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On Weakly S-2-Absorbing Submodules

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Abstract. Let R be a commutative ring with identity and let M be a unitary R-module. In this paper, we introduce the notion of weakly S-2-absorbing submodule. Suppose that S is a multiplicatively closed subset of R. A submodule P of M with $(P :_R M) \cap S = \emptyset$ is said to be a weakly S-2-absorbing submodule if there exists an element $s \in S$ such that whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in P$, then $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. We give the characterizations, properties and examples of weakly S-2-absorbing submodules.

1. Introduction

Throughout this paper, R denotes a commutative ring with non zero identity and M is an R module. Prime ideals and submodules have vital role in ring and module theory. Of course a proper submodule P of M is called prime if $am \in P$ for $a \in R$ and $m \in M$ implies $a \in (P :_R M)$ or $m \in P$ where $(P :_R M) = \{r \in R : rM \subseteq P\}$. Several generalizations of these concepts have been studied extensively by many authors [9], [13], [6], [16], [3], [11], [14], [5].

In 2007, Atani and Farzalipour introduced the concept of weakly prime submodules as a generalization of prime submodules. A proper submodule P of M is defined as weakly prime if for $a \in R$ and $m \in M$, whenever for $0 \neq am \in P$ implies $a \in (P :_R M)$ or $m \in P$ as in [5].

A new kind of generalization of prime submodule has been introduced and studied by Sengelen sevim et. al. in 2019 in [14]. For a multiplicatively closed subset S of R, that is, S satisfies the following conditions: (i) $1 \in S$ and (ii) $s_1s_2 \in S$ for each $s_1, s_2 \in S$, a proper submodule P of an R-module M with $(P :_R M) \cap S = \emptyset$ is called an S-prime submodule if there exists $s \in S$ such that for $a \in R$ and $m \in M$, if $am \in P$ then either $sa \in (P :_R M)$ or $sm \in P$. In particular an ideal I of R is called

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as S-prime ideal if I is an S-prime submodule of an R-module R, [10].

After that, the concept of weakly S-prime submodule was introduced as a generalization of S-prime submodules in [11]. Here, for a multiplicatively closed subset S of R, they called a submodule P of an R-module M with $(P :_R M) \cap S = \emptyset$ a weakly S-prime submodule if there exists $s \in S$ such that for $a \in R$ and $m \in M$, if $0 \neq am \in P$ then either $sa \in (P :_R M)$ or $sm \in P$. In particular, a proper ideal I of R disjoint with S is said to be weakly S-prime if there exists $s \in S$ such that for $a, b \in R$ and $0 \neq ab \in I$ then either $sa \in I$ or $sb \in I$ [3].

One of the important generalizations of prime submodule is the concept of 2-absorbing submodule. In 2011, Darani and Soheilnia [6] introduced the concepts of 2-absorbing and weakly 2-absorbing submodules of modules over commutative rings with identities. A proper submodule P of a module M over a commutative ring R with identity is said be a 2-absorbing submodule (weakly 2-absorbing submodule) of M if whenever $a, b \in R$ and $m \in M$ with $abm \in P(0 \neq abm \in P)$, then $abM \subseteq P$ or $am \in P$ or $bm \in P$. Predictably, a proper ideal I of R is 2-absorbing ideal if and only if I is a 2-absorbing submodule of R-module R.

Recently, the concept of S-2-absorbing submodules was introduced in [16] which is a generalization of S-prime submodules and 2-absorbing submodules. A submodule P of M is said to be an S-2-absorbing submodule if $(P :_R M) \cap S = \emptyset$ and there exists a fixed $s \in S$ such that for $a, b \in R$ and $m \in M$, if $abm \in P$ then either $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. In particular, an ideal I of R is an S-2-absorbing ideal if I is an S-2-absorbing submodule of R-module R.

Our objective in this paper is to define and study the concept of weakly S-2-absorbing submodule as an extension of the above concepts. A submodule P of M is said to be a weakly S-2-absorbing submodule if $(P :_R M) \cap S = \emptyset$ and there exists an element $s \in S$ such that for $a, b \in R$ and $m \in M$, if $0 \neq abm \in P$ then either $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. In this case, we say that P is associated to s. In particular, an ideal I of R is a weakly S-2-absorbing ideal if I is a weakly S-2-absorbing submodule of R-module R.

Some characterizations of weakly S-2-absorbing submodules are obtained. Besides, we investigate relationships between S-2-absorbing submodule and weakly S-2-absorbing submodule and also between weakly S-prime and weakly S-2-absorbing submodules of modules over commutative rings.

2. Characterizations of weakly S-2-absorbing submodules

We start with the definitions and relationships of the main concepts of the paper.

