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Payne-Sperb-Stakgold Type Inequality for a Wedge-Like Membrane

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Abstract. For a bounded domain contained in a wedge, we give a new Payne-Sperb-Stakgold type inequality for the solution of a semi-linear equation. The result is isoperimetric in the sense that the sector is the unique extremal domain.

1. Introduction

For a two-dimensional bounded domain D, Payne and Rayner proved [9, 10] that the eigenfunction u of the Dirichlet Laplacian corresponding to the fondamental eigenvalue $\lambda(D)$ satisfies the following inequality

$$\int_D u^2 da \leq \frac{\lambda(D)}{4\pi} \left(\int_D u \, da \right)^2 , \tag{1.1}$$

where da denotes the Lebesgue measure. Equality is achieved if, and only if, D is a disk. The importance of this inequality is that it is a reverse Cauchy-Schwarz type inequality for the first eigenfunction .

This inequality was extended to higher dimension by kohler Kohler-Jobin [5, 6]. Her inequality states that

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$$\int_D u^2 \, da \leq \frac{\lambda^{d/2}}{2d\, C_d\, j^{d-2}_{d/2-1,1}} \left(\int_D u \, da \right)^2 \tag{1.2}$$

where D is a bounded domain in $\mathbb{R}^d$, C_d denotes the volume of the unit ball in $\mathbb{R}^d$, and $j_{d/2-1,1}$ is the first positive zero of the Bessel function $J_{d/2-1}$. Using the comparison method due to Giorgio Talenti, Chiti [1] proved that

$$\left(\int_D u^q \, da \right)^{\frac{1}{q}} \leq K(p,q,d)\, \lambda^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \left(\int_D u^p \, da \right)^{\frac{1}{p}} \text{ for } q \geq p > 0. \tag{1.3}$$

Here

$$K(p,q,d) = (d\, C_d)^{\frac{1}{q}-\frac{1}{p}}\, j^{d\,(\frac{1}{q}-\frac{1}{p})}_{\frac{d}{2}-1,1} \frac{\left(\int_0^1 r^{d-1+q(1-\frac{d}{2})} J^q_{\frac{d}{2}-1}(j_{\frac{d}{2}-1,1}\, r) dr \right)^{\frac{1}{q}}}{\left(\int_0^1 r^{d-1+p(1-\frac{d}{2})} J^p_{\frac{d}{2}-1}(j_{\frac{d}{2}-1,1}\, r) dr \right)^{\frac{1}{p}}}.$$

Equality holds if and only if D is a ball.

A more interesting inequality in the spirit of the above has been proved by Payne, Sperb and Stakgold [11] for the following nonlinear problem

$$\begin{aligned} \Delta\, u + f(u) &= 0 \text{ in } \Omega \subset \mathbb{R}^2, & (1.4)\\ u &> 0 \text{ in } \Omega \subset \mathbb{R}^2, & (1.5)\\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

for a given continuous function $f(t)$, with $f(0) = 0$. This includes Dirichlet eigenvalue problem for the Laplace operator when $f(t) = \lambda t$. For this problem, the Payne-Rayner inequality takes the form

$$\left(\int_\Omega f(u)\, dx \right)^2 \geq 8\pi \int_\Omega F(u)\, dx \tag{1.6}$$

where $F(u) = \int_0^u f(t) dt$. Finally, Mossino [7] prove a generalization of the latest inequality for the p-Laplacian and the case of equality was discussed by Kesavan and Pacella [4]. Our aims is to give a version of Payne-Sperb - Stakgold inequality for the case of wedge like domains.

