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Application of the F-Expansion Method for Solving the Fokas-Lenells Equation

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Abstract. By the aid of traveling wave hypothesis, the F-expansion method has been implemented in this paper to obtain Jacobian-Elliptic function solutions for the optical Fokas-Lenells model. The hyperbolic-function solutions are derived as special cases from the Jacobian-Elliptic function solutions. The present approach is straightforward to determine the exact solutions for the Fokas-Lenells equation. The existence criteria of the obtained solutions are also reported.

1. Introduction

In the field of telecommunications engineering, the optical soliton perturbation and the PDEs are the most active areas of research [1-15]. The dynamics of soliton molecules are administered by a variety of nonlinear evolution equations. The nonlinear Schrödinger's equation is the most studied model in this context. Although the nonlinear Schrödinger's model has been extensively studied by many authors with different forms of nonlinearity, the present work analyzes the pulse propagation engineering through optical fibers and PCF with a newly established model known as the Fokas–Lenells equation (FLE) [12, 13]. Such model has been studied to obtain various kinds of soliton solutions by using the complex–amplitude ansatz and other approaches [12, 13]. The objective of this paper is to apply the F-expansion method by the aid of the traveling wave hypothesis to deduct more general solution as well as soliton solutions. The paper is organized as follows. In the next subsection, the FLE is presented. In section 2, the traveling wave hypothesis is introduced. Section 3 is devoted to the application of the F-expansion method on the current model. The exact solutions in terms of Jacobian-Elliptic functions are displayed in section 4. The hyperbolic-function solutions are derived

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from the Jacobian-Elliptic function solutions in section 5. The paper is ended by a conclusion section 6 and two appendices.

1.1. **Governing model.** The dimensionless form of perturbed FLE that has been proposed takes the form [16, 17]:

$$iq_t + a_1 q_{xx} + a_2 q_{xt} + |q|^2 \left(B_1 q + i\sigma q_x\right) = i\left[\alpha q_x + \lambda\left(|q|^{2m} q\right)_x + \mu\left(|q|^{2m}\right)_x q\right], \tag{1}$$

where the right hand side represents all the perturbation terms. In equation (1), the independent variables are x and t that represent spatial and temporal variables respectively while $q(x,t)$ is the complex–valued wave profile representing the soliton profile. Here, a_1 is the coefficient of group velocity dispersion while a_2 is the coefficient of spatio–temporal dispersion that was proposed to be included a few years ago. Then, σ is the coefficient of nonlinear dispersion. On the right hand side, α is the coefficient of inter–modal dispersion while λ accounts for self–steepening effect and finally μ gives another form of nonlinear dispersion. The parameter B_1 indicates the effect of self-phase modulation while the parameter m refers to the full nonlinearity.

2. Traveling wave hypothesis

The solutions of (1) may be supposed as

$$q(x,t) = e^{i\theta(x,t)}\ u(\omega), \tag{2}$$

where $\omega = x - \gamma t$ and the phase $\theta(x,t) = -kx + \beta t + \theta_0$, $u(\omega)$ is the amplitude component of the wave and γ is its speed. k is the soliton frequency, β is its wave-number and θ_0 is the phase constant. Equation (1) can be decomposing into real and imaginary parts yields a pair of relations. The real and imaginary parts of Eq. (1) are respectively

$$(a_1 - a_2)u'' + (a_2\omega k - a_1 k^2 - \alpha k - \omega)u + (B_1 + k\sigma)u^3 - ku[(2m+1)\lambda + 2m\mu]u^{2m} = 0, \tag{3}$$

and

$$\gamma + 2ka_1 - a_2(\gamma k + \omega) - \sigma u^2 + \alpha + [(2m+1)\lambda + 2m\mu]u^{2m} = 0. \tag{4}$$

We notice from (4) that

$$[\lambda(2m+1) + 2m\mu]u^{2m} = -\gamma - 2ka_1 + a_2(\gamma k + \omega) + \sigma u^2 - \alpha. \tag{5}$$

Setting

$$(2m+1)\lambda + 2m\mu = 0, \quad \sigma = 0, \tag{6}$$

in (5), then

$$\lambda = \frac{-2m\mu}{(2m+1)}, \tag{7}$$

and

$$\gamma = \frac{2ka_1 - a_2\omega + \alpha}{(a_2k - 1)}. \tag{8}$$

Accordingly, Eq. (3) reduces to

$$(a_1 - a_2\gamma)u'' + (a_1k^2 - \omega + k\gamma(1 - a_2k))u + B_1u^3 = 0. \tag{9}$$

