Int. J. Anal. Appl. (2022), 20:24

International Journal of Analysis and Applications

$K-g-$Duals in Hilbert C^*-Modules

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Abstract. Generalized frames with adjointable operators called $K-g-$frame is a generalization of a $g-$frame. In this paper, we give some results of dual $K-g-$bessel sequence, finally we obtain a new properties of approximate $K-g-$duals in Hilbert C^*-module.

1. introduction

Frames, introduced by Duffin and Schaefer [1] in 1952 to analyse some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [2] for signal processing. In 2000, Frank-larson [4] introduced the concept of frames in Hilbet C^*-modules as a generalization of frames in Hilbert spaces. The basic idea was to consider modules over C^*-algebras of linear spaces and to allow the inner product to take values in the C^*-algebras [5]. Many generalizations of the concept of frame have been defined in Hilbert C^*-modules [3, 6, 8–10, 12–15].

Throughout this paper, H is considered to be a countably generated Hilbert $\mathcal{A}-$module. Let $\{H_j\}_{j\in J}$ be the collection of submodules of H where I is a finite or countable index set. $End^*_{\mathcal{A}}(H,H_j)$ is the set of all adjointable operator from H to H_j. In particular $End^*_{\mathcal{A}}(H)$ denote the set of all adjointable operators on H.

Received: Feb. 28, 2022.

2010 *Mathematics Subject Classification.* Primary 41A58, Secondary 42C15.

Key words and phrases. g-Frame, K-g-Frame, C^*-algebra, Hilbert $\mathcal{A}$-modules.

https://doi.org/10.28924/2291-8639-20-2022-24

ISSN: 2291-8639

Define the module

$$l^2(\{H_j\}_{j\in J}) = \{\{x_j\}_{j\in J} : x_j \in H_j, \|\sum_{j\in J}\langle x_j, x_j\rangle\| < \infty\}$$

with $\mathcal{A}-$valued inner product $\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle$, where $x = \{x_j\}_{j\in J}$ and $y = \{y_j\}_{j\in J}$, clearly $l^2(\{H_j\}_{j\in J})$ is a Hilbert $\mathcal{A}-$module.

In the following we briefly recall some definitions and basic properties.

For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. *[7]. Let $\mathcal{A}$ be a unital C^*-algebra and H be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and U are compatible. H is a pre-Hilbert $\mathcal{A}$-module if H is equipped with an $\mathcal{A}$-valued inner product $\langle ., .\rangle : H \times H \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,*

(i) $\langle x, x\rangle \geq 0$ *for all* $x \in H$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$.
(ii) $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$ *for all* $a \in \mathcal{A}$ *and* $x, y, z \in H$.
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ *for all* $x, y \in H$.

For $x \in H$, we define $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$. If H is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on H is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}$ for $x \in H$.

Definition 1.2. *Suppose that X and Y are Hilbert $\mathcal{A}-$modules and $T \in End^*_{\mathcal{A}}(X, Y)$. The Moore-Penrose inverse of T (if it exists) is an element T^+ of $End^*_{\mathcal{A}}(Y, X)$ satisfying*

$TT^+T = T$, $T^+TT^+ = T^+$, $(TT^+)^* = TT^+$, $(T^+T)^* = T^+T$

Lemma 1.1. *Let X and Y be two Hilbert $\mathcal{A}-$modules and $T \in End^*_{\mathcal{A}}(X, Y)$ Then the Moore-Penrose inverse T^+ exixts if and only if T has a closed range.*

Definition 1.3. *Suppose that $K \in End^*_{\mathcal{A}}(H)$ and $\{\Lambda_j\}_{j\in J}$ is a $K - g-$frame for H. A $g-$bessel sequence $\{\Gamma_j\}_{j\in J}$ for H is said to be a dual $K - g-$bessel sequence of $\{\Lambda_j\}_{j\in J}$ if*

$$Kx = \sum_{j\in J}\Lambda_j^*\Gamma_j x, \quad \forall x \in H.$$

Definition 1.4. *[3] A sequence $\{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j), j \in J\}$ is called a $g-$frame with respect to $\{H_j\}_{j\in J}$ if there exist constants $C, D > 0$ such that for every $x \in H$*

$$C\langle x, x\rangle \leq \sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle \leq D\langle x, x\rangle. \tag{1.1}$$

As usual C and D are $g-$frame bounds of $\{\Lambda_j\}_{j\in J}$. If only upper inequality of (1.1) holds, then $\{\Lambda_j\}_{j\in J}$ is called $g-$bessel sequence for H.