2. Preliminary Tools and main result

Before stating our result, we give some notation . Let $\alpha \geq 1$ and $\mathcal{W}$ be the wedge defined in polar coordinates (r, θ) by

$$\mathcal{W} = \left\{ (r,\theta) \;\middle|\; r > 0,\ 0 < \theta < \frac{\pi}{\alpha} \right\}. \tag{2.1}$$

Whenever pertinent, the arc length element will be denoted by $ds^2 = dr^2 + r^2 d\theta^2$ while the element of area is denoted by $da = r dr d\theta$, and we let

$$v(r,\theta) = r^\alpha \sin \alpha\theta. \tag{2.2}$$

Then, v is a positive harmonic function in $\mathcal{W}$ which is zero on the boundary $\partial\mathcal{W}$.

We are interested in the solution u of the following quasi-linear problem:

$$\mathcal{P}_1 : \begin{cases} \Delta u + f(\frac{u}{v})v &= 0 \quad \text{in } D \\ u &> 0 \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D, \end{cases}$$

where D is a sufficiently smooth bounded domain completely contained in the wedge $\mathcal{W}$ and the $g((r,\theta),t) = f(\frac{t}{v(r,\theta)})v(r,\theta)$ is locally Hölder continuous and satisfies the following hypotheses.
(H1) There exists $A \in L^1(D)$ and $C > 0$ such that
$|g((r,\theta),t)| \leq A(r,\theta) + C|t|^p, \forall((r,\theta),t) \in D \times \mathbb{R}$, where $p > 0$.
(H2) For $t > 0$, we have $g((r,\theta),t) > 0$.
The role of hypothesis (H1)is to ensure that every weak solution of the problem $(\mathcal{P}_1)$ is a C^2-solution of $(\mathcal{P}_1)$. Notice that, The problem$(\mathcal{P}_1)$ includes the eigenvalue problem for the Laplace operator with Dirichlet boundary condition, when we take $f(\frac{u}{v}) = \lambda\frac{u}{v}$. Now, if we write the solution of $(\mathcal{P}_1)$ as $u = vw$, then the problem above transforms to

$$\mathcal{P}_2 : \begin{cases} -div(v^2\nabla w) &= f(w)v^2 \quad \text{in } D \\ v &> 0 \quad \text{in } D \\ v &= 0 \quad \text{on } \partial D \cap \mathcal{W}. \end{cases}$$

The solution w may be interpreted as a solution of the nonlinear classical problem $(\mathcal{P}_1)$ for the 4-dimensional domain symmetric about the x_2-axis when $\alpha = 1$ and for the 6-dimensional domain bi-axially symmetric about the x_1-axis and the x_2-axis when $\alpha = 2$, see [8] and [2]. Now, we need to introduce some notations and definitions. Let μ denoted measure defined by $d\mu = v^2 da$. Then, the weighted unidimensional decreasing rearrangement of the function w with respect to measure μ is the function

$$w^* : [0, \mu(D)] \to [0, +\infty)$$

defined by

$$w^*(0) = \sup w,$$

$$w^*(\xi) = \inf\left\{t \geq 0; \quad m_w(t) < \xi\right\}, \quad \forall \xi \in (0, \mu(D)],$$

where

$$m_w(t) = \mu\big(\left\{(r,\theta) \in D; \quad w(r,\theta) > t\right\}\big), \quad \forall t \in [0, \sup w]. \tag{2.3}$$

The main result is given in the following theorem.

Theorem 2.1. *Let D be a smooth bounded domain completely contained in the wedge. Assume that (H1) and (H2) are satisfied. Let F be the primitive of f such that $F(0) = 0$. Then the solution u of the problem $(\mathcal{P}_1)$ satisfies the inequality*

$$4(2\alpha+2)(2\alpha+1)\left(\frac{\pi}{2\alpha(2\alpha+2)}\right)^{\frac{1}{\alpha+1}} \int_0^{\mu(D)} \xi^{\frac{\alpha}{\alpha+1}} F\left((\frac{u}{v})^*(\xi)\right) d\xi \leq \int_D F(\frac{u}{v})v^2\, da.$$

Equality holds if and only if D is a perfect sector $\mathcal{S}_R$.

The proof of this inequality and the equality case will be discussed in the next section .