Using

$$n_1 = (a_1 - a_2\gamma), \quad n_2 = (a_1k^2 - \omega + k\gamma(1 - a_2k)), \tag{10}$$

hence (9) gives

$$n_1u^{''} + n_2u + B_1u^3 = 0. \tag{11}$$

3. Application the F-expansion method to the FLE

Assume that the solution of (11) is in the form [18]

$$u(\omega) = \sum_{i=0}^{n} a_iF^i(\omega), \tag{12}$$

where a_i, $i = 0, 1, 2, \dots, n$, are constants to be determined, n is a positive integer which can be evaluated by balancing the highest-order linear term $u^{''}$ and nonlinear term u^3, this gives $n = 1$. Moreover, $F(\omega)$ satisfies the following auxiliary equation

$$F'(\omega) = \pm\sqrt{PF^4(\omega) + QF^2(\omega) + R}, \tag{13}$$

where P, Q, and R are constants. Eq. (13) for $F(\omega)$ leads to

$$\begin{cases} F'' = 2PF^3 + QF, \\ F''' = (6PF^2 + Q)F', \\ F'''' = 24P^2F^5 + 20PQF^3 + (12PR + Q^2)F \\ . \\ . \\ . \end{cases} \tag{14}$$

In Appendix A, we present 46 forms of exact solutions for Eq. (13), (see ref. [18] for details). In fact, these exact solutions can be used to construct more exact solutions for Eq. (11). According to $n = 1$, Eq. (12) becomes

$$u(\omega) = a_0 + a_1F(\omega). \tag{15}$$

Substituting (15) into (11), we obtain the following system of algebraic equations

$$\begin{cases} B_1 a_0^3 + a_0 n_2 = 0, \\ 3B_1 a_0^2 a_1 + Q a_1 n_1 + a_1 n_2 = 0, \\ 3B_1 a_1^2 a_0 = 0, \\ B_1 a_1^3 + 2P a_1 n_1 = 0. \end{cases} \tag{16}$$

Solving the last system (16), we have

$$a_0 = 0, \quad a_1 = \pm\sqrt{\frac{-2Pn_1}{B_1}}, \quad Q = -\frac{n_2}{n_1}. \tag{17}$$

With these values of a_0, a_1 and Q, the exact solution of (11) can be obtained by 46 forms depending on the values of P, Q and R with the corresponding solution $F(\omega)$ in Appendix A. However, some selected cases are presented in the following sections.

4. Jacobian-elliptic function solutions

Case: 1:

Let us consider the inputs of case 1 in Appendix A with implementing (15) and (17), thus

$$\begin{cases} P = m^2, \ Q = -\frac{n_2}{n_1} = -(1+m^2), \ R = 1, \\ F(\omega) = \mathrm{sn}\omega, \\ u = \pm\sqrt{\frac{-2n_1 m^2}{B_1}}\ \mathrm{sn}\omega, \ n_2 = (1+m^2)n_1, \quad n_1 < 0, \\ q(x,t) = \pm\sqrt{\frac{-2n_1 m^2}{B_1}}\ e^{i\ (-kx+\beta t+\theta_0)}\ \mathrm{sn}(x-\gamma t). \end{cases} \tag{18}$$

This general solution has not been reported in [16, 17]. Moreover, it will be demonstrated in section 5 that the solution (18) and other solutions of this section reduce to hyperbolic forms as special cases.

Case: 2:

The solution of this case (case 2 in Appendix A) is expressed in terms of another kind of Jacobian-Elliptic functions as

$$\begin{cases} P = m^2, \ Q = -\frac{n_2}{n_1} = -(1+m^2), \ R = 1, \\ F(\omega) = \mathrm{cd}\omega, \\ u = \pm\sqrt{\frac{-2n_1 m^2}{B_1}}\ \mathrm{cd}\omega, \ n_2 = (1+m^2)n_1, \quad n_1 < 0, \\ q(x,t) = \pm\sqrt{\frac{-2n_1 m^2}{B_1}}\ e^{i\ (-kx+\beta t+\theta_0)}\ \mathrm{cd}(x-\gamma t). \end{cases} \tag{19}$$

which was not also reported in Refs. [16, 17].

