J}}\left\{\ddot{z}-\mu\left(\hat{J},\ddot{z}\right)\right\} + \frac{m\left(\ddot{z},\check{k}\right)}{\check{k}-\ddot{z}}\left\{\ddot{z}-\mu\left(\ddot{z},\check{k}\right)\right\}\right)\frac{\left\|f'\right\|_{\infty}}{2} & \text{for } f' \in L_{\infty}\left[\hat{J},\check{k}\right] \\ \left(\frac{\check{k}-\hat{J}}{\ddot{q}+1}\right)^{\frac{1}{\ddot{q}}}\frac{\left\|f'\right\|_{p}\varpi(\ddot{z})}{2} & \text{for } f' \in L_{p}\left[\hat{J},\check{k}\right] \\ \zeta\left[1+\frac{|\eta|}{\zeta}\right]\frac{\left\|f'\right\|_{1}}{4} & \text{for } f' \in L_{1}\left[\hat{J},\check{k}\right] \end{cases} \quad (2.6)$$

where

$$\begin{aligned}\tau\left(\varpi;\ddot{z};\epsilon,\epsilon\right) &:= \frac{1}{2}\left[\left(\frac{1}{\ddot{z}-\hat{J}}m\left(\hat{J},\ddot{z}\right)+\frac{1}{\check{k}-\ddot{z}}m\left(\ddot{z},\check{k}\right)\right)f\left(\ddot{z}\right)\right. \\ &\left.-\left\{\ddot{M}\left(f;\varpi;\hat{J},\ddot{z}\right)+\ddot{M}\left(f;\varpi;\ddot{z},\check{k}\right)\right\}\right],\end{aligned}$$

$$\zeta = \frac{1}{\left(\ddot{z}-\hat{J}\right)\left(\check{k}-\ddot{z}\right)}\left[m\left(\hat{J},\ddot{z}\right)\left(\check{k}-\ddot{z}\right)+m\left(\ddot{z},\check{k}\right)\left(\ddot{z}-\hat{J}\right)\right]$$

and

$$\eta = \frac{1}{\left(\ddot{z}-\hat{J}\right)\left(\check{k}-\ddot{z}\right)}\left[m\left(\hat{J},\ddot{z}\right)\left(\check{k}-\ddot{z}\right)-m\left(\ddot{z},\check{k}\right)\left(\ddot{z}-\hat{J}\right)\right].$$

Proof. The result is readily obtained on allowing $\epsilon = \delta$ in (2.3) so that the left hand side is $\tau\left(\varpi;\ddot{z};\epsilon,\epsilon\right)$ from (2.4). □

Corollary 2.2. *According to Theorem 2.2, then mid point $\left(\ddot{z}=\check{D}\Rightarrow\frac{\hat{J}+\check{k}}{2}\right)$ inequality from (2.2)*

$$\left|\tau\left(\varpi;\check{D},\epsilon,\delta\right)\right| \leq \begin{cases} \frac{2}{\check{k}-\hat{J}}\left[\epsilon m\left(\hat{J},\check{D}\right)\left\{\check{D}-\mu\left(\hat{J},\check{D}\right)\right\} + \delta m\left(\check{D},\check{k}\right)\left\{\check{D}-\mu\left(\check{D},\check{k}\right)\right\}\right]\frac{\left\|f'\right\|_{\infty}}{\epsilon+\delta} & \text{for } f' \in L_{\infty}\left[\hat{J},\check{k}\right] \\ \left[\epsilon^{\ddot{q}}+\delta^{\ddot{q}}\right]^{\frac{1}{\ddot{q}}}\left(\frac{\check{k}-\hat{J}}{2(\ddot{q}+1)}\right)^{\frac{1}{\ddot{q}}}\frac{\left\|f'\right\|_{p}\varpi(\check{D})}{(\epsilon+\delta)} & \text{for } f' \in L_{p}\left[\hat{J},\check{k}\right] \\ \frac{\varrho}{(\epsilon+\delta)}\left[1+\frac{|\Psi|}{\varrho}\right]\frac{\left\|f'\right\|_{1}}{2} & \text{for } f' \in L_{1}\left[\hat{J},\check{k}\right] \end{cases} \quad (2.7)$$

where

$$\varrho = \frac{1}{\check{k}-\hat{J}}\left[\epsilon m\left(\hat{J},\check{D}\right)+\delta m\left(\check{D},\check{k}\right)\right]$$

and

$$\Psi = \frac{1}{\check{k} - \hat{J}} \left[\epsilon m \left(\hat{J}, \check{D} \right) - \delta m \left(\check{D}, \check{k} \right) \right].$$