Definition 1.5. *Let $K \in End_{\mathcal{A}}^*(H)$ and $\{\Lambda_j \in End_{\mathcal{A}}^*(H, H_j), j \in J\}$. A sequence $\{\Lambda_j\}_{j\in J}$ is called $K-g-$frame for H with respect to $\{H_j\}_{j\in J}$, if there exist constants $0 < A \leq B < \infty$ such that*

$$A\langle K^*x, K^*x\rangle \leq \sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle \leq B\langle x, x\rangle, \forall x \in H.$$

A $K-g-$frame $\{\Lambda_j\}_{j\in J}$ is said to be tight if there exists a constant $A > 0$ such that

$$\sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle = A\langle K^*x, K^*x\rangle, \forall x \in H.$$

2. Main Results

Theorem 2.1. *Suppose $K \in End_{\mathcal{A}}^*(H)$ has closed range and $\{\Lambda_j \in End_{\mathcal{A}}^*(H, H_j)\}$ is a $K-g-$frame for H. for each $j \in J$, let $\Gamma_j \in End_{\mathcal{A}}^*(H, H_j)$. Then the following statement are equivalent*

(1) *The sequence $\{\Gamma_j\}_{j\in J}$ is a dual $K-g-$bessel sequence of $\{\Lambda_j\}_{j\in J}$.*
(2) *For each $a \in Ker(K)$, $\sum_{j\in J}\Lambda_j^*\Gamma_j a = 0$. the sequence $\{\Gamma_j\}_{j\in J}$ is a $g-$bessel sequence for H, which can naturally generate a $g-$bessel sequence $\{\theta_j\}_{j\in J}$ for H in the forme $\theta_j = \Gamma_j K^+ P_{range(K)}$ for each $j \in J$ such that*

$$h = \sum_{j\in J}\Lambda_j^*\theta_j h = \sum_{j\in J}\theta_j^*\Lambda_j h, \quad \forall h \in range(K) \tag{2.1}$$

where $P_{range(K)}$ denotes the orthogonal projection onto $range(K)$.

(1) $\implies$ (2) Assume that $\{\Gamma_j\}_{j\in J}$ is a dual $K-g-$bessel sequence of $\{\Lambda_j\}_{j\in J}$, then for each $a \in Ker(K)$, $\sum_{j\in J}\Lambda_j^*\Gamma_j a = Ka = 0$ and we have for any $h \in range(K)$

$$h = KK^+h = \sum_{j\in J}\Lambda_j^*\Gamma_j K^+h = \sum_{j\in J}\Lambda_j^*\Gamma_j K^+P_{range(K)}h,$$

put $\theta_j = \Gamma_j K^+P_{range(K)}$, for each $j \in J$

so for any $x \in H$

$$\begin{aligned}\sum_{j\in J}\langle\theta_j x, \theta_j x\rangle &= \sum_{j\in J}\langle \Gamma_j K^+P_{range(K)}x, \Gamma_j K^+P_{range(K)}x\rangle \\ &\leq D_\Gamma\langle K^+x, K^+x\rangle \\ &\leq D_\Gamma\|K^+\|^2\langle x, x\rangle,\end{aligned}$$

hence $\{\theta_j\}_{j\in J}$ is $g-$bessel sequence for H.