Case: 3:

The solution of this case (by using the inputs of case 26 in Appendix A) is expressed as

$$\begin{cases} P>0,\ Q=-\frac{n_2}{n_1}<0,\ R=\frac{m^2Q^2}{(1+m^2)^2P}, \\ F(\omega)=\sqrt{\frac{-m^2Q}{(1+m^2)P}}\ \mathrm{sn}\left(\sqrt{\frac{-Q}{1+m^2}}\ \omega\right), \\ u=\pm\sqrt{\frac{-2m^2n_2}{(1+m^2)B_1}}\ \mathrm{sn}\left(\sqrt{\frac{n_2}{(1+m^2)n_1}}\ \omega\right),\ n_2n_1>0,,\ n_2B_1<0, \\ q(x,t)=\pm\sqrt{\frac{-2m^2n_2}{(1+m^2)B_1}}e^{i\ (-kx+\beta t+\theta_0)}\ \mathrm{sn}\left(\sqrt{\frac{n_2}{(1+m^2)n_1}}\ (x-\gamma t)\right). \end{cases} \tag{20}$$

Case: 4:

The solution of this case (by using the inputs of case 27 in Appendix A) is expressed as

$$\begin{cases} P<0,\ Q=-\frac{n_2}{n_1}>0,\ R=\frac{(1-m^2)Q^2}{(m^2-2)^2P}, \\ F(\omega)=\sqrt{\frac{-Q}{(2-m^2)P}}\ \mathrm{dn}\left(\sqrt{\frac{Q}{2-m^2}}\ \omega\right), \\ u=\pm\sqrt{\frac{2n_2}{(2-m^2)B_1}}\ \mathrm{dn}\left(\sqrt{\frac{-n_2}{(2-m^2)n_1}}\ \omega\right),\ n_2n_1<0,,\ n_2B_1>0, \\ q(x,t)=\pm\sqrt{\frac{2n_2}{(2-m^2)B_1}}e^{i\ (-kx+\beta t+\theta_0)}\ \mathrm{dn}\left(\sqrt{\frac{-n_2}{(2-m^2)n_1}}\ (x-\gamma t)\right). \end{cases} \tag{21}$$

which was not reported in Refs. [16, 17].

Case: 5:

On using the inputs of case 28 in Appendix A, we have

$$\begin{cases} P<0,\ Q=-\frac{n_2}{n_1}>0,\ R=\frac{m^2(m^2-1)Q^2}{(2m^2-1)^2P}, \\ F(\omega)=\sqrt{\frac{-m^2Q}{(2m^2-1)P}}\ \mathrm{cn}\left(\sqrt{\frac{Q}{2m^2-1}}\ \omega\right), \\ u=\pm\sqrt{\frac{2m^2n_2}{(2m^2-1)B_1}}\ \mathrm{cn}\left(\sqrt{\frac{-n_2}{(2m^2-1)n_1}}\ \omega\right),\ n_2n_1<0,,\ n_2B_1>0, \\ q(x,t)=\pm\sqrt{\frac{2m^2n_2}{(2m^2-1)B_1}}\ e^{i\ (-kx+\beta t+\theta_0)}\ \mathrm{cn}\left(\sqrt{\frac{-n_2}{(2m^2-1)n_1}}\ (x-\gamma t)\right). \end{cases} \tag{22}$$

5. Hyperbolic-function solutions

Some soliton−like solutions of Eq. (11) can be obtained in the limited case when the modulus $m\to 1$ (see Appendix B), as:

Case: 1:

$$\begin{cases} P=1,\ Q=-\frac{n_2}{n_1}=-2,\ R=1, \\ F(\omega)=\tanh\omega, \\ u=\pm\sqrt{\frac{-2n_1}{B_1}}\ \tanh\omega,\ n_2=2n_1,\quad n_1<0, \\ q(x,t)=\pm\sqrt{\frac{-2n_1}{B_1}}\ e^{i\ (-kx+\beta t+\theta_0)}\ \tanh(x-\gamma t). \end{cases} \tag{23}$$

Case: 2:

$$\begin{cases} P=1,\ Q=-\frac{n_2}{n_1}=-2,\ R=1, \\ F(\omega)=1, \\ u=\pm\sqrt{\frac{-2n_1}{B_1}},\ n_2=2n_1, \quad n_1<0, \\ q(x,t)=\pm\sqrt{\frac{-2n_1}{B_1}}\ e^{i\ (-kx+\beta t+\theta_0)}. \end{cases} \tag{24}$$

