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A New Ostrowski's Type Inequality for Quadratic Kernel

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Abstract. From the past few decades, the integral inequalities have been extensively researched. Integral inequalities are applied in innumerable mathematical problems. In current paper, we obtain new versions of Ostrowski's type integral inequalities by implementing proposed general form of 7-step quadratic kernel. Applications for cumulative distribution are also provided.

1. Introduction

In 1938, a Ukrainian Mathematician A. M. Ostrowski [1] discovered the integral inequality. Many articles and research books have been dedicated to inequalities and its applications [3]-[7]. In this paper we present 7-step quadratic kernel that further generalize many earlier results contained in [8]-[12]. Several authors have recently addressed the generalization of the Ostrowski's type inequalities. Qayyum et.al [9]-[10] applied a 5-step kernel to generalize some Ostrowski's type inequalities.

2. Main Findings

Theorem 2.1. *Let* $f:\left[\check{c},\check{d}\right]\to\mathbb{R}$ *be differentiable on* $\left(\check{c},\check{d}\right)$, f' *is absolutely continuous on* $\left[\check{c},\check{d}\right]$ *and* $\gamma\le f''(\text{ţ})\le\Gamma,\forall\ \text{ţ}\in\left[\check{c},\check{d}\right]$ *,then* $\forall\ \ddot{y}\in\left[\check{c},\frac{\check{c}+\check{d}}{2}\right]$ *, we get*

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$$\begin{aligned}
&\Bigg|\frac{1}{4}\Bigg[f(\ddot{y})+f(\check{c}+\check{d}-\ddot{y})+\frac{1}{2}f\left(\frac{\check{c}+\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{3\check{c}+\ddot{y}}{4}\right)\\
&+\frac{1}{2}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right)\\
&\times\left\{f'(\check{c}+\check{d}-\ddot{y})-f'(\ddot{y})+\frac{1}{4}f'\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)-\frac{1}{4}f'\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}\\
&+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)\left\{f'\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f'\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\Bigg]\\
&+\frac{f'(\check{d})-f'(\check{c})}{(\check{d}-\check{c})^2}\left\{\frac{1}{192}(\ddot{y}-\check{c})^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3\right.\\
&\left.-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f(ţ)\,dţ\Bigg|\\
&\leq\omega(\ddot{y})(\check{d}-\check{c})(S-\gamma)
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
&\Bigg|\frac{1}{4}\Bigg[f(\ddot{y})+f(\check{c}+\check{d}-\ddot{y})+\frac{1}{2}f\left(\frac{\check{c}+\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{3\check{c}+\ddot{y}}{4}\right)\\
&+\frac{1}{2}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right)\\
&\times\left\{f'(\check{c}+\check{d}-\ddot{y})-f'(\ddot{y})+\frac{1}{4}f'\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)-\frac{1}{4}f'\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}\\
&+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)\left\{f'\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f'\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\Bigg]\\
&+\frac{f'(\check{d})-f'(\check{c})}{(\check{d}-\check{c})^2}\left\{\frac{1}{192}(\ddot{y}-\check{c})^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3\right.\\
&\left.-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f(ţ)\,dţ\Bigg|\\
&\leq\omega(\ddot{y})(\check{d}-\check{c})(\Gamma-S),
\end{aligned}\tag{2.2}$$

where

$$S=\frac{f'(\check{d})-f'(\check{c})}{\check{d}-\check{c}}$$

and

$$\begin{aligned}
&\omega(\ddot{y})\\
&=\frac{1}{768}\max\left\{\left|-65\check{c}^2+231\check{c}\ddot{y}+165\check{d}\ddot{y}-101\check{c}\check{d}\right.\right.\\
&\left.-32\check{d}^2-198\ddot{y}^2\right|,\left|-35\check{c}^2+135\check{c}\ddot{y}+117\check{d}\ddot{y}-65\check{c}\check{d}-26\check{d}^2-126\ddot{y}^2\right|,
\end{aligned}$$