Definition 2.1. *Let S be a multiplicatively closed subset of R. A submodule P of an R-module M is called a weakly S-2-absorbing submodule if $(P :_R M) \cap S = \emptyset$ and there exists an element $s \in S$ such that, whenever $a, b \in R$ and $m \in M$, $0 \neq abm \in P$ implies $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. In this case, we say that P is associated to s. In particular, an ideal I of R is a weakly S-2-absorbing ideal if I is a weakly S-2-absorbing submodule of R-module R*

Example 2.1. *Consider the Z-module $M = Z \times Z_6$ and let $P = 2Z\times <\bar{3}>$. Then P is a weakly S-2-absorbing submodule of M where $S = \{2^n : n \in \mathbb{N} \cup \{0\}\}$. Indeed, let $(0, \bar{0}) \neq r_1 r_2(r', m) \in P$ for r_1,r_2,$r' \in Z$ and $m \in Z_6$ such that $2r_1r_2 \notin (P : M) = 6Z$. Then $r_1r_2m \in <\bar{3}>$ with r_1, $r_2 \notin 3Z$ and so $m \in <\bar{3}>$ also $r' \in 2Z$. Thus, $2r_1(r', m) \in P$ as needed.*

Example 2.2. *Consider the submodule $P =< 6 >$ of the Z-module Z and the multiplicatively closed subset $S = \{5^n : n \in \mathbb{N} \cup \{0\}\}$. Then P is a weakly S-2-absorbing submodule.*

It is clear that every S-2-absorbing submodule is a weakly S-2-absorbing submodule. Since the zero submodule is (by definition) a weakly S-2-absorbing submodule of any R-module, hence the converse is not true in general and the following example shows this.

Example 2.3. *Consider $R = Z$, $M = Z/30Z$, $P = 0$ and $S = Z - \{0\}$. Then $2.3(5 + 30Z) = 0 \in P$ while $1.2.3 \notin (P : M)$, $1.2(5 + 30Z) \notin P$ and $1.3(5 + 30Z) \notin P$. Therefore P is not S-2-absorbing while it is weakly S-2-absorbing.*

Every weakly 2-absorbing submodule P of an R-module M satisfying $(P : M) \cap S = \emptyset$ is a weakly S-2-absorbing submodule of M and the two concepts coincide if $S \subseteq U(R)$ where $U(R)$ denotes the set of units in R. The following example shows that the converse need not be true.

Example 2.4. *Suppose that $M = Z \times Z$ is an $R = Z \times Z$-module and $P = pZ \times \{0\}$ is a submodule of M where p is prime. Then P is weakly S-2-absorbing submodule of M where $S = Z - \{0\} \times \{0\}$. Indeed, let $(0, 0) \neq (r_1, r_2)(r_3, r_4)(m_1, m_2) \in P$ for (r_1, r_2), $(r_3, r_4) \in Z \times Z$ and $(m_1, m_2) \in M$ such that $s(r_1, r_2)(r_3, r_4) \notin (P : M) = 0$. Then either r_1 or r_3 or m_1 must be p and either r_2 or r_4 or m_2 must be 0. Thus $s(p, r_2)(m_1, m_2) \in P$ as needed.*

On the other hand, P is not a weakly 2-absorbing submodule since $(0, 0) \neq (p, 1)(1, 0)(1, 1) \in P$ but neither $(p, 1)(1, 0) \in (P : M)$ nor $(p, 1)(1, 1) \in P$ nor $(1, 0)(1, 1) \in P$. Hence P is not weakly 2-absorbing.

Lemma 2.1. *Let S be a multiplicatively closed subset of R and P be a submodule of M. If P is weakly S-prime, then there exists an element $s \in S$ of P such that $0 \neq abm \in P$ for all a, $b \in R$ and $m \in M$ implies $sbM \subseteq P$ whenever $sam \notin P$.*

Proof. Let a, $b \in R$ and $m \in M$. Assume that $0 \neq abm \in P$. Then $0 \neq b(am) \in P$. Since P is weakly S-prime, there exists $s \in S$ of P such that $sb \in (P : M)$ or $sam \in P$. Hence if $sam \notin P$, then we get $sbM \subseteq P$.

Proposition 2.1. *Let S be a multiplicatively closed subset of R and P be a submodule of M. If P is weakly S-prime, then it is weakly S-2-absorbing.*

Proof. Let a, $b \in R$ and $m \in M$ be such that $0 \neq abm \in P$. Since P is weakly S-prime, there exists $s \in S$ of P such that $sa \in (P : M)$ or $sbm \in P$. If $sbm \in P$, then we are done. Suppose

$sbm \notin P$, then by Lemma2.1, we get $saM \subseteq P$ and consequently $sabM \subseteq P$. Hence P is weakly S-2-absorbing.

The converse of the previous proposition need not be true, is illustrated in the following example.