3. The weighted version of Payne-sperb-stackgold inequality

To beginning, we introduce the space $W(D, d\mu)$ of measurable functions φ that possess weak gradient denoted by $|\nabla\varphi|$ and satisfy the following conditions

(i) $\int_D |\nabla\varphi|^2 d\mu + \int_D |\varphi|^2 d\mu < +\infty$

(ii) There exists a sequence of functions $\varphi_n \in C^1(\overline{D})$ such that $\varphi_n(r,\theta) = 0$ on $\partial D \cap \mathcal{W}$ and

$$\lim_{n\to+\infty} \int_D |\nabla(\varphi - \varphi_n)|^2 d\mu + \int_D |\varphi - \varphi_n|^2 d\mu = 0. \tag{3.1}$$

Using the fact that v is harmonic and the divergence theorem , we see

$$\int_D |\nabla u|^2 da = \int_D |\nabla(wv)|^2 da = \int_D |\nabla w|^2 v^2 da = \int_D |\nabla w|^2 d\mu.$$

Thus w satisfies the first condition (i). Since u is a smooth solution of the problem P_1, then w is also smooth and by the boundary condition in P_2,we conclude that w satisfies the second condition (ii). Then w is in the space $W(D, d\mu)$. We introduce now the function

$$\Phi(t) = \int_{D_t} f(w) d\mu. \tag{3.2}$$

Since w and w^* are equimeaserable then we have

$$\Phi(t) = \int_{D_t} f(w) d\mu = \int_0^{m(t)} f(w^*) d\xi. \tag{3.3}$$

To proceed further, we need to show that $m(t)$ is absolutely continuous on $(0, M)$. Indeed, assume that $\mu(\{w = t\})$ is positive. Recall that $w \in W(D, d\mu)$ and proceeding as in the proof of Stampacchia's theorem [3] to conclude that $\nabla w = 0$ almost everywhere on the set $\{w = t\}$. Substitute this into P_2, we obtain $f(w) = 0$ on and so $g((r,\theta), u) = f(\frac{u}{v})v = 0$ on this set, which contradicts the hypothesis H_2. Thus, w is continuous on $(0, M)$ and By the fact that w^* is the left inverse of $m(t)$,we get

$$\Phi'(t) = f(w^*(m(t))m'(t) = f(t)m'(t). \tag{3.4}$$

By a weak solution to the problem P_2 we mean a function w belong to $W(D, d\mu)$ and satisfies the equality

$$\int_D \nabla w \cdot \nabla\varphi d\mu = \int_D f(w)\varphi d\mu, \tag{3.5}$$

for every φ in $C^1(\overline{D})$, such that $\varphi = 0$ on $\partial D \cap \mathcal{W}$. Choose the test function φ defined by

$$\varphi(r,\theta) = \begin{cases} \big(w(r,\theta) - t\big), & \text{if } w(r,\theta) > t \\ 0, & \text{otherwise ,} \end{cases} \tag{3.6}$$

where $0 \leq t < M$. Plugging (3.6) into (3.5) we get

$$\int_{w>t} |\nabla w|^2 d\mu = \int_{w>t} f(w)(w-t)d\mu. \tag{3.7}$$

Then, for $\epsilon > 0$, we have

$$\frac{1}{\epsilon}\left(\int_{w>t}|\nabla w|^2 d\mu - \int_{w>t+\epsilon}|\nabla w|^2 d\mu\right) = \int_{w>t} f(w)d\mu + \int_{t<w\leq t+\epsilon} f(w)(\frac{w-t-\epsilon}{\epsilon})d\mu, \tag{3.8}$$

which, on letting ϵ go to zero, gives,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_{w>t}|\nabla w|^2 d\mu - \int_{w>t+\epsilon}|\nabla w|^2 d\mu\right) = \int_{w>t} f(w)d\mu. \tag{3.9}$$