$$
\begin{aligned}
&\left|-83\check{c}^2+255\check{c}\ddot{y}+141\check{c}\ddot{y}-89\check{c}\check{d}-26\check{d}^2-198\ddot{y}^2\right|,\\
&\left|127\check{c}^2-297\check{c}\ddot{y}-27\check{d}\ddot{y}+43\check{c}\check{d}-8\check{d}^2+162\ddot{y}^2\right|,\\
&\left|7\check{c}^2-105\check{c}\ddot{y}-219\check{d}\ddot{y}+91\check{c}\check{d}+64\check{d}^2+162\ddot{y}^2\right|,\\
&\left|-65\check{c}^2+183\check{c}\ddot{y}+69\check{d}\ddot{y}-53\check{c}\check{d}-8\check{d}^2-126\ddot{y}^2\right|,\\
&\left|89\check{c}^2-279\check{c}\ddot{y}-165\check{d}\ddot{y}+101\check{c}\check{d}+32\check{d}^2+222\ddot{y}^2\right|\Big\}.
\end{aligned}
$$

Proof. To prove our required results, first of all we introduce a mapping. Let $f:\left[\check{c},\check{d}\right]\to\mathbb{R}$ be such that f' is absolutely continuous on $\left[\check{c},\check{d}\right]$. Define the kernel $K\left(\ddot{y},\text{ţ}\right)$ as:

$$
K\left(\ddot{y},\text{ţ}\right)=\begin{cases}
\frac{1}{2}\left(\text{ţ}-\check{c}\right)^2 & \text{ţ}\in\left(\check{c},\frac{3\check{c}+\ddot{y}}{4}\right]\\[2ex]
\frac{1}{2}\left(\text{ţ}-\frac{7\check{c}+\check{d}}{8}\right)^2 & \text{ţ}\in\left(\frac{3\check{c}+\ddot{y}}{4},\frac{\check{c}+\ddot{y}}{2}\right]\\[2ex]
\frac{1}{2}\left(\text{ţ}-\frac{3\check{c}+\check{d}}{4}\right)^2 & \text{ţ}\in\left(\frac{\check{c}+\ddot{y}}{2},\ddot{y}\right]\\[2ex]
\frac{1}{2}\left(\text{ţ}-\frac{\check{c}+\check{d}}{2}\right)^2 & \text{ţ}\in\left(\ddot{y},\check{c}+\check{d}-\ddot{y}\right]\\[2ex]
\frac{1}{2}\left(\text{ţ}-\frac{\check{c}+3\check{d}}{4}\right)^2 & \text{ţ}\in\left(\check{c}+\check{d}-\ddot{y},\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right]\\[2ex]
\frac{1}{2}\left(\text{ţ}-\frac{\check{c}+7\check{d}}{8}\right)^2 & \text{ţ}\in\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2},\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right]\\[2ex]
\frac{1}{2}\left(\text{ţ}-\check{d}\right)^2 & \text{ţ}\in\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4},\check{d}\right]
\end{cases}
\tag{2.3}
$$

$\forall\ \ddot{y}\in\left[\check{c},\frac{\check{c}+\check{d}}{2}\right]$. Then the following identity

$$
\begin{aligned}
&\int_{\check{c}}^{\check{d}}K\left(\ddot{y},\text{ţ}\right)df'\left(\text{ţ}\right) \qquad (2.4)\\
&=\int_{\check{c}}^{\check{d}}f\left(\text{ţ}\right)d\text{ţ}-\frac{\check{d}-\check{c}}{4}\left[f\left(\ddot{y}\right)+f\left(\check{c}+\check{d}-\ddot{y}\right)+\frac{1}{2}f\left(\frac{\check{c}+\ddot{y}}{2}\right)\right.\\
&+\frac{1}{2}f\left(\frac{3\check{c}+\ddot{y}}{4}\right)+\frac{1}{2}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)\\
&+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right)\times\left\{f'\left(\check{c}+\check{d}-\ddot{y}\right)-f'\left(\ddot{y}\right)+\frac{1}{4}f'\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)-\frac{1}{4}f'\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}\\
&\left.+\frac{1}{8}\left(\ddot{y}-\frac{3a+\check{d}}{4}\right)\left\{f'\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f'\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\right]
\end{aligned}
$$