Let $g, h \in range(K)$

$$\begin{aligned}\langle g, h\rangle &= \langle \sum_{j\in J} \Lambda_j^* \theta_j g, h\rangle \\ &= \sum_{j\in J} \langle \theta_j g, \Lambda_j h\rangle \\ &= \sum_{j\in J} \langle \Gamma_j K^+ P_{range(K)} g, \Lambda_j h\rangle \\ &= \sum_{j\in J} \langle g, (\Gamma_j K^+ P_{range(K)})^* \Lambda_j h\rangle \\ &= \langle g, \sum_{j\in J} \theta_j^* \Lambda_j h\rangle,\end{aligned}$$

therefore, $\sum_{j\in J} \theta_j^* \Lambda_j h = h$ (2) $\implies$ (1) We have $range(K^+) = (Ker(K))^\perp$ then each $x \in H$ can be expressed as $x = x_1 + x_2$, where $x_1 \in range(K^+)$ and $x_2 \in (range(K^+))^\perp = Ker(K)$, so

$$\begin{aligned}Kx_1 &= \sum_{j\in J} \Lambda_j^* \Gamma_j K^+ P_{range(K)} K x_1 \\ &= \sum_{j\in J} \Lambda_j^* \Gamma_j K^+ K x_1 \\ &= \sum_{j\in J} \Lambda_j^* \Gamma_j x_1.\end{aligned}$$

And we have, $Kx_2 = \sum_{j\in J} \Lambda_j^* \Gamma_j x_2 = 0$, we obtain

$$\begin{aligned}Kx &= K(x_1 + x_2) \\ &= Kx_1 \\ &= \sum_{j\in J} \Lambda_j^* \Gamma_j x_1 \\ &= \sum_{j\in J} \Lambda_j^* \Gamma_j (x_1 + x_2) \\ &= \sum_{j\in J} \Lambda_j^* \Gamma_j x.\end{aligned}$$

Hence, $\{\Gamma_j\}_{j\in J}$ is a dual $K-g-$bessel sequence of $\{\Lambda_j\}_{j\in J}$.

Theorem 2.2. *Suppose that $K \in End_{\mathcal{A}}^*(H)$, $\Lambda_j, \Lambda_j^{'} \in End_{\mathcal{A}}^*(H, H_j)$ for each $j \in J$ and that $\{\Gamma_j \in End_{\mathcal{A}}^*(H_j, W_{ji})\}_{i\in I_j}$ is a $g-$frame for H_j with bounds C_j, D_j such that $0 < C = \inf_{j\in J} C_j \leq \sup_{j\in J} D_j = D < \infty$.*

Let $\{\Gamma_{ji}^{'} \in End_{\mathcal{A}}^*(H_j, W_{ji})\}_{i\in I_j}$ be a dual $g-$frame of $\{\Gamma_{ji}\}_{i\in I_j}$ for each $j \in J$. Then the following conditions are equivalent

1. *The pair $\{\Lambda_j\}_{j\in J}$ and $\{\Lambda_j^{'}\}_{j\in J}$ are a dual $K-g-$frame pair.*
2. *The pair $\{\Gamma_{ji}\Lambda_j\}_{j\in J, i\in I_j}$ and $\{\Gamma_{ji}^{'}\Lambda_j^{'}\}_{j\in J, i\in I_j}$ are a dual $K-g-$frame pair.*

Proof. Suppose that $\{\Lambda_j\}_{j\in J}$ is a $K-g-$frame for H with bounds D_Λ and C_Λ.

For each $x \in H$, we have

$$\begin{aligned}\sum_{j\in J}\sum_{i\in I_j}\langle \Gamma_{ji}\Lambda_j x, \Gamma_{ji}\Lambda_j x\rangle &\leq \sum_{j\in J} D_j\langle \Lambda_j x, \Lambda_j x\rangle \\ &\leq DD_\Lambda\langle x, x\rangle.\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{j\in J}\sum_{i\in I_j}\langle \Gamma_{ji}\Lambda_j x, \Gamma_{ji}\Lambda_j x\rangle &\geq \sum_{j\in J} C_j\Lambda_j x, \Lambda_j x\rangle \\ &\geq CC_\Lambda\langle K^*x, K^*x\rangle.\end{aligned}$$