Case: 3:

$$\begin{cases} P>0,\ Q=-\frac{n_2}{n_1}<0,\ R=\frac{Q^2}{4P}, \\ F(\omega)=\sqrt{\frac{-Q}{2P}}\ \tanh\left(\sqrt{\frac{-Q}{2}}\ \omega\right), \\ u=\pm\sqrt{\frac{-n_2}{B_1}}\ \tanh\left(\sqrt{\frac{n_2}{2n_1}}\ \omega\right),\ n_2n_1>0,,\ n_2B_1<0, \\ q(x,t)=\pm\sqrt{\frac{-n_2}{B_1}}e^{i\ (-kx+\beta t+\theta_0)}\ \tanh\left(\sqrt{\frac{n_2}{2n_1}}\ (x-\gamma t)\right). \end{cases} \tag{25}$$

Cases: 4:

$$\begin{cases} P<0,\ Q=-\frac{n_2}{n_1}>0,\ R=0, \\ F(\omega)=\sqrt{\frac{-Q}{P}}\ \text{sech}\left(\sqrt{Q}\ \omega\right), \\ u=\pm\sqrt{\frac{2n_2}{B_1}}\ \text{sech}\left(\sqrt{\frac{-n_2}{n_1}}\ \omega\right),\ n_2n_1<0,,\ n_2B_1>0, \\ q(x,t)=\pm\sqrt{\frac{2n_2}{B_1}}e^{i\ (-kx+\beta t+\theta_0)}\ \text{sech}\left(\sqrt{\frac{-n_2}{n_1}}\ (x-\gamma t)\right). \end{cases} \tag{26}$$

Case: 5:

This case leads to the same hyperbolic-function solution given in (26).

Here, it should be noted that the solutions presented in the previous section in terms of the Jacobian-Elliptic function are more general than those previously obtained in the relevant literature. In addition, the obtained hyperbolic-function solutions were derived as special cases from our Jacobian-Elliptic function solutions. Moreover, some of the present solutions have not been reported in previous works [16, 17] which analyzed the same Fokas-Lenells equation. As a final observation is that all of the current solutions are obtained by using only one method, however, three different methods have been applied in [16, 17] to obtain only three solutions. Finally, several kinds of soliton solutions such as singular soliton solution and dark-singular combo soliton solution can be derived by considering more inputs of the 46 cases in Appendix A with the aid of Appendix B.

6. Conclusions

This paper revealed new types of exact solutions for the perturbed FLE, where the perturbation terms are of Hamiltonian type and appeared with full nonlinearity. The F-expansion method was applied in this paper to obtain several kinds of Jacobian-Elliptic function solutions for the optical Fokas-Lenells model. In special cases, the solito-like solutions in terms of the hyperbolic-functions are derived from the Jacobian-Elliptic function solutions. The results have not been reported in previous works in relevant literatures. Several kinds of soliton solutions such as singular soliton solution and dark-singular combo soliton solution can be derived by further investigations of the suggested method.

Appendix A

Relations between values of $(P,\ Q,\ R)$ and corresponding $F(\omega)$ in Eq. (13), where A, B and C are arbitrary constants and $m_1 = \sqrt{1-m^2}$.