Example 2.5. *Suppose that $M = Z \times Z$ is an $R = Z \times Z$-module and $P = 2Z \times \{0\}$ is a submodule of M. Then P is weakly S-2-absorbing where $S = (2Z + 1) \times \{0\}$. Indeed, let $(0, 0) \neq (r_1, r_2)(r_3, r_4)(m_1, m_2) \in P$ for (r_1, r_2), $(r_3, r_4) \in Z \times Z$ and $(m_1, m_2) \in M$ such that $s(r_1, r_2)(r_3, r_4) \notin (P : M) = 0$. Then either r_1 or r_3 or m_1 must be in $2Z$. Without loss of generality, assume that $r_1 \in 2Z$. Then $s(r_1, r_2)(m_1, m_2) \in 2Z \times \{0\}$ as needed. On the other hand, we have $(0, 0) \neq (2, 0)(1, 1) \in P$. Now neither $s(2, 0) \in (P : M)$ nor $s(1, 1) \in P$. Hence P is not weakly S-prime.*

Let R be a ring and $S \subseteq R$ a multiplicatively closed subset of R. The saturation S^* of S is defined as $S^*=\{r \in R\colon \frac{r}{1}$ is a unit of $S^{-1}R$ $\}$. Note that S^* is a multiplicatively closed subset containing S.

Proposition 2.2. *If M is an R-module and S is a mltiplicatively closed subset of R. Then the following statements hold.*

(i) Suppose that $S_1 \subseteq S_2$ are multiplicatively closed subsets of R. If P is a weakly S_1-2-absorbing submodule and $(P : M) \cap S_2 = \emptyset$, then P is a weakly S_2-2-absorbing submodule.

(ii) A submodule P of M is a weakly S-2-absorbing submodule if and only if it is a weakly S^*-2-absorbing submodule.

(iii) If P is a weakly S-2-absorbing submodule of M, then $S^{-1}P$ is a weakly 2-absorbing submodule of $S^{-1}M$.

Proof. (i): It is clear.

(ii):Let P be weakly S-2-absorbing. Suppose $(P : M) \cap S^* \neq \emptyset$. Then we have $t \in (P : M) \cap S^*$ and this implies that $\frac{t}{1}.\frac{a}{s} = 1$ for some $a \in R$ and $s \in S$ as $\frac{t}{1}$ is a unit of $S^{-1}R$. Thus $ta = s \in S$ implies $ta \in S$ and so $(P : M) \cap S \neq \emptyset$ which is a contradiction. Hence $(P : M) \cap S^* = \emptyset$. By (i), P is a weakly S^*-2-absorbing submodule as $S \subseteq S^*$.

Conversely, let a, $b \in R$ and $m \in M$ such that $0 \neq abm \in P$. Since P is weakly S^*-2-absorbing, there exists $s'' \in S^*$ of P such that $s''ab \in (P : M)$ or $s''am \in P$ or $s''bm \in P$. Since $s'' \in S^*$, we have $\frac{s''}{1}.\frac{t}{s} = 1$ for some $t \in R$, $s \in S$. Then $s''t = s \in S$ and so $s''t \in S$. Then $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. Thus P is weakly S-2-absorbing.

(iii) Let $\frac{a}{s_1}$, $\frac{b}{s_2} \in S^{-1}R$ and $\frac{m}{s_3} \in S^{-1}M$ be such that $\frac{0_M}{S} \neq \frac{a}{s_1}\frac{b}{s_2}\frac{m}{s_3} \in S^{-1}P$. Then we get $0_M \neq sabm \in P$ for some $s \in S$. By assumption, there exists $s_4 \in S$ of P such that $s_4(sa)b \in (P : M)$ or $s_4(sa)m \in P$ or $s_4bm \in P$. Then $\frac{a}{s_1}\frac{b}{s_2} = \frac{s_4s}{s_4s}\frac{ab}{s_1s_2} \in S^{-1}(P : M) \subseteq (S^{-1}P : S^{-1}M)$ or $\frac{a}{s_1}\frac{m}{s_3} = \frac{s_4s}{s_4s}\frac{am}{s_1s_3} \in S^{-1}P$ or $\frac{b}{s_2}\frac{m}{s_3} = \frac{s_4}{s_4}\frac{bm}{s_2s_3} \in S^{-1}P$. Hence $S^{-1}P$ is weakly 2-absorbing submodule of $S^{-1}M$.

The converse of (iii) in the above proposition need not be true is shown by the following example.