Assume now that $\{\Gamma_{ji}\Lambda_j\}_{j\in J, i\in I_j}$ is a $K-g-$frame for H with bounds A, B. For each $x \in H$

$$C_j\langle \Lambda_j x, \Lambda_j x\rangle \leq \sum_{i\in I_j}\langle \Gamma_{ji}\Lambda_j x, \Gamma_{ji}\Lambda_j x\rangle \leq D_j\langle \Lambda_j x, \Lambda_j x\rangle,$$

then

$$\begin{aligned}\sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle &\leq \sum_{j\in J}\frac{1}{C_j}\sum_{i\in I_j}\langle \Gamma_{ji}\Lambda_j x, \Gamma_{ji}\Lambda_j x\rangle \\ &\leq \frac{1}{c}\sum_{j\in J}\sum_{i\in I_j}\langle \Gamma_{ji}\Lambda_j x, \Gamma_{ji}\Lambda_j x\rangle \\ &\leq \frac{B}{C}\langle x, x\rangle.\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle &\geq \sum_{j\in J}\frac{1}{D_j}\sum_{i\in I_j}\langle \Gamma_{ji}\Lambda_j x, \Gamma_{ji}\Lambda_j x\rangle \\ &\geq \frac{1}{D}\sum_{j\in J}\sum_{i\in I_j}\langle \Gamma_{ji}\Lambda_j x, \Gamma_{ji}\Lambda_j x\rangle \\ &\geq \frac{A}{D}\langle K^*x, K^*x\rangle.\end{aligned}$$

Therefore $\{\Lambda_j\}_{j\in J}$ being a $K-g-$frame for H is equivalent to $\{\Gamma_{ji}\Lambda_j\}_{j\in J, i\in I_j}$ being a $K-g-$frame for H, it remains only to prove the duality, then for each $x \in H$, we have

$$\begin{aligned}\sum_{j\in J}\Lambda_j^*\Lambda_j' x &= \sum_{j\in J}\Lambda_j^*\big(\sum_{i\in I_j}\Gamma_{ji}^*\Gamma_{ji}'\Lambda_j' x\big) \\ &= \sum_{j\in J}\sum_{i\in I_j}\Lambda_j^*\Gamma_{ji}^*\Gamma_{ji}'\Lambda_j' x \\ &= \sum_{j\in J}\sum_{i\in I_j}(\Gamma_{ji}\Lambda_j)^*\Gamma_{ji}'\Lambda_j' x,\end{aligned}$$

so $\{\Lambda_j'\}_{j\in J}$ is a dual $K-g-$bessel sequence of $\{\Lambda_j\}_{j\in J}$ if and only if $\{\Gamma_{ji}'\Lambda_j'\}_{j\in J, i\in I_j}$ is a dual $K-g-$bessel sequence of $\{\Gamma_{ji}\Lambda_j\}_{j\in J, i\in I_j}$. □

Theorem 2.3. *Let $\{\Lambda_j\}_{j\in J}$ be a Parseval $K-g-$frame for H where $K \in End_{\mathcal{A}}^*(H)$ has closed range. Then $\{\Lambda_j(K^+)^*\}_{j\in J}$ is a dual $K-g-$bessel sequence of $\{\Lambda_j\}_{j\in J}$.*

Proof. For each $x \in H$, we have

$$\begin{aligned}\sum_{j\in J}\langle \Lambda_j(K^+)^*x, \Lambda_j(K^+)^*x\rangle &= \langle K^*(K^+)^*x, K^*(K^+)^*x\rangle \\ &\leq \|K\|^2\|K^+\|^2\langle x, x\rangle,\end{aligned}$$

then $\{\Lambda_j(K^+)^*\}_{j\in J}$ is $g-$bessel sequence for H. And we have for each $g \in range(K^*)$, $g = K^*(K^*)^+g = K^*(K^+)^*g$, so

$$Kg = KK^*(K^+)^*g = \sum_{j\in J}\Lambda_j^*\Lambda_j(K^+)^*g.$$

If $h \in (range(K^*))^\perp = Ker(K)$ we obtain $h \in Ker((K^+)^*)$ implying that $\sum_{j\in J}\Lambda_j^*\Lambda_j(K^+)^*h = 0 = Kh$, altogether we have $Kf = \sum_{j\in J}\Lambda_j^*\Lambda_j(K^+)^*f$ for each $f \in H$. □