Case	P	Q	R	$F(\omega)$
1	m^2	$-(1+m^2)$	1	$\text{sn}\omega$
2	m^2	$-(1+m^2)$	1	$\text{cd}\omega\text{=cn}\omega/\text{dn}\omega$
3	$-m^2$	$2m^2-1$	$1-m^2$	$\text{cn}\omega$
4	-1	$2-m^2$	m^2-1	$\text{dn}\omega$
5	1	$-(1+m^2)$	m^2	$\text{ns}\omega = (\text{sn}\omega)^{-1}$
6	1	$-(1+m^2)$	m^2	$\text{dc}\omega = \text{dn}\omega/\text{cn}\omega$
7	$1-m^2$	$2m^2-1$	$-m^2$	$\text{nc}\omega = (\text{cn}\omega)^{-1}$
8	m^2-1	$2-m^2$	-1	$\text{nd}\omega = (\text{dn}\omega)^{-1}$
9	$1-m^2$	$2-m^2$	1	$\text{sc}\omega = \text{sn}\omega/\text{cn}\omega$
10	$-m^2(1-m^2)$	$2m^2-1$	1	$\text{sd}\omega = \text{sn}\omega/\text{dn}\omega$
11	1	$2-m^2$	$1-m^2$	$\text{cs}\omega = \text{cn}\omega/\text{sn}\omega$
12	1	$2m^2-1$	$-m^2(1-m^2)$	$\text{ds}\omega = \text{dn}\omega/\text{sn}\omega$
13	$1/4$	$(1-2m^2)/2$	$1/4$	$\text{ns}\omega \pm \text{cs}\omega$
14	$(1-m^2)/4$	$(1+m^2)/2$	$(1-m^2)/4$	$\text{nc}\omega \pm \text{sc}\omega$
15	$1/4$	$(m^2-2)/2$	$m^2/4$	$\text{ns}\omega \pm \text{ds}\omega$
16	$m^2/4$	$(m^2-2)/2$	$m^2/4$	$\text{sn}\omega \pm i\text{cn}\omega$
17	$m^2/4$	$(m^2-2)/2$	$m^2/4$	$\sqrt{1-m^2}\text{sd}\omega \pm \text{cd}\omega$
18	$1/4$	$(1-m^2)/2$	$1/4$	$m\ \text{cd}\omega \pm i\sqrt{1-m^2}\text{nd}\omega$
19	$1/4$	$(1-2m^2)/2$	$1/4$	$m\ \text{sn}\omega \pm i\text{dn}\omega$
20	$1/4$	$(1-m^2)/2$	$1/4$	$\sqrt{1-m^2}\text{sc}\omega \pm \text{dc}\omega$
21	$(m^2-1)/4$	$(m^2+1)/2$	$(m^2-1)/4$	$m\ \text{sd}\omega \pm \text{nd}\omega$
22	$m^2/4$	$(m^2-2)/2$	$1/4$	$\frac{\text{sn}\omega}{1\pm\text{dn}\omega}$
23	$-1/4$	$(m^2+1)/2$	$(1-m^2)^2/4$	$m\ \text{cn}\omega \pm \text{dn}\omega$
24	$(1-m^2)^2/4$	$(m^2+1)/2$	$1/4$	$\text{ds}\omega \pm \text{cs}\omega$
25	$\frac{m^4(1-m^2)}{2(2-m^2)}$	$\frac{2(1-m^2)}{m^2-2}$	$\frac{1-m^2}{2(2-m^2)}$	$\text{dc}\omega \pm \sqrt{1-m^2}\text{nc}\omega$
26	$P>0$	$Q<0$	$\frac{m^2Q^2}{(m^2+1)^2P}$	$\sqrt{\frac{-m^2Q}{(m^2+1)P}}\ \text{sn}\left(\sqrt{\frac{-Q}{m^2+1}}\omega\right)$
27	$P<0$	$Q>0$	$\frac{(1-m^2)Q^2}{(m^2-2)^2P}$	$\sqrt{\frac{-Q}{(2-m^2)P}}\ \text{dn}\left(\sqrt{\frac{Q}{2-m^2}}\omega\right)$