Proof. Placing $\ddot{z} = \check{D} \Rightarrow \frac{\hat{J}+\check{k}}{2}$ in (2.4) and (2.3) produces the results as stated in (2.7). □

Corollary 2.3. *When the conditions of Theorem 2.2 hold and $\epsilon = \delta$ using in (2.7) is evaluated at mid point $\left(\ddot{z} = \check{D} \Rightarrow \frac{\hat{J}+\check{k}}{2} \right)$*

$$\left| \tau \left(\varpi; \check{D}, \epsilon, \epsilon \right) \right| \leq \begin{cases} \begin{array}{l} \left[m \left(\hat{J}, \check{D} \right) \left\{ \check{D} - \mu \left(\hat{J}, \check{D} \right) \right\} \right. \\ \left. + m \left(\check{D}, \check{k} \right) \left\{ \check{D} - \mu \left(\check{D}, \check{k} \right) \right\} \right] \frac{\left\| f' \right\|_\infty}{\check{k} - \hat{J}} \end{array} & \text{for } f' \in L_\infty \left[\hat{J}, \check{k} \right] \\ \left(\frac{\check{k} - \hat{J}}{\ddot{q}+1} \right)^{\frac{1}{\ddot{q}}} \frac{\left\| f' \right\|_p \varpi(\check{D})}{2} & \text{for } f' \in L_p \left[\hat{J}, \check{k} \right] \\ \frac{\zeta}{2} \left[1 + \frac{|\eta|}{\zeta} \right] \frac{\left\| f' \right\|_1}{2} & \text{for } f' \in L_1 \left[\hat{J}, \check{k} \right] \end{cases} \tag{2.8}$$

where

$$\zeta = \frac{2}{\left(\check{k} - \hat{J} \right)} \left[m \left(\hat{J}, \check{D} \right) + m \left(\check{D}, \check{k} \right) \right]$$

and

$$\eta = \frac{2}{\left(\check{k} - \hat{J} \right)} \left[m \left(\hat{J}, \check{D} \right) - m \left(\check{D}, \check{k} \right) \right].$$

Proof. Putting $\epsilon = \delta$ in (2.7), we get (2.8). □

Remark 2.3. *For $\varpi(\ddot{z}) = 1$ in (2.3) and (2.5) − (2.8), we get Cerone's results [3].*

3. Application for Some Special Means

Now we discuss application for some special means by taking different weights.

Remark 3.1. *For Uniform (Legendre) mean:*

Let $\varpi(\check{r}) = 1$ put in (2.3) and in (2.4), we get Cerone's results [3].

Remark 3.2. *For Logarithm mean:*

Let

$$\varpi(\check{r}) = \ln(1/\check{r}), \quad \hat{J} = 0, \ \check{k} = 1,$$

then $\mu\left(\hat{J}, \check{k}\right)$ is

$$\mu(0,1) = \frac{1}{4},$$

then

$$\left|\left[\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\right]\left(f(\ddot{z})-\int_0^1 \ln(1/\check{r}) f(\check{r})\,d\check{r}\right)\right|$$
$$\leq \begin{cases} \left[\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\right]\left(\ddot{z}-\frac{1}{4}\right)\left\|f'\right\|_\infty & \textit{for } f'\in L_\infty\left[\hat{J},\check{k}\right] \\ \frac{\left\|f'\right\|_p \ln(1/\check{r})}{(\epsilon+\delta)(\ddot{q}+1)^{\frac{1}{\ddot{q}}}}\left[\epsilon^{\ddot{q}}\left(\ddot{z}-\hat{J}\right)+\delta^{\ddot{q}}\left(\check{k}-\ddot{z}\right)\right]^{\frac{1}{\ddot{q}}} & \textit{for } f'\in L_p\left[\hat{J},\check{k}\right] \\ \frac{\|f''\|_1}{2(\epsilon+\delta)}\left(\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right)\left|\frac{\epsilon}{\ddot{z}-\hat{J}}-\frac{\delta}{\check{k}-\ddot{z}}\right| & \textit{for } f'\in L_1\left[\hat{J},\check{k}\right]. \end{cases}$$

holds.