holds. We know that

$$
\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f''\left(\text{ţ}\right)d\text{ţ}=\frac{f'(\check{d})-f'(\check{c})}{\check{d}-\check{c}}
\tag{2.5}
$$

$$\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},ţ)\,dţ=\frac{1}{\check{d}-\check{c}}\left[\frac{1}{192}(\ddot{y}-\check{c})^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right] \tag{2.6}$$

and implies that

$$\begin{aligned}&\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},ţ)f''(ţ)\,dţ-\frac{1}{(\check{d}-\check{c})^2}\int_{\check{c}}^{\check{d}}K(\ddot{y},ţ)\,dţ\int_{\check{c}}^{\check{d}}f''(ţ)\,dţ \\ &=\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f(ţ)\,dţ-\frac{1}{4}\left[f(\ddot{y})+f(\check{c}+\check{d}-\ddot{y})+\frac{1}{2}f\left(\frac{\check{c}+\ddot{y}}{2}\right)\right. \\ &+\frac{1}{2}f\left(\frac{3\check{c}+\ddot{y}}{4}\right)+\frac{1}{2}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right) \\ &+\frac{1}{2}f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right) \\ &\times\left\{f'(\check{c}+\check{d}-\ddot{y})-f'(\ddot{y})+\frac{1}{4}f'\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)-\frac{1}{4}f'\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\} \\ &\left.+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)\left\{f'\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f'\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\right] \\ &-\frac{f'(\check{d})-f'(\check{c})}{(\check{d}-\check{c})^2}\left\{\frac{1}{192}(\ddot{y}-\check{c})^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3\right. \\ &\left.-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}.\end{aligned} \tag{2.7}$$

We suppose that

$$\begin{aligned}&R_n(\ddot{y}) \\ &=\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},ţ)f''(ţ)\,dt-\frac{1}{(\check{d}-\check{c})^2}\int_{\check{c}}^{\check{d}}K(\ddot{y},ţ)\,dt\int_{\check{c}}^{\check{d}}f''(ţ)\,dt.\end{aligned} \tag{2.8}$$

If $C\in R$ is an arbitray constant, then we have

$$\begin{aligned}&R_n(\ddot{y}) \\ &=\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}\left(f''(ţ)-C\right)\left[K(\ddot{y},ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},s)\,ds\right]dţ.\end{aligned} \tag{2.9}$$

Furthermore, we have

$$\begin{aligned}&|R_n(\ddot{y})| \\ &\leq\frac{1}{\check{d}-\check{c}}\max_{ţ\in[\check{c},\check{d}]}\left|K(\ddot{y},ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},s)\,ds\right|\int_{\check{c}}^{\check{d}}\left|f''(ţ)-C\right|dţ.\end{aligned} \tag{2.10}$$

Now

$$\max\left|K(\ddot{y},ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},s)\,ds\right| \tag{2.11}$$

$$=\max\left\{\left|\frac{1}{2}\left(\frac{\ddot{y}-\check{c}}{4}\right)^2-\frac{\beta(\ddot{y})}{\check{d}-\check{c}}\right|,\left|\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^2-\frac{\beta(\ddot{y})}{\check{d}-\check{c}}\right|,\right.$$

$$\left|\frac{1}{32}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^2-\frac{\beta(\ddot{y})}{\check{d}-\check{c}}\right|,\left|\frac{1}{2}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^2-\frac{\beta(\ddot{y})}{\check{d}-\check{c}}\right|,$$

$$\left.\left|\frac{1}{2}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^2-\frac{\beta(\ddot{y})}{\check{d}-\check{c}}\right|,\left|\frac{1}{8}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^2-\frac{\beta(\ddot{y})}{\check{d}-\check{c}}\right|,\frac{\beta(\ddot{y})}{\check{d}-\check{c}}\right\}$$