Example 2.6. *Consider the Z-module $M = Q^3$ and $S = Z - \{0\}$. Let $P = \{(r_1, r_2, 0) : r_1, r_2 \in Z\}$. Note that $(P : M) = 0$ and $(P : M) \cap S = \emptyset$. If $a = 2$, $b = 3$ and $m = (\frac{1}{2}, \frac{1}{3}, 0)$, then $(0, 0, 0) \neq 2.3(\frac{1}{2}, \frac{1}{3}, 0) = (3, 2, 0) \in P$. If we take $s = 5 \in S$, then clearly $5.2.3 \notin (P : M)$, $5.2(\frac{1}{2}, \frac{1}{3}, 0) \notin P$, $5.3(\frac{1}{2}, \frac{1}{3}, 0) \notin P$. Thus P is not weakly S-2-absorbing. From the fact that $S^{-1}M$ is a vectorspace over the field $S^{-1}Z$ that is Q and the proper subspace $S^{-1}P$ is 2-absorbing [16], we have $S^{-1}P$ is a weakly 2-absorbing submodule by [6].*

Proposition 2.3. *Let S be a multiplicatively closed subset of R and M be an R-module. Then the intersection of two weakly S-prime submodule is a weakly S-2-absorbing submodule.*

Proof. Let P_1, P_2 be two weakly S-prime submodules of M and $P = P_1 \cap P_2$. Let a, $b \in R$ and $m \in M$ be such that $0 \neq abm \in P$. Since P_1 is weakly S-prime and $0 \neq a(bm) \in P_1$, there exists $s_1 \in S$ of P_1 such that $s_1 a \in (P_1 : M)$ or $s_1 bm \in P_1$. Again as P_2 is weakly S-prime and $0 \neq bam \in P_2$ there exists $s_2 \in S$ of P_2 such that $s_2 b \in (P_2 : M)$ or $s_2 am \in P_2$. Now consider the following four cases.

Case 1: $s_1 a \in (P_1 : M)$ and $s_1 bm \notin P_1$
$s_2 b \in (P_2 : M)$ and $s_2 am \notin P_2$.

Now, put $s = s_1 s_2 \in S$. Then $sab \in (P_1 : M)$ and $sab \in (P_2 : M)$ and so $sabM \subseteq P_1 \cap P_2 = P$. Hence $sab \in (P : M)$.

Case 2: $s_1 a \in (P_1 : M)$ and $s_1 bm \notin P_1$
$s_2 am \in P_2$ and $s_2 b \notin (P_2 : M)$.

Then $s_1 am \in s_1 aM \subseteq P_1$ and $s_2 am \in P_2$ implies that $sam \in P$ where $s = s_1 s_2 \in S$.

Case 3: $s_1 bm \in P_1$ and $s_1 a \notin (P_1 : M)$
$s_2 am \notin P_2$ and $s_2 b \in (P_2 : M)$

Then clearly $sbm \in P$ where $s = s_1 s_2 \in S$.

Case 4: $s_1 bm \in P_1$ and $s_1 a \notin (P_1 : M)$
$s_2 am \in P_2$ and $s_2 b \notin (P_2 : M)$

As P_1 is weakly S-prime and $0 \neq abm \in P_1$ and also $s_1 am \notin P_1$ gives that $s_1 bM \subseteq P_1$ by Lemma 2.1. For the same reason, we get $s_2 aM \subseteq P_2$. Then clearly $sab \in (P : M)$ where $s = s_1 s_2 \in S$. Hence P is weakly S-2-absorbing.

The following result provides some condition under which a weakly S-2-absorbing submodule is S-2-absorbing.

Theorem 2.1. *Let S be a multiplicatively closed subset of R and P be a weakly S-2-absorbing submodule of M. If P is not S-2-absorbing, then $(P : M)^2 P = 0$.*

Proof. By our assmption, there exists $s \in S$ of P such that, whenever x, $y \in R$ and $m \in M$, $0 \neq xym \in P$ implies $sxy \in (P : M)$ or $sxm \in P$ or $sym \in P$. Suppose $(P : M)^2 P \neq 0$, we claim that P is S-2-absorbing. Let a, $b \in R$ and $m \in M$ be such that $abm \in P$. If $abm \neq 0$, then

$sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. So assume that $abm = 0$.

Now, first we assume that $abP \neq 0$. Then $abp_0 \neq 0$ for some $p_0 \in P$ implies $0 \neq abp_0 = ab(m + p_0) \in P$. Then $sab \in (P : M)$ or $sa(m + p_0) \in P$ or $sb(m + p_0) \in P$ by our assumption. Hence $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. Hence we may assume that $abP = 0$.

If $a(P : M)m \neq 0$, then $aq_0m \neq 0$ for some $q_0 \in (P : M)$. Then $0 \neq aq_0m = a(b+q_0)m \in P$. Then, we get $sa(b+q_0) \in (P : M)$ or $sam \in P$ or $s(b+q_0)m \in P$. Hence $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. So we can assume that $a(P : M)m = 0$. In the same manner, we can assume that $b(P : M)m = 0$. Since $(P : M)^2P \neq 0$, there exists x_0, $y_0 \in (P : M)$ and $m_0 \in P$ with $x_0y_0m_0 \neq 0$.