Remark 3.3. *For Jacobi mean:*

Let

$$\varpi(\check{r})=1/\sqrt{\check{r}}, \quad \hat{J}=0,\ \check{k}=1,$$

in (1.4), we get

$$\mu(0,1)=\frac{1}{3},$$

then

$$\left|\left[\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\right]\left(2f(\ddot{z})-\int_0^1 f(\check{r})\,1/\sqrt{\check{r}}\,d\check{r}\right)\right|$$
$$\leq \begin{cases} \left[\frac{2}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\right]\left(\ddot{z}-\frac{1}{3}\right)\left\|f'\right\|_\infty & \textit{for } f'\in L_\infty\left[\hat{J},\check{k}\right] \\ \frac{\left\|f'\right\|_p 1/\sqrt{\check{r}}}{(\epsilon+\delta)(\ddot{q}+1)^{\frac{1}{\ddot{q}}}}\left[\epsilon^{\ddot{q}}\left(\ddot{z}-\hat{J}\right)+\delta^{\ddot{q}}\left(\check{k}-\ddot{z}\right)\right]^{\frac{1}{\ddot{q}}} & \textit{for } f'\in L_p\left[\hat{J},\check{k}\right] \\ \frac{\|f''\|_1}{(\epsilon+\delta)}\left(\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right)\left|\frac{\epsilon}{\ddot{z}-\hat{J}}-\frac{\delta}{\check{k}-\ddot{z}}\right| & \textit{for } f'\in L_1\left[\hat{J},\check{k}\right]. \end{cases}$$

Remark 3.4. *For Chebyshev mean:*

Let

$$\varpi(\check{r})=1/\sqrt{1-\check{r}^2}, \quad \hat{J}=-1,\ \check{k}=1,$$

then

$$\mu(-1,1)=0,$$

thus

$$\left|\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\left(\pi f\left(\ddot{z}\right)-\int_{-1}^{1}1/\sqrt{1-\check{r}^{2}}f\ \ \left(\check{r}\right)d\check{r}\right)\right|$$

$$\leq\begin{cases}\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\left(\pi\ddot{z}\right)\left\|f'\right\|_{\infty} & \text{for } f'\in L_{\infty}\left[\hat{J},\check{k}\right]\\ \frac{\left\|f'\right\|_{p}{}_{1/\sqrt{1-\check{r}^{2}}}}{(\epsilon+\delta)(\ddot{q}+1)^{\frac{1}{\ddot{q}}}}\left[\epsilon^{\ddot{q}}\left(\ddot{z}-\hat{J}\right)+\delta^{\ddot{q}}\left(\check{k}-\ddot{z}\right)\right]^{\frac{1}{\ddot{q}}} & \text{for } f'\in L_{p}\left[\hat{J},\check{k}\right]\\ \frac{\pi\|\ f''\|_{1}}{2(\epsilon+\delta)}\left(\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right)\left|\frac{\epsilon}{\ddot{z}-\hat{J}}-\frac{\delta}{\check{k}-\ddot{z}}\right| & \text{for } f'\in L_{1}\left[\hat{J},\check{k}\right].\end{cases}$$