where

$$\beta(\ddot{y})=\frac{1}{192}(\ddot{y}-\check{c})^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3.$$

and

$$\omega(\ddot{y}) \tag{2.12}$$

$$=\frac{1}{768}\max\left\{\left|-65\check{c}^2+231\check{c}\ddot{y}+165\check{d}\ddot{y}-101\check{c}\check{d}\right.\right.$$

$$\left.-32\check{d}^2-198\ddot{y}^2\right|,$$

$$\left|-35\check{c}^2+135\check{c}\ddot{y}+117\check{d}\ddot{y}-65\check{c}\check{d}-26\check{d}^2-126\ddot{y}^2\right|,$$

$$\left|-83\check{c}^2+255\check{c}\ddot{y}+141\check{c}\ddot{y}-89\check{c}\check{d}-26\check{d}^2-198\ddot{y}^2\right|,$$

$$\left|127\check{c}^2-297\check{c}\ddot{y}-27\check{d}\ddot{y}+43\check{c}\check{d}-8\check{d}^2+162\ddot{y}^2\right|,$$

$$\left|7\check{c}^2-105\check{c}\ddot{y}-219\check{d}\ddot{y}+91\check{c}\check{d}+64\check{d}^2+162\ddot{y}^2\right|,$$

$$\left|-65\check{c}^2+183\check{c}\ddot{y}+69\check{d}\ddot{y}-53\check{c}\check{d}-8\check{d}^2-126\ddot{y}^2\right|,$$

$$\left.\left|89\check{c}^2-279\check{c}\ddot{y}-165\check{d}\ddot{y}+101\check{c}\check{d}+32\check{d}^2+222\ddot{y}^2\right|\right\}.$$

We also have

$$\int_{\check{c}}^{\check{d}}\left|f''(ţ)-\gamma\right|dt=(S-\gamma)\left(\check{d}-\check{c}\right) \tag{2.13}$$

and

$$\int_{\check{c}}^{\check{d}}\left|f''(ţ)-\Gamma\right|dţ=(\Gamma-S)\left(\check{d}-\check{c}\right). \tag{2.14}$$

So, we attain (2.1) and (2.2) by using (2.5) to (2.14) and taking $C=\gamma$ and $C=\Gamma$ in (2.10) respectively.

Corollary 2.1. *By replacing $\ddot{y}=\check{c}$ in* (2.1) *and* (2.2)*, we get*

$$\left|\frac{f(\check{c})+f(\check{d})}{2}-\left(\check{d}-\check{c}\right)\frac{f'(\check{d})-f'(\check{c})}{12}-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f(ţ)dţ\right|$$

$$\leq\frac{1}{96}\left(\check{d}-\check{c}\right)^3(S-\gamma),$$

$$\left|\frac{f(\check{c})+f(\check{d})}{2}-\left(\check{d}-\check{c}\right)\frac{f'(\check{d})-f'(\check{c})}{12}-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f(ţ)dţ\right|$$
$$\leq\frac{1}{96}\left(\check{d}-\check{c}\right)^{3}(\Gamma-S).$$

□

Now some new perturbed Ostrowski type inequalities are presented by working with differentiable mapping whose first derivatives are absolutely continuous and the second derivatives belong to $f''' \in L^2$ the usual Lebesgue spaces which refine and generalize some previous inequalities of this domain.