If $ay_0m_0 \neq 0$, then $0 \neq ay_0m_0 = a(b+y_0)(m+m_0) \in P$ since $abm = 0$, $abm_0 \in abP = 0$ and $ay_0m = amy_0 \in am(P : M) = 0$. Hence, by our assumption $sa(b+y_0) \in (P : M)$ or $sa(m+m_0) \in P$ or $s(b + y_0)(m + m_0) \in P$ and so $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. So we can assume that $ay_0m_0 = 0$. In the same manner, we can assume that $x_0y_0m = 0$ and $x_0bm_0 = 0$.

Since $x_0y_0m_0 \neq 0$, we have $0 \neq x_0y_0m_0 = (a + x_0)(b + y_0)(m + m_0) \in P$ since $abm = 0$, $abm_0 \in abP = 0$ and $ay_0m = amy_0 \in am(P : M) = 0$. Then, $s(a + x_0)(b + y_0) \in (P : M)$ or $s(a + x_0)(m + m_0) \in P$ or $s(b + y_0)(m + m_0) \in P$. Hence $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. Hence P is S-2-absorbing.

Recall that an R-module M is said to be a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. If N_1, N_2 are two submodules of M, then $N_1 = AM$ and $N_2 = BM$ for some ideals A, B of R. The product of N_1 and N_2 is defined as $N_1N_2 = ABM$ [4]. Also note that this product is independent of the presentations of submodules N_1 and N_2 of M [4, Theorem 3.4]. A submodule N of an R-module M is called a nilpotent submodule if $(N : M)^kN = 0$ for some positive integer k [1].

Corollary 2.1. *Let S be a multiplicatively closed subset of R and P be a submodule of M. Assume that P is a weakly S-2-absorbing submodule of M that is not S-2-absorbing, then*

1, P is nilpotent.

2, If M is a multiplication module, then $P^3 = 0$.

Proof. 1. Immediate from the definition of nilpotent submodule and by Theorem 2.1.

2. By Theorem 2.1, $(P : M)^2P = 0$. Then $(P : M)^3M = (P : M)^2(P : M)M = 0$. Thus $P^3 = 0$.

If N is a proper submodule of a non-zero R-module M. Then the M-radical of N, denoted by M-rad$N$ is defined as the intersection of all prime submodules of M containing N [12], [8]. If A is an ideal of the ring R then the M-radical of A (considered as a submodule of the R-module R) is denoted by $\sqrt{A}$ and consists of all elements r of R such that $r^n \in A$ for some positive integer n [8]. Also it is shown in [8, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then M-rad$N = (\sqrt{(N : M)})M$.

Proposition 2.4. *Assume that M is a faithful multiplication R-module, S is a multiplicatively closed subset of R and P is a submodule of M. Let P be a weakly S-2-absorbing submodule of M. If P is not S-2-absorbing, then $P \subseteq M$-rad0.*

Proof. Suppose P is not S-2-absorbing. By Theorem 2.1, $(P : M)^2P = 0$. Since $(P : M)^2(P : M)M \subseteq (P : M)^2P$, we have $(P : M)^3 \subseteq ((P : M)^2P : M) = (0 : M) = 0$. Let $a \in (P : M)$, then $a^3 = 0$ and so $a \in \sqrt{0}$. Thus $(P : M) \subseteq \sqrt{0}$. Hence $P = (P : M)M \subseteq \sqrt{0}M = M$-rad0.

Proposition 2.5. *If S is a multiplicatively closed subset of R and P is a submodule of a cyclic faithful R-module M, then P is a weakly S-2-absorbing submodule of M if and only if $(P : M)$ is a weakly S-2-absorbing ideal of R.*

Proof. Let P be a weakly S-2-absorbing submodule of M. Assume that $M = Rm$ for some $m \in M$ and let $0 \neq abc \in (P : M)$ for some $a, b, c \in R$. Then $abcm \in P$. If $abcm \neq 0$, then their exists an element $s \in S$ of P such that $sab \in (P : M)$ or $sacm \in P$ or $sbcm \in P$. If $sab \in (P : M)$, then we are done. If $sacm \in P$, then $sac \in (P : m) = (P : M)$ as M is cyclic. Likewise, if $sbcm \in P$, then $sbc \in (P : M)$. Then, assume that $abcm = 0$, we get $abc \in (0 : m) = (0 : M)$. As M is faithful, we have $abc = 0$, a contradiction. Hence $(P : M)$ is a weakly S-2-absorbing ideal of R.