Remark 3.5. *For Laguerre mean:*

Let

$$\varpi\left(\check{r}\right)=e^{-\check{r}}\qquad\hat{J}=0,\quad\check{k}=\infty,$$

then

$$\mu\left(0,\infty\right)=1,$$

and

$$\left|\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\left(f\left(\ddot{z}\right)-\int_{0}^{\infty}e^{-\check{r}}f\ \ \left(\check{r}\right)d\check{r}\right)\right|$$

$$\leq\begin{cases}\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\left\{\ddot{z}-1\right\}\left\|f'\right\|_{\infty} & \text{for } f'\in L_{\infty}\left[\hat{J},\check{k}\right]\\ \frac{\left\|f'\right\|_{p}{}_{e^{-\check{r}}}}{(\epsilon+\delta)(\ddot{q}+1)^{\frac{1}{\ddot{q}}}}\left[\epsilon^{\ddot{q}}\left(\ddot{z}-\hat{J}\right)+\delta^{\ddot{q}}\left(\check{k}-\ddot{z}\right)\right]^{\frac{1}{\ddot{q}}} & \text{for } f'\in L_{p}\left[\hat{J},\check{k}\right]\\ \frac{\|\ f''\|_{1}}{2(\epsilon+\delta)}\left(\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right)\left|\frac{\epsilon}{\ddot{z}-\hat{J}}-\frac{\delta}{\check{k}-\ddot{z}}\right| & \text{for } f'\in L_{1}\left[\hat{J},\check{k}\right].\end{cases}$$

holds.

Remark 3.6. *For Hermite mean:*

Let

$$\varpi\left(\check{r}\right)=e^{-\check{r}^{2}}\qquad\hat{J}=-\infty,\ \ \check{k}=\infty,$$

then

$$\mu\left(-\infty,\infty\right)=0,$$

and

$$\left|\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\left(\sqrt{\pi}f\left(\ddot{z}\right)-\int_{-\infty}^{\infty}e^{-\check{r}^{2}}f\;\left(\check{r}\right)d\check{r}\right)\right|$$
$$\leq\begin{cases}\frac{1}{\epsilon+\delta}\left\{\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right\}\left(\sqrt{\pi}\ddot{z}\right)\left\|f'\right\|_{\infty} & \text{for } f'\in L_{\infty}\left[\hat{J},\check{k}\right]\\ \frac{\left\|f'\right\|_{p}e^{-\check{r}^{2}}}{(\epsilon+\delta)(\ddot{q}+1)^{\frac{1}{\ddot{q}}}}\left[\epsilon^{\ddot{q}}\left(\ddot{z}-\hat{J}\right)+\delta^{\ddot{q}}\left(\check{k}-\ddot{z}\right)\right]^{\frac{1}{\ddot{q}}} & \text{for } f'\in L_{p}\left[\hat{J},\check{k}\right]\\ \frac{\sqrt{\pi}\|\,f''\|_{1}}{2(\epsilon+\delta)}\left(\frac{\epsilon}{\ddot{z}-\hat{J}}+\frac{\delta}{\check{k}-\ddot{z}}\right)\left|\frac{\epsilon}{\ddot{z}-\hat{J}}-\frac{\delta}{\check{k}-\ddot{z}}\right| & \text{for } f'\in L_{1}\left[\hat{J},\check{k}\right].\end{cases}$$

4. Perturbed Results For Weighted Ostrowski Type Inequalities

Perturbed versions of the results of the previous section may be obtained by using Grüss type results involving Chebychev functional

$$\check{T}\left(f,g;\varpi\right)=\ddot{M}\left(fg;\varpi\right)-\ddot{M}\left(f;\varpi\right)\ddot{M}\left(g;\varpi\right), \tag{4.1}$$

where $\ddot{M}\left(f;\varpi\right)$ is the weighted integral mean as defined in (1.2).

For $f,g:\left[\hat{J},\check{k}\right]\longrightarrow\mathbb{R}$ and integrable on $\left[\hat{J},\check{k}\right]$, as is their product, then

$$\begin{aligned}&\left|\check{T}\left(f,g\right)\right|\\&\leq\check{T}^{\frac{1}{2}}\left(f,f\right)\check{T}^{\frac{1}{2}}\left(g,g\right),\ \text{Dragomir [4] for } f,\ g\in L_{2}\left[\hat{J},\check{k}\right]\\&\leq\frac{\Gamma-\gamma}{2}\check{r}^{\frac{1}{2}}\left(f,f\right),\ \text{Matic et al. [8] for } \gamma\leq g\left(\check{r}\right)\leq\Gamma,\ \check{r}\in\left[\hat{J},\check{k}\right]\\&\leq\frac{\left(\Gamma-\gamma\right)\left(\Phi-\phi\right)}{4},\ \text{Grüss [5] for } \phi\leq g\left(\check{r}\right)\leq\Psi,\ \check{r}\in\left[\hat{J},\check{k}\right].\end{aligned} \tag{4.2}$$