Theorem 2.2. *Let $f : [\check{c},\check{d}] \to \mathbb{R}$ be three times differentiable function on $(\check{c},\check{d})$. If $f''' \in L^2[\check{c},\check{d}]$, then for all $\ddot{y} \in \left[\check{c},\frac{\check{c}+\check{d}}{2}\right]$, we have*

$$\begin{aligned}&\left|\frac{1}{4}\left[f(\ddot{y})+f\left(\check{c}+\check{d}-\ddot{y}\right)+\frac{1}{2}f\left(\frac{\check{c}+\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right.\right.\\&+\frac{1}{2}f\left(\frac{\check{c}+2b-\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right)\\&\times\left\{f'\left(\check{c}+\check{d}-\ddot{y}\right)-f'(\ddot{y})+\frac{1}{4}f'\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)-\frac{1}{4}f'\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}\\&\left.+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)\left\{f'\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f'\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\right]\\&+\frac{f'(\check{d})-f'(\check{c})}{\left(\check{d}-\check{c}\right)^{2}}\left\{\frac{1}{192}(\ddot{y}-\check{c})^{3}+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^{3}\right.\\&\left.\left.-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^{3}\right\}-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f(ţ)\,dţ\right|\\&\leq\frac{1}{\pi}\left\|f'''\right\|_{2}\left[\frac{1}{10240}(\ddot{y}-\check{c})^{5}+\frac{33}{320}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^{5}-\frac{1057}{10240}\right.\\&\times\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^{5}-\frac{1}{\check{d}-\check{c}}\left\{\frac{1}{192}(\ddot{y}-\check{c})^{3}+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^{3}\right.\\&\left.\left.-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^{3}\right\}^{2}\right]^{\frac{1}{2}}.\end{aligned}\tag{2.15}$$

Proof. Let $R_n(x)$ be defined by (2.7) and (2.8), If we take $C = f''\left(\frac{\check{c}+\check{d}}{2}\right)$ in (2.9) by applying the Cauchy Inequality, then

$$\begin{aligned}|R_n(\ddot{y})|\leq\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}&\left|\left(f''(ţ)-f''\left(\frac{\check{c}+\check{d}}{2}\right)\right)\right|\\&\times\left|K(\ddot{y},ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},s)\,ds\right|dţ.\end{aligned}\tag{2.16}$$

$$\leq \frac{1}{\check{d}-\check{c}}\left[\int_{\check{c}}^{\check{d}}\left(f''(ţ)-f''\left(\frac{\check{c}+\check{d}}{2}\right)\right)^{2}dţ\right]^{\frac{1}{2}}$$
$$\times\left[\int_{\check{c}}^{\check{d}}\left(K(\ddot{y},ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},s)\,ds\right)^{2}dţ\right]^{\frac{1}{2}}.$$

We apply the Diaze-Metcalf inequality from [16] to get

$$\int_{\check{c}}^{\check{d}}\left(f''(ţ)-f''\left(\frac{\check{c}+\check{d}}{2}\right)\right)^{2}dţ \leq \frac{\left(\check{d}-\check{c}\right)^{2}}{\pi^{2}}\left\|f'''\right\|_{2}^{2}$$

and

$$\int_{\check{c}}^{\check{d}}\left(K(\ddot{y},ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},s)\,ds\right)^{2}dţ \tag{2.17}$$
$$=\int_{\check{c}}^{\check{d}}K(\ddot{y},ţ)^{2}\,dţ-\frac{1}{\check{d}-\check{c}}\left\{\frac{1}{192}(\ddot{y}-\check{c})^{3}+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^{3}\right.$$
$$\left.-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^{3}\right\}^{2}$$
$$=\frac{1}{10240}(\ddot{y}-\check{c})^{5}+\frac{33}{320}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^{5}-\frac{1057}{10240}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^{5}$$
$$-\frac{1}{\check{d}-\check{c}}\left\{\frac{1}{192}(\ddot{y}-\check{c})^{3}+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^{3}-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^{3}\right\}^{2}.$$

So, by using the above relations (2.18)-(2.19) , we attain (2.17) .