Conversely, let $0 \neq abm' \in P$ for some $a, b \in R$ and $m' \in M$. Then $m' = cm$ for some $c \in R$ and we get $0 \neq abcm \in P$. This implies $abc \in (P : m) = (P : M)$. If $abc \neq 0$, then there exists an element $s' \in S$ of $(P : M)$ such that $s'ab \in (P : M)$ or $s'bc \in (P : M)$ or $s'ac \in (P : M)$. If $s'ab \in (P : M)$, then we are done. If $s'bc \in (P : M)$, then $s'bc \in (P : m)$ and so $s'bm' \in P$. Likewise if $s'ac \in (P : M)$, then $s'am' \in P$. Now, assume that $abc = 0$, then $abcm = 0.m = 0$, a contradiction. Hence P is weakly S-2-absorbing.

Proposition 2.6. *If S is a multiplicatively closed subset of R and P is a submodule of a cyclic R-module M, then P is an S-2-absorbing submodule of M if and only if $(P : M)$ is an S-2-absorbing ideal of R.*

After recalling the concepts of triple-zero in various papers like [9], [7], we give the following result which is an analogue of [9, Theorem 3.10].

Theorem 2.2. *Let S be a multiplicatively closed subset of R and let P be a weakly S-2-absorbing submodule of M. If $a, b \in R$, $m \in M$ with $abm = 0$ and $sab \notin (P : M)$, $sam \notin P$, $sbm \notin P$ for any $s \in S$, then*

(1) $abP = a(P : M)m = b(P : M)m = 0$

(2) $a(P : M)P = b(P : M)P = (P : M)^2m = 0$

Proof. (1). If $abP \neq 0$, then for some $p \in P$, $abp \neq 0$. Since $0 \neq abp = ab(m + p) \in P$, then by assumption there exists $s \in S$ of P such that $sab \in (P : M)$ or $sa(m + p) \in P$ or $sb(m + p) \in P$.

Hence $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$, which is not possible by our assumption. Hence $abP = 0$.

If $a(P : M)m \neq 0$, then for some $r \in (P : M)$, $arm \neq 0$. Since $0 \neq arm = a(r + b)m \in P$, then there exists $s \in S$ of P such that $sa(r + b) \in (P : M)$ or $sam \in P$ or $s(r + b)m \in P$. That is $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$, which is not possible by our assumption. Thus $a(P : M)m = 0$. The similar argument prove that $b(P : M)m = 0$.

(2). Assume that $a(P : M)P \neq 0$. Then for some $r \in (P : M)$, $p \in P$, $0 \neq arp \in P$. As $0 \neq arp = a(b + r)(m + p)$. By (1), we get $0 \neq a(b + r)(m + p) \in P$, then there exists $s \in S$ of P such that $sa(b + r) \in (P : M)$ or $sa(m + p) \in P$ or $s(b + r)(m + p) \in P$. Hence $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$, a contradiction by our assumption. Hence $a(P : M)P = 0$.

Now, if $(P : M)^2m \neq 0$, then for some r_1, $r_2 \in (P : M)$, $0 \neq r_1r_2m \in P$. Since by (1), $0 \neq r_1r_2m = (a+r_1)(b+r_2)m \in P$, then there exists $s \in S$ of P such that $s(a+r_1)(b+r_2) \in (P : M)$ or $s(a + r_1)m \in P$ or $s(b + r_2)m \in P$ and so $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$, a contradiction by our assumption. Hence $(P : M)^2m = 0$.

We recall that if N is a submodule of an R-module M and A is an ideal of R, then the residual of N by A is the set $(N :_M A) = \{m \in M : Am \subseteq N\}$. It is clear that $(N :_M A)$ is a submodule of M containing N. More generally, for any subset $B \subseteq R$, $(N :_M B)$ is a submodule of M containing N.

Proposition 2.7. *Let S be a multiplicatively closed subset of R. For a submodule P of an R-module M with $(P : M) \cap S = \emptyset$, the following assertions are equivalent.*

(1) P is a weakly S-2-absorbing submodule of M.

(2) For any a, $b \in R$, there exists $s \in S$ such that, if $sabM \nsubseteq P$, then $(P : ab) = (0 : ab)$ or $(P : ab) \subseteq (P : sa)$ or $(P : ab) \subseteq (P : sb)$

(3) For any a, $b \in R$ and for any submodule K of M, there exists $s \in S$ such that, if $0 \neq abK \subseteq P$ then $sab \in (P : M)$ or $saK \subseteq P$ or $sbK \subseteq P$.

Proof. (1) $\Longrightarrow$ (2) Let a, $b \in R$. Let $m \in (P : ab)$. If $abm = 0$, then clearly $m \in (0 : ab)$. If $abm \neq 0$, that is if $0 \neq abm \in P$, then by (1), there exist $s \in S$ of P such that $sab \in (P : M)$ or $sam \in P$ or $sbm \in P$. Clearly, if $sabM \nsubseteq P$, we conclude that either $sam \in P$ or $sbm \in P$. As $(0 : ab) \subseteq (P : ab)$, we get $(P : ab) = (0 : ab)$ or $(P : ab) \subseteq (P : sa)$ or $(P : ab) \subseteq (P : sb)$.