28	$P<0$	$Q>0$	$\frac{m^2(m^2-1)Q^2}{(2m^2-1)^2P}$	$\sqrt{-\frac{m^2Q}{(2m^2-1)P}}\ \mathrm{cn}\left(\sqrt{\frac{Q}{2m^2-1}}\omega\right)$
29	1	$2-4m^2$	1	$\frac{\mathrm{sn}\omega\mathrm{dn}\omega}{\mathrm{cn}\omega}$
30	m^4	2	1	$\frac{\mathrm{sn}\omega\mathrm{cn}\omega}{\mathrm{dn}\omega}$
31	1	m^2+2	$1-2m^2+m^4$	$\frac{\mathrm{dn}\omega\mathrm{cn}\omega}{\mathrm{sn}\omega}$
32	$\frac{A^2(m-1)^2}{4}$	$\frac{m^2+1}{2}+3m$	$\frac{(m-1)^2}{4A^2}$	$\frac{\mathrm{dn}\omega\mathrm{cn}\omega}{A(1+\mathrm{sn}\omega)(1+m\mathrm{sn}\omega)}$
33	$\frac{A^2(m+1)^2}{4}$	$\frac{m^2+1}{2}-3m$	$\frac{(m+1)^2}{4A^2}$	$\frac{\mathrm{dn}\omega\mathrm{cn}\omega}{A(1+\mathrm{sn}\omega)(1-m\mathrm{sn}\omega)}$
34	$-\frac{4}{m}$	$6m-m^2-1$	$-2m^3+m^4+m^2$	$\frac{m\mathrm{cn}\omega\mathrm{dn}\omega}{m\mathrm{sn}^2\omega+1}$
35	$\frac{4}{m}$	$-6m-m^2-1$	$2m^3+m^4+m^2$	$\frac{m\mathrm{cn}\omega\mathrm{dn}\omega}{m\mathrm{sn}^2\omega-1}$
36	$1/4$	$\frac{1-2m^2}{2}$	$1/4$	$\frac{\mathrm{sn}}{1\pm\mathrm{cn}\omega}$
37	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\frac{\mathrm{cn}\omega}{1\pm\mathrm{sn}\omega}$
38	$4m_1$	$2+6m_1-m^2$	$2+2m_1-m^2$	$\frac{m^2\mathrm{sn}\omega\mathrm{cn}\omega}{m_1-\mathrm{dn}^2\omega}$
39	$-4m_1$	$2-6m_1-m^2$	$2-2m_1-m^2$	$-\frac{m^2\mathrm{sn}\omega\mathrm{cn}\omega}{m_1+\mathrm{dn}^2\omega}$
Case	P	Q	R	$F(\omega)$
40	$\frac{2-m^2-2m_1}{4}$	$\frac{m^2}{2}-1-3m_1$	$\frac{2-m^2-2m_1}{4}$	$\frac{m^2\mathrm{sn}\omega\mathrm{cn}\omega}{\mathrm{sn}^2\omega+(1+m_1)\mathrm{dn}\omega-1-m_1}$
41	$\frac{2-m^2+2m_1}{4}$	$\frac{m^2}{2}-1+3m_1$	$\frac{2-m^2+2m_1}{4}$	$\frac{m^2\mathrm{sn}\omega\mathrm{cn}\omega}{\mathrm{sn}^2\omega+(-1+m_1)\mathrm{dn}\omega-1+m_1}$
42	$\frac{C^2m^4-(B^2+C^2)m^2+B^2}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4(C^2m^2-B^2)}$	$\frac{\sqrt{\frac{(B^2-C^2)}{(B^2-C^2m^2)}}+\mathrm{sn}\omega}{B\ \mathrm{cn}\omega+C\ \mathrm{dn}\omega}$
43	$\frac{B^2+C^2m^2}{4}$	$\frac{1}{2}-m^2$	$\frac{1}{4(C^2m^2+B^2)}$	$\frac{\sqrt{\frac{(C^2m^2+B^2-C^2)}{(B^2+C^2m^2)}}+\mathrm{cn}\omega}{B\ \mathrm{sn}\omega+C\ \mathrm{dn}\omega}$
44	$\frac{B^2+C^2}{4}$	$\frac{m^2}{2}-1$	$\frac{m^4}{4(C^2+B^2)}$	$\frac{\sqrt{\frac{(B^2+C^2-C^2m^2)}{(B^2+C^2)}}+\mathrm{dn}\omega}{B\ \mathrm{sn}\omega+C\ \mathrm{cn}\omega}$
45	$-(m^2+2m+1)B^2$	$2m^2+2$	$\frac{2m-m^2-1}{B^2}$	$\frac{m\ \mathrm{sn}^2\omega-1}{B(m\ \mathrm{sn}^2\omega+1)}$
46	$-(m^2-2m+1)B^2$	$2m^2+2$	$-\frac{2m+m^2+1}{B^2}$	$\frac{m\ \mathrm{sn}^2\omega+1}{B(m\ \mathrm{sn}^2\omega-1)}$

Appendix B

The Jacobi−elliptic functions degenerate into hyperbolic functions when $m \to 1$ as:

$$\mathrm{sn}\omega \to \tanh\omega,\ \{\mathrm{cn}\omega,\ \mathrm{dn}\omega\} \to \mathrm{sech}\omega,\ \{\mathrm{sc}\omega,\ \mathrm{sd}\omega\} \to \sinh\omega,$$
$$\{\mathrm{ds}\omega,\ \mathrm{cs}\omega\} \to \mathrm{csch}\omega,\ \{\mathrm{nc}\omega,\ \mathrm{nd}\omega\} \to \cosh\omega,\ \mathrm{ns}\omega \to \coth\omega,\ \{\mathrm{cd}\omega,\ \mathrm{dc}\omega\} \to 1.$$

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