Corollary 2.2. *By replacing* $\ddot{y}=\check{c}$ *in* (2.17) , *we get*

$$\left|\frac{f(\check{c})+f(\check{d})}{2}-\left(\check{d}-\check{c}\right)\frac{f'(\check{c})+f'(\check{d})}{12}-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f(ţ)dţ\right|$$
$$\leq\frac{1}{\pi}\left\|f'''\right\|_{2}\left(\check{d}-\check{c}\right)^{\frac{5}{2}}\frac{1}{12}\frac{1}{\sqrt{5}}.$$

□

Theorem 2.3. *Let* $f:\left[\check{c},\check{d}\right]\to\mathbb{R}$ *be an absolutely continuous function on* $\left(\check{c},\check{d}\right)$, *with* $f''\in L^{2}\left[\check{c},\check{d}\right]$. *Then*

$$\left|\frac{1}{4}\left[f(\ddot{y})+f\left(\check{c}+\check{d}-\ddot{y}\right)+\frac{1}{2}f\left(\frac{\check{c}+\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right.\right. \tag{2.18}$$
$$+\frac{1}{2}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)+\frac{1}{2}f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right)$$
$$\times\left\{f'\left(\check{c}+\check{d}-\ddot{y}\right)-f'(\ddot{y})+\frac{1}{4}f'\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)-\frac{1}{4}f'\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}$$
$$\left.+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)\left\{f'\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f'\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\right]$$

$$+\frac{f'(\check{d})-f'(\check{c})}{\left(\check{d}-\check{c}\right)^2}\left\{\frac{1}{192}\left(\ddot{y}-\check{c}\right)^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3\right.$$

$$\left.-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f\left(ţ\right)dţ\Bigg|$$

$$\leq\frac{\sqrt{\sigma(f'')}}{\check{d}-\check{c}}\left[\frac{1}{10240}\left(\ddot{y}-\check{c}\right)^5+\frac{33}{320}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^5\right.$$

$$-\frac{1057}{10240}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^5-\frac{1}{\check{d}-\check{c}}\left\{\frac{1}{192}\left(\ddot{y}-\check{c}\right)^3\right.$$

$$\left.\left.+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}^2\right]^{\frac{1}{2}}$$

$\forall\ddot{y}\in\left[\check{c},\frac{\check{c}+\check{d}}{2}\right]$, *where*

$$\sigma(f'')=\left\|f''\right\|_2^2-\frac{\left(f'(\check{d})-f'(\check{c})\right)^2}{\check{d}-\check{c}}=\left\|f''\right\|_2^2-S^2\left(\check{d}-\check{c}\right)\tag{2.19}$$

where

$$S=\frac{f'(\check{d})-f'(\check{c})}{\check{d}-\check{c}}.$$

Proof. Let $R_n(x)$ be defined by (2.7) and (2.8) .If we take $C=\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f''(s)\,ds$ in (2.9) and applying the Cauchy Inequality, then we have

$$|R_n(\ddot{y})|$$

$$\leq\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}\left|\left(f''(ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f''(s)\,ds\right)\right|$$

$$\times\left|K(\ddot{y},ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},s)\,ds\right|dţ.$$

$$\leq\frac{1}{\check{d}-\check{c}}\left[\int_{\check{c}}^{\check{d}}\left(f''(ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f''(s)\,ds\right)^2dţ\right]^{\frac{1}{2}}$$

$$\times\left[\int_{\check{c}}^{\check{d}}\left(K(\ddot{y},ţ)-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}K(\ddot{y},s)\,ds\right)^2dţ\right]^{\frac{1}{2}}$$

$$=\frac{\sqrt{\sigma(f'')}}{\check{d}-\check{c}}\left[\frac{1}{10240}\left(\ddot{y}-\check{c}\right)^5+\frac{33}{320}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^5\right.$$

$$-\frac{1057}{10240}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^5-\frac{1}{\check{d}-\check{c}}\left\{\frac{1}{192}\left(\ddot{y}-\check{c}\right)^3\right.$$

$$+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^{3}-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^{3}\Bigg\}^{2}\Bigg]^{\frac{1}{2}}.$$