The same computation for $-\epsilon$ gives the same value of the limit. Thus

$$-\frac{d}{dt}\int_{w>t}|\nabla w|^2 d\mu = \int_{w>t} f(w)d\mu. \tag{3.10}$$

Now, applying the Cauchy Schwarz inequality

$$\left(\frac{1}{\epsilon}\int_{t<w\leq t+\epsilon}|\nabla w|d\mu\right)^2 \leq \left(\frac{1}{\epsilon}\int_{t<w\leq t+\epsilon}|\nabla w|^2 d\mu\right)\left(\frac{1}{\epsilon}\int_{t<w\leq t+\epsilon} d\mu\right) \tag{3.11}$$

and letting ϵ go to zero, we get

$$\left(-\frac{d}{dt}\int_{w>t}|\nabla w|d\mu\right)^2 \leq -m'(t)\Phi(t). \tag{3.12}$$

From the coarea formula, we have

$$-\frac{d}{dt}\int_{w>t}|\nabla w|d\mu = \int_{\partial\{w>t\}} v^2 ds. \tag{3.13}$$

Then, an application of the Payne-Weinberger isoperimetric inequality for the wedge-like membrane [12] leads to

$$(\frac{\pi}{2\alpha})^2\left(\frac{4\alpha(\alpha+1)}{\pi}m(t)\right)^{\frac{2\alpha+1}{\alpha+1}} \leq \left(\int_{\partial\{w>t\}} v^2 ds\right)^2 \leq -m'(t)\Phi(t). \tag{3.14}$$

By appealing to (3.13), we obtain

$$(\frac{\pi}{2\alpha})^{\frac{1}{\alpha+1}}(2\alpha+2)^{\frac{2\alpha+1}{\alpha+1}}(m(t))^{\frac{2\alpha+1}{\alpha+1}}f(t) \leq -\Phi'(t)\Phi(t). \tag{3.15}$$

Integrating both sides of the last relation from 0 to M, then we have

$$(\frac{\pi}{2\alpha})^{\frac{1}{\alpha+1}}(2\alpha+2)^{\frac{2\alpha+1}{\alpha+1}}\int_0^M(m(t))^{\frac{2\alpha+1}{\alpha+1}}f(t) \leq \frac{1}{2}\Phi^2(0), \tag{3.16}$$

since $\Phi(M)=0$. But on the left hand side we have

$$\begin{aligned}\int_0^M (m(t))^{\frac{2\alpha+1}{\alpha+1}} f(t)dt &= \int_0^M \frac{2\alpha+1}{\alpha+1} f(t) \int_0^{m(t)} \xi^{\frac{\alpha}{\alpha+1}} d\xi dt \\ &= \frac{2\alpha+1}{\alpha+1} \int_0^M f(t) \int_0^{\mu(D)} \xi^{\frac{\alpha}{\alpha+1}} \chi_{\{w^*>t\}}(\xi) d\xi dt \\ &= \frac{2\alpha+1}{\alpha+1} \int_0^{\mu(D)} \int_0^{w^*(\xi)} f(t) \xi^{\frac{\alpha}{\alpha+1}} dt d\xi \\ &= \frac{2\alpha+1}{\alpha+1} \int_0^{\mu(D)} F(w^*(\xi)) \xi^{\frac{\alpha}{\alpha+1}} dt d\xi.\end{aligned} \tag{3.17}$$

Substituting the last result into (3.16), the desired inequality in Theorem 2.1 follows. Moreover, if equality is achieved in Theorem 2.1, then obviously inequality (3.15) reduces to equality. Since $\Phi'(t) = f(t)m'(t)$ and $f(t) > 0$, then equality in (3.15) implies equality in (3.14) and so Payne-Weinberger Lemma [12] implies that almost all level sets D_t are concentric sectors with fixed angle $\frac{\pi}{\alpha}$. Since $D = \{w > 0\}$ is the increasing union of such sectors then D is a sector as well.

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