Theorem 2.4. *Suppose that $K \in End_{\mathcal{A}}^*(H)$ has closed range and that $\{\Lambda_j \in End_{\mathcal{A}}^*(H, H_j)\}_{j\in J}$ is a Parseval $K-g-$frame for H. Then the following condition hold.*

1. *For any dual $K-g-$bessel sequence $\{\theta_j\}_{j\in J}$ of $\{\Lambda_j\}_{j\in J}$ we have $T_{\tilde{\Lambda}}^*T_{\tilde{\Lambda}} = T_{\tilde{\Lambda}}^*T_\theta$, where $T_{\tilde{\Lambda}}$ is the analysis operator of $\{\Lambda_j(K^+)^*\}_{j\in J}$.*
2. *If $\{\Gamma_j \in End_{\mathcal{A}}^*(H, H_j)\}_{j\in J}$ is also a Parseval $K-g-$frame for H such that $T_\Lambda^*T_\Gamma = 0$, then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ admit a common dual $K-g-$bessel sequence $\{\Lambda_j(K^+)^* + \Gamma_j(K^+)^*\}_{j\in J}$.*

Proof. Assume that $\{\theta_j\}_{j\in J}$ is a dual $K-g-$bessel sequence of $\{\Lambda_j\}_{j\in J}$. So

$$\begin{aligned}T_\Lambda^*(T_{\tilde{\Lambda}}x - T_\theta x) &= \sum_{j\in J}\Lambda_j^*\Lambda_j(K^+)^*x - \sum_{j\in J}\Lambda_j^*\theta_j x \\ &= Kx - Kx = 0,\end{aligned}$$

then,

$$\langle T_{\tilde{\Lambda}}^*(T_{\tilde{\Lambda}} - T_\theta)x, y\rangle = \langle K^+T_\Lambda^*(T_{\tilde{\Lambda}} - T_\theta)x, y\rangle = 0,$$

hence, $T_{\tilde{\Lambda}}^*(T_{\tilde{\Lambda}} - T_\theta)x = 0$, so $T_{\tilde{\Lambda}}^*T_{\tilde{\Lambda}} = T_{\tilde{\Lambda}}^*T_\theta$. (2) Since $T_\Lambda^*T_\Gamma = 0$, then

$$\begin{aligned}\sum_{j\in J}\Lambda_j^*(\Lambda_j(K^+)^* + \Gamma_j(K^+)^*)x &= \sum_{j\in J}\Lambda_j^*\Lambda_j(K^+)^*x \\ &= Kx \\ &= \sum_{j\in J}\Gamma_j^*\Gamma_j(K^+)^*x \\ &= \sum_{j\in J}\Gamma_j^*(\Lambda_j(K^+)^* + \Gamma_j(K^+)^*)x\end{aligned}$$

□

Theorem 2.5. *Let $\{\Lambda_j \in End_{\mathcal{A}}^{*}(H, H_j); j \in J\}$ be a $K-g-$frame for H with bound A and B. Then $\{\Lambda_j\}_{j\in J}$ is a $g-$frame for H if K^* is bounded below.*

Proof. Since K^* is bounded below, then there exists $C > 0$ such that

$$\langle K^*x, K^*x\rangle \leq C\langle x, x\rangle, \forall x \in H.$$

And we have

$$A\langle K^*x, K^*x\rangle \leq \sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle \leq B\langle x, x\rangle, \forall x \in H.$$

So,

$$AC\langle x, x\rangle \leq A\langle K^*x, K^*x\rangle \leq \sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle \leq B\langle x, x\rangle, \forall x \in H.$$

Hence, $\{\Lambda_j\}_{j\in J}$ is a $g-$frame for H. □

Theorem 2.6. *If $\{\Lambda_j \in End_{\mathcal{A}}^{*}(H, H_j); j \in J\}$ is a tight $K-g-$frame with bounded A, then $\{\Lambda_j\}_{j\in J}$ is a tight $g-$frame with bounded B if and only if the right inverse of the operator K is $\frac{A}{B}K^*$.*