Hence proved (2.21) □

Corollary 2.3. *By replacing $\ddot{y}=\check{c}$ in* (2.21) *we get*

$$\left|\frac{f(\check{c})+f(\check{d})}{2}-\left(\check{d}-\check{c}\right)\frac{f'(\check{c})+f'(\check{d})}{12}-\frac{1}{\check{d}-\check{c}}\int_{\check{c}}^{\check{d}}f(\c{t})d\c{t}\right|$$
$$\leq\sqrt{\sigma(f'')}\left(\check{d}-\check{c}\right)^{\frac{3}{2}}\frac{1}{12}\frac{1}{\sqrt{5}}.$$

3. An Application to Cumulative Distribution Function

Consider X is a random variable taking values in the finite interval $\left[\check{c},\check{d}\right]$ with the probability density function $f:\left[\check{c},\check{d}\right]\to[0,1]$ and cumulative distributive function

$$F\left(\ddot{y}\right)=\Pr\left(X\leq\ddot{y}\right)=\int_{\check{c}}^{\ddot{y}}f(\c{t})d\c{t}, \tag{3.1}$$

$$F\left(\check{d}\right)=\Pr\left(X\leq\check{d}\right)=\int_{\check{c}}^{\check{d}}f(u)du=1. \tag{3.2}$$

In this section, we may use the inequalities (2.1)-(2.2), (2.17) and (2.21) to get useful application for cumulative distribution function by using probability density function with smaller error than that which may be obtained by the classical results.

Theorem 3.1. *Under the assumption of Theorem 2.1, we get the following inequality which holds*

$$\Bigg|\frac{\check{d}-E(X)}{\check{d}-\check{c}}-\frac{1}{4}\Bigg[F\left(\ddot{y}\right)+F\left(\check{c}+\check{d}-\ddot{y}\right)+\frac{1}{2}F\left(\frac{\check{c}+\ddot{y}}{2}\right) \tag{3.3}$$
$$+\frac{1}{2}F\left(\frac{3\check{c}+\ddot{y}}{4}\right)+\frac{1}{2}F\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)$$
$$+\frac{1}{2}F\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right)$$
$$\times\left\{f\left(\check{c}+\check{d}-\ddot{y}\right)-f\left(\ddot{y}\right)+\frac{1}{4}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)-\frac{1}{4}f\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}$$
$$+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)\left\{f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\Bigg]$$
$$-\frac{f(\check{d})-f(\check{c})}{\left(\check{d}-\check{c}\right)^{2}}\left\{\frac{1}{192}\left(\ddot{y}-\check{c}\right)^{3}+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^{3}\right.$$
$$\leq\omega(\ddot{y})(\check{d}-\check{c})(S-\gamma)$$

and

$$\Bigg|\frac{\check{d}-E(X)}{\check{d}-\check{c}}-\frac{1}{4}\Bigg[F\left(\ddot{y}\right)+F\left(\check{c}+\check{d}-\ddot{y}\right)+\frac{1}{2}F\left(\frac{\check{c}+\ddot{y}}{2}\right) \tag{3.4}$$
$$+\frac{1}{2}F\left(\frac{3\check{c}+\ddot{y}}{4}\right)+\frac{1}{2}F\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)$$
$$+\frac{1}{2}F\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right)$$
$$\times\left\{f\left(\check{c}+\check{d}-\ddot{y}\right)-f\left(\ddot{y}\right)+\frac{1}{4}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)-\frac{1}{4}f\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}$$
$$+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)\left\{f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\Bigg]$$
$$-\frac{f(\check{d})-f(\check{c})}{\left(\check{d}-\check{c}\right)^2}\left\{\frac{1}{192}\left(\ddot{y}-\check{c}\right)^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3\right.$$
$$\left.-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}\Bigg|\le\omega(\ddot{y})(\check{d}-\check{c})(\Gamma-S)$$

for all $\ddot{y}\in\left[\check{c},\frac{\check{c}+\check{d}}{2}\right]$.