We obtain following theorem:

Theorem 4.1. *Let $f:\left[\hat{J},\check{k}\right]\longrightarrow\mathbb{R}$ be an absolutely continuous mapping and $\epsilon\geq0$, $\delta\geq0$, $\epsilon+\delta\neq0$, then*

$$\begin{aligned}&\left|\tau\left(\varpi;\ddot{z};\epsilon,\delta\right)-\frac{\left(\ddot{z}-\gamma\right)}{2}S\right|\\&\leq\left(\check{k}-\hat{J}\right)\kappa\left(\ddot{z}\right)\left[\frac{1}{\check{k}-\hat{J}}\left\|f'\right\|_{2}^{2}-S^{2}\right]^{\frac{1}{2}},\quad f'\in L_{2}\left[\hat{J},\check{k}\right]\\&\leq\left(\check{k}-\hat{J}\right)\kappa\left(\ddot{z}\right)\frac{\Gamma-\gamma}{2},\ \gamma\leq f'\left(\check{r}\right)\leq\Gamma,\quad\check{r}\in\left[\hat{J},\check{k}\right]\\&\leq\left(\check{k}-\hat{J}\right)\frac{\Gamma-\gamma}{4}.\end{aligned} \tag{4.3}$$

The constant $\frac{1}{4}$ *is the best possible, where* $\tau(\varpi;\ddot{z};\epsilon,\delta)$ *is as given in* (2.2),

$$\gamma = \frac{\epsilon\hat{J}+\delta\check{k}}{\epsilon+\delta}, \quad S = \frac{f(\check{k})-f(\hat{J})}{\check{k}-\hat{J}}, \tag{4.4}$$

$$\kappa^2 = \varpi(\ddot{z})^2\left[\frac{1}{3}\left(\left(\frac{\epsilon}{\epsilon+\delta}\right)^2(\ddot{z}-\hat{J})+\left(\frac{\delta}{\epsilon+\delta}\right)^2(\check{k}-\ddot{z})\right)\right.$$
$$\left.-\left(\frac{(\ddot{z}-\gamma)}{2(\check{k}-\hat{J})}\right)^2\right]. \tag{4.5}$$

Proof. Associating $f(\check{r})$ with $\dot{G}(\ddot{z},\check{r})$ and $g(\check{r})$ with $f'(\check{r})$, then from (2.1) and (4.1), we get

$$\check{T}\left(\dot{G}(\ddot{z},.),f'(.)\right)$$
$$=\ddot{M}\left(\dot{G}(\ddot{z},.),f'(.)\right)-\ddot{M}\left(\dot{G}(\ddot{z},.)\right)\ddot{M}\left(,f'(.)\right).$$

By using (2.1)

$$(\check{k}-\hat{J})\check{T}\left(\dot{G}(\ddot{z},.),f'(.)\right) \tag{4.6}$$
$$=\tau(\varpi;\ddot{z};\epsilon,\delta)-(\check{k}-\hat{J})\ddot{M}\left(\dot{G}(\ddot{z},.)\right)S.$$

Now from (2.1)

$$(\check{k}-\hat{J})\ddot{M}\left(\dot{G}(\ddot{z},.)\right)$$
$$=\int_{\hat{J}}^{\check{k}}\dot{G}(\ddot{z},\check{r})\,d\check{r}=\frac{\varpi(\ddot{z})}{\epsilon+\delta}\left[\frac{\epsilon}{\ddot{z}-\hat{J}}\frac{(\ddot{z}-\hat{J})^2}{2}-\frac{\delta}{\check{k}-\ddot{z}}\frac{(\check{k}-\ddot{z})^2}{2}\right]$$
$$=\frac{\varpi(\ddot{z})}{2}(\ddot{z}-\gamma) \tag{4.7}$$

(4.7) and (4.5) gives the left hand side of (4.3).