Proof. Now if $\{\Lambda_j\}_{j\in J}$ is a tight $g-$frame with bounded B, then

$$\sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle = B\langle x, x\rangle, \forall x \in H.$$

Since $\{\Lambda_j\}_{j\in J}$ is a tight $K-g-$frame with bounded A then,

$$A\langle K^*x, K^*x\rangle = B\langle x, x\rangle, \forall x \in H,$$

hence,

$$\langle KK^*x, x\rangle = \langle \frac{B}{A}x, x\rangle,$$

therefore,

$$KK^* = \frac{B}{A}I_H,$$

so, $\frac{A}{B}K^*$ is the right inverse of the operator K.

Conversely, assume that $\frac{A}{B}K^*$ is the right inverse of the operator K, then

$$KK^* = \frac{B}{A}I_H$$

so,

$$A\langle KK^*x, x\rangle = B\langle x, x\rangle,$$

hence,

$$A\langle K^*x, K^*x\rangle = B\langle x, x\rangle,$$

since, $\{\Lambda_j\}_{j\in J}$ is tight $K-g-$frame with bound A, then

$$B\langle x, x\rangle = \sum_{j\in J}\langle \Lambda_j x, \Lambda_j x\rangle, \forall x \in H.$$

□

Theorem 2.7. *Let $I \subset J$ be given. Suppose that $\{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j), j \in J\}$ is a $K-g-$frame with bounds A, B and $K-g-$frame operator $S_{\Lambda,J}$. Then the following statements are equivalent:*

(1) *$I_{range(K)} - S^{-1}_{\Lambda,J} S_{\Lambda,I}$ is boundedly invertible on $range(K)$,*
(2) *The sequence $\{\Lambda_j\}_{j\in J-I}$ is a $K-g-$frame for H with lower $K-g-$frame bound $\frac{B^{-1}}{\|S^{-1}_{\Lambda,J}\|^2 \|K^*(I_{range(K)} - S^{-1}_{\Lambda,J} S_{\Lambda,I})^{-1}\|^2}$.*

Proof. Denote the frame operator of the $K-g-$frame $\{\Lambda_j\}_{j\in J-I}$ by $S_{\Lambda,J-I}$. We have

$$S_{\Lambda,J-I} = S_{\Lambda,J} - S_{\Lambda,I} = S_{\Lambda,J}(I_{range(K)} - S^{-1}_{\Lambda,J} S_{\Lambda,I}),$$

then, $\{\Lambda_j\}_{j\in J-I}$ is a $K-g-$frame if and only if $I_{range(K)} - S^{-1}_{\Lambda,J} S_{\Lambda,I}$ is boundedly invertible.

Suppose that $I_{range(K)} - S^{-1}_{\Lambda,J} S_{\Lambda,I}$ is invertible. for any $x \in H$

$$\begin{aligned} x &= S^{-1}_{\Lambda,J} S_{\Lambda,J} x \\ &= S^{-1}_{\Lambda,J}\Big(\sum_{j\in J} \Lambda_j^* \Lambda_j x + \sum_{j\in J-I} \Lambda_j^* \Lambda_j x\Big) \\ &= S^{-1}_{\Lambda,J} S_{\Lambda,I} x + \sum_{j\in J-I} S^{-1}_{\Lambda,J} \Lambda_j^* \Lambda_j x. \end{aligned}$$

So,

$$(I_{range(K)} - S^{-1}_{\Lambda,J} S_{\Lambda,I})x = \sum_{j\in J-I} S^{-1}_{\Lambda,J} \Lambda_j^* \Lambda_j x.$$