(2) $\Longrightarrow$ (3) Let a, $b \in R$ and K be a submodule of M such that $0 \neq abK \subseteq P$ and, for the element $s \in S$ of (2), we have to claim that $sab \in (P : M)$ or $saK \subseteq P$ or $sbK \subseteq P$. If $sab \in (P : M)$, then there is nothing to prove. Suppose $sab \notin (P : M)$. As $abK \subseteq P$, we have $K \subseteq (P : ab)$ and by (2), we have $K \subseteq (0 : ab)$ or $K \subseteq (P : sa)$ or $K \subseteq (P : sb)$. If $K \subseteq (0 : ab)$, then $abK = 0$, a contradiction. If $K \subseteq (P : sa)$, then $saK \subseteq P$ as required.

(3) $\Longrightarrow$ (1) Let a, $b \in R$ and $m \in M$ with $0 \neq abm \in P$. Clearly $ab < m > \subseteq P$. If

$ab < m > \neq 0$, by (3), $sab \in (P : M)$ or $sam \in sa < m > \subseteq P$ or $sbm \in sb < m > \subseteq P$. If $ab < m >= 0$, then $abm \in ab < m >= 0$, a contradiction.

Theorem 2.3. *Let S be a multiplicatively closed subset of R and P be a submodule of an R-module M. If P is a weakly S-2-absorbing submodule of M. Then*

(1) There exists an $s \in S$ such that for any a, $b \in R$, if $abK \subseteq P$ and $0 \neq 2abK$ for some submodule K of M, then $sab \in (P : M)$ or $saK \subseteq P$ or $sbK \subseteq P$.

(2) There exists an $s \in S$ such that for an ideal I of R and a submodule K of M, if $aIK \subseteq P$ and $0 \neq 4aIK$, where $a \in R$, then $saI \in (P : M)$ or $saK \subseteq P$ or $sIK \subseteq P$.

(3) There exists an $s \in S$ such that for all ideals I, J of R and submodule K of M, if $0 \neq IJK \subseteq P$ and $0 \neq 8(IJ + (I + J)(P : M))(K + P)$, then $sIJ \subseteq (P : M)$ or $sIK \subseteq P$ or $sJK \subseteq P$. In particular this holds if the group $(M, +)$ has no elements of order 2.

Proof. (1) By our assmption, there exists $s \in S$ of P such that, whenever x, $y \in R$ and $m \in M$, $0 \neq xym \in P$ implies $sxy \in (P : M)$ or $sxm \in P$ or $sym \in P$. Let a, $b \in R$ such that $abK \subseteq P$ and $0 \neq 2abK$ for some submodule K of M. Now, we will show that $sab \in (P : M)$ or $saK \subseteq P$ or $sbK \subseteq P$. Suppose $sab \notin (P : M)$. Then proving that $saK \subseteq P$ or $sbK \subseteq P$ is enough. Let k be an arbitrary element of K. As $abk \in abK \subseteq P$, if $abk \neq 0$, then $sab \in (P : M)$ or $sak \in P$ or $sbk \in P$. Thus we have $k \in (P : sa)$ or $k \in (P : sb)$ since $sab \notin (P : M)$. Hence $saK \subseteq P$ or $sbK \subseteq P$.

If $abk = 0$. Since $0 \neq 2abK$, for some $k_1 \in K$, we get $0 \neq 2abk_1$ and clearly $0 \neq abk_1 \in P$. Then we get $sak_1 \in P$ or $sbk_1 \in P$ since $sab \notin (P : M)$. Put $k_2 = k + k_1$ and so $0 \neq abk_2 \in P$. Then $sak_2 \in P$ or $sbk_2 \in P$ since $sab \notin (P : M)$. This leads to the following cases.

Case 1: $sak_1 \in P$ and $sbk_1 \in P$

Since $sak_2 \in P$ or $sbk_2 \in P$, we have $sak \in P$ or $sbk \in P$. Thus $saK \in P$ or $sbK \in P$.

Case 2: $sak_1 \in P$ and $sbk_1 \notin P$

Suppose $sak \notin P$ and $sbk \notin P$. Then $sak_2 = sak_1 + sak \notin P$ and so $sbk_2 \in P$. Hence $sa(k_2 + k_1) \notin P$ and similarly $sb(k_2 + k_1) \notin P$. As P is weakly S-2-absorbing and $sab \notin (P : M)$, hence $ab(k_2 + k_1) = 0$. But $ab(k_2 + k_1) = ab(k_1 + k + k_1) = 2abk_1$, a contradiction as $2abk_1 \neq 0$. Thus $sak \in P$ or $sbk \in P$ and so $saK \subseteq P$ or $sbK \subseteq P$.