Proof. By (3.1) and (3.2) on taking $f=F$ and by applying the fact

$$E(X)=\int_{\check{c}}^{\check{d}}ţ dF(ţ)=\check{d}-\int_{\check{c}}^{\check{d}}F(ţ)dţ \tag{3.5}$$

we attain (3.3) and (3.4). ∎

Theorem 3.2. *By using Theorem 2.1, we get the following inequality which holds*

$$\Bigg|\frac{\check{d}-E(X)}{\check{d}-\check{c}}-\frac{1}{4}\Bigg[F\left(\ddot{y}\right)+F\left(\check{c}+\check{d}-\ddot{y}\right)+\frac{1}{2}F\left(\frac{\check{c}+\ddot{y}}{2}\right) \tag{3.6}$$
$$+\frac{1}{2}F\left(\frac{3\check{c}+\ddot{y}}{4}\right)+\frac{1}{2}F\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)$$
$$+\frac{1}{2}F\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right)$$
$$\times\left\{f\left(\check{c}+\check{d}-\ddot{y}\right)-f\left(\ddot{y}\right)+\frac{1}{4}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right)\right.$$
$$\left.-\frac{1}{4}f\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)$$
$$\times\left\{f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\Bigg]$$

$$-\frac{f(\check{d})-f(\check{c})}{(\check{d}-\check{c})^2}\left\{\frac{1}{192}(\ddot{y}-\check{c})^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3 -\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}\Bigg|$$

$$\leq \frac{1}{\pi}\left\|f'''\right\|_2\left[\frac{1}{10240}(\ddot{y}-\check{c})^5+\frac{33}{320}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^5 -\frac{1057}{10240}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^5-\frac{1}{\check{d}-\check{c}}\left\{\frac{1}{192}(\ddot{y}-\check{c})^3 +\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}^2\right]^{\frac{1}{2}}.$$

Proof. Using (4.36) and by the conditions that we used in above Theorem , we get (3.6). □

Theorem 3.3. *With the statement of Theorem 2.1, we have the following inequality which holds*

$$\Bigg|\frac{\check{d}-E(X)}{\check{d}-\check{c}}-\frac{1}{4}\left[F(\ddot{y})+F\left(\check{c}+\check{d}-\ddot{y}\right)+\frac{1}{2}F\left(\frac{\check{c}+\ddot{y}}{2}\right) +\frac{1}{2}F\left(\frac{3\check{c}+\ddot{y}}{4}\right)+\frac{1}{2}F\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right) +\frac{1}{2}F\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)+\left(\ddot{y}-\frac{5\check{c}+3\check{d}}{8}\right) \times\left\{f\left(\check{c}+\check{d}-\ddot{y}\right)-f(\ddot{y})+\frac{1}{4}f\left(\frac{\check{c}+2\check{d}-\ddot{y}}{2}\right) -\frac{1}{4}f\left(\frac{\check{c}+\ddot{y}}{2}\right)\right\}+\frac{1}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right) \times\left\{f\left(\frac{\check{c}+4\check{d}-\ddot{y}}{4}\right)-f\left(\frac{3\check{c}+\ddot{y}}{4}\right)\right\}\right] -\frac{f(\check{d})-f(\check{c})}{(\check{d}-\check{c})^2}\left\{\frac{1}{192}(\ddot{y}-\check{c})^3+\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}\Bigg|$$
$$\leq\frac{\sqrt{\sigma(f'')}}{\check{d}-\check{c}}\left[\frac{1}{10240}(\ddot{y}-\check{c})^5+\frac{33}{320}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^5 -\frac{1057}{10240}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^5-\frac{1}{\check{d}-\check{c}}\left\{\frac{1}{192}(\ddot{y}-\check{c})^3 +\frac{3}{8}\left(\ddot{y}-\frac{3\check{c}+\check{d}}{4}\right)^3-\frac{73}{192}\left(\ddot{y}-\frac{\check{c}+\check{d}}{2}\right)^3\right\}^2\right]^{\frac{1}{2}}. \tag{3.7}$$

Proof. Applying (4.43) and by set of conditions that we used in Theorem 3.1, we get (3.7). □

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