Hence,

$$\begin{aligned} \|(I_{range(K)} - S^{-1}_{\Lambda,J} S_{\Lambda,I})x\| &= \|\sum_{j\in J-I} S^{-1}_{\Lambda,J} \Lambda_j^* \Lambda_j x\| \\ &= \sup_{\|y\|=1} \Big\|\langle \sum_{j\in J-I} S^{-1}_{\Lambda,J} \Lambda_j^* \Lambda_j x, y\rangle\Big\| \\ &= \sup_{\|y\|=1} \Big\| \sum_{j\in J-I} \langle \Lambda_j x, \Lambda_j S^{-1}_{\Lambda,J} y\rangle\Big\| \\ &\leq \sup_{\|y\|=1} \Big\| \sum_{j\in J-I} \langle \Lambda_j x, \Lambda_j x\rangle\Big\|^{\frac{1}{2}} \Big\| \sum_{j\in J-I} \langle \Lambda_j S^{-1}_{\Lambda,J} y, \Lambda_j S^{-1}_{\Lambda,J} y\rangle\Big\|^{\frac{1}{2}} \\ &\leq \sqrt{B}\|S^{-1}_{\Lambda,J}\| \Big\| \sum_{j\in J-I} \langle \Lambda_j x, \Lambda_j x\rangle\Big\|^{\frac{1}{2}}, \end{aligned}$$

therefore,

$$\|(I_{range(K)} - S^{-1}_{\Lambda,J} S_{\Lambda,I})x\| \leq \sqrt{B}\|S^{-1}_{\Lambda,J}\| \Big\| \sum_{j\in J-I} \langle \Lambda_j x, \Lambda_j x\rangle\Big\|^{\frac{1}{2}},$$

then $I_{range(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I}$ is well defined. If $I_{range(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I}$ is invertible on H, then for any $x \in H$ we have

$$\|K^*x\| \leq \|K^*(I_{range(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I})^{-1}\|\|(I_{range(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I})x\|$$

hence,

$$\frac{B^{-1}}{\|S_{\Lambda,J}^{-1}\|^2\|K^*(I_{range(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I})^{-1}\|^2}\|K^*x\|^2 \leq \left\|\sum_{j\in J-I}\langle\Lambda_j x, \Lambda_j x\rangle\right\|.$$

□

Definition 2.1. *Consider two g−bessel sequences* $\{\Lambda_j \in End_{\mathcal{A}}^*(H, H_j), j \in J\}$ *and* $\{\theta_j \in End_{\mathcal{A}}^*(H, H_j), j \in J\}$*. The sequence* $\{\Lambda_j\}_{j\in J}$ *and* $\{\theta_j\}_{j\in J}$ *are said to be approximately* $K - g-$*dual frames if* $\|I_{range(K)} - T_\Lambda T_\theta^*\| < 1$*. In this case, we say that* $\{\theta_j\}_{j\in J}$ *is an approximate* $K - g-$*dual of* $\{\Lambda_j\}_{j\in J}$*.*

Theorem 2.8. *If* $\{\theta_j\}_{j\in J}$ *is an approximate* $K - g-$*dual of* $\{\Lambda_j\}_{j\in J}$*, then* $\{\theta_j(T_\Lambda T_\theta^*)^{-1}\}_{j\in J}$ *is a* $K - g-$*dual of* $\{\Lambda_j\}_{j\in J}$*.*

Proof. it is easy to see that $\{\theta_j(T_\Lambda T_\theta^*)^{-1}\}_{j\in J}$ is a $g-$bessel sequence and

$$\begin{aligned} x &= (T_\Lambda T_\theta^*)(T_\Lambda T_\theta^*)^{-1}x \\ &= \sum_{j=0}^{\infty}\Lambda_j^*\theta_j(T_\Lambda T_\theta^*)^{-1}x \\ &= \sum_{j=0}^{\infty}\Lambda_j^*\left(\theta_j\sum_{j=0}^{\infty}(I_{range(K)} - -T_\Lambda T_\theta^*)^n x\right). \end{aligned}$$

Then, $\theta_j(T_\Lambda T_\theta^*)^{-1} = \{\theta_j\sum_{j=0}^{\infty}(I_{range(K)} - -T_\Lambda T_\theta^*)^n\}_{j\in J}$ is a $K - g-$dual of $\{\Lambda_j\}_{j\in J}$. □

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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