Now, for the bounds on (4.6) from (4.2), we have to find $\check{T}^{\frac{1}{2}}\left(\dot{G}(\ddot{z},.),\dot{G}(\ddot{z},.)\right)$ and $\phi\le\dot{G}(\ddot{z},.)\le\Phi$. Firstly, we note however that

$$0\le\check{T}^{\frac{1}{2}}\left(f'(.),f'(.)\right)$$
$$=\left[\ddot{M}\left(f'(.)\right)^2-\ddot{M}^2\left(f'(.)\right)\right]^{\frac{1}{2}}$$
$$=\left[\frac{1}{\check{k}-\hat{J}}\int_{\hat{J}}^{\check{k}}\left[f'(\check{r})\right]^2d\check{r}-\left(\frac{1}{\check{k}-\hat{J}}\int_{\hat{J}}^{\check{k}}f'(\check{r})\,d\check{r}\right)^2\right]^{\frac{1}{2}}$$
$$=\left[\frac{1}{\check{k}-\hat{J}}\left\|f'(\check{r})\right\|_2^2-S^2\right]^{\frac{1}{2}}$$
$$\le\frac{\Gamma-\gamma}{2},\ \text{where } \gamma\le f'(\check{r})\le\Gamma,\quad \check{r}\in\left[\hat{J},\check{k}\right]. \tag{4.8}$$

Now from (2.1), the definition of $\dot{G}(\ddot{z},\check{r})$, we have

$$\kappa(\ddot{z})^2=\check{T}\left(\dot{G}(\ddot{z},.),\dot{G}(\ddot{z},.)\right)=\ddot{M}\left(\dot{G}^2(\ddot{z},.)\right)-\ddot{M}^2\left(\dot{G}(\ddot{z},.)\right), \tag{4.8-1}$$

from (4.7)

$$\ddot{M}\left(\dot{G}(\ddot{z},.)\right)=\frac{\varpi(\ddot{z})(\ddot{z}-\gamma)}{2\left(\check{k}-\hat{J}\right)}$$

and

$$\begin{aligned}
&\ddot{M}\left(\dot{G}^2(\ddot{z},.)\right)\\
&=\left(\frac{\epsilon}{(\epsilon+\delta)\left(\ddot{z}-\hat{J}\right)}\right)^2\int_{\hat{J}}^{\ddot{z}}\left(\int_{\hat{J}}^{\check{r}}\varpi(u)\,du\right)^2 d\check{r}\\
&+\left(\frac{\delta}{(\epsilon+\delta)\left(\check{k}-\ddot{z}\right)}\right)^2\int_{\ddot{z}}^{\check{k}}\left(\int_{\check{k}}^{\check{r}}\varpi(u)\,du\right)^2 d\check{r}\\
=&\frac{\varpi(\ddot{z})^2}{3}\left[\left(\frac{\epsilon}{\epsilon+\delta}\right)^2\left(\ddot{z}-\hat{J}\right)+\left(\frac{\delta}{\epsilon+\delta}\right)^2\left(\check{k}-\ddot{z}\right)\right].
\end{aligned}$$

By substituting the derived results into (4.8-1), gives

$$0\le\kappa(\ddot{z})=\check{T}^{\frac{1}{2}}\left(\dot{G}(\ddot{z},.),\dot{G}(\ddot{z},.)\right), \tag{4.9}$$

which is given explicitly by (4.5). We observe from (2.1), that for $\epsilon,\delta\ge0$ and both are not zero at a time give

$$\Phi=\sup_{\check{r}\in[\hat{J},\check{k}]}\dot{G}(\ddot{z},\check{r})\qquad\text{and}\qquad\phi=\inf_{\check{r}\in[\hat{J},\check{k}]}\dot{G}(\ddot{z},\check{r}),$$

giving $\Phi=\frac{\epsilon}{\epsilon+\delta}$ and $\phi=\frac{\delta}{\epsilon+\delta}$.

Hence, proved the result.

□

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