Case 3: $sak_1 \notin P$ and $sbk_1 \in P$

The proof is same as that of Case 2.

(2) By our assmption, there exists $s \in S$ of P such that, whenever x, $y \in R$ and $m \in M$, $0 \neq xym \in P$ implies $sxy \in (P : M)$ or $sxm \in P$ or $sym \in P$. Let I be an ideal of R and K be a submodule of M such that $aIK \subseteq P$ and $0 \neq 4aIK$, where $a \in R$. We have to prove that $saI \in (P : M)$ or $saK \subseteq P$ or $sIK \subseteq P$. Suppose $saI \nsubseteq (P : M)$, for some $i \in I$ we have $sai \notin (P : M)$. Let us first prove that there exists $b \in I$ such that $0 \neq 4abK$ and $sab \notin (P : M)$.

Since $0 \neq 4aIK$, for some $i' \in I$, $0 \neq 4ai'K$. Suppose $sai' \notin (P : M)$ or $0 \neq 4aiK$, if we put $b = i'$, we get $sab \notin (P : M)$ and $0 \neq 4abK$ and if we put $b = i$, we get $0 \neq 4abK$ and

$sab \notin (P:M)$. From the above, clearly by putting $b = i'$ or $b = i$, we get the result. Hence assume that $sai' \in (P:M)$ and $4aiK = 0$. Hence $0 \neq 4a(i+i')K \subseteq P$ and $sa(i+i') \notin (P:M)$. Thus we find $b \in I$ such that $0 \neq 4abK$ and $sab \notin (P:M)$.

As $0 \neq 4abK$, we get $0 \neq 2abK$ and by (1), since $abK \subseteq aIK \subseteq P$ and $sab \notin (P:M)$, we get $saK \subseteq P$ or $sbK \subseteq P$. If $saK \subseteq P$, there we are done. Thus assume that $saK \nsubseteq P$ and so $sbK \subseteq P$.

Now to exhibit that $saI \in (P:M)$ or $sIK \subseteq P$. Let $i" \in I$. If $2ai"K \neq 0$, then by (1), $sai" \in (P:M)$ or $si"K \subseteq P$ since $saK \nsubseteq P$. Thus we get $i" \in ((P:M):sa)$ or $i" \in (P:sK)$. Therefore $I \subseteq ((P:M):sa)$ or $I \subseteq (P:sK)$. Then we are done.

If $2ai"K = 0$, then clearly $0 \neq 2a(b+i")K$ and $a(b+i")K \subseteq P$, by (1) $sa(b+i") \in (P:M)$ or $s(b+i")K \subseteq P$ since $saK \nsubseteq P$, $(b+i") \in (P:sK)$ or $(b+i") \in ((P:M):sa)$.

(i): If $(b+i") \in (P:sK)$, then $si"K \subseteq P$ as $sbK \subseteq P$. Hence $i" \in (P:sK)$.

(ii): Now assume $(b+i") \in ((P:M):sa)$ and $(b+i") \notin (P:sK)$. Consider $0 \neq 4abK = 2a(b+i"+b)K$ and $a(b+i"+b)K \subseteq P$. By (1), $sa(b+i"+b) \in (P:M)$ or $s(b+i"+b)K \subseteq P$ since $saK \nsubseteq P$. As $sab \notin (P:M)$, we have $sa(b+i"+b) \notin (P:M)$. Then we have $s(b+i"+b)K \subseteq P$. Since $(b+i") \notin (P:sK)$, we have $s(b+i"+b)K \nsubseteq P$. Therefore $(b+i") \in (P:sK)$. Since $sbK \subseteq P$, we have $si"K \subseteq P$ and so $i" \in (P:sK)$. Consequently $I \subseteq ((P:M):sa)$ or $I \subseteq (P:sK)$ and hence as $saI \nsubseteq (P:M)$, we get $sIK \subseteq P$.

(3) Let I, J be the ideals of R and K be a submodule of M such that $0 \neq IJK \subseteq P$ and $0 \neq 8(IJ+(I+J)(P:M))(K+P)$. Since $0 \neq 8(IJ+(I+J)(P:M))(K+P) = 8IJK + 8I(P:M)K + 8J(P:M)K + 8IJP + 8I(P:M)P + 8J(P:M)P$. As a result, one of the types listed below has been satisfied.

Type 1: $0 \neq 8IJK$. Then for some $j \in J$, $0 \neq 8jIK$ and so $0 \neq 4jIK$. As $jIK \subseteq P$, by (2), there exists $s \in S$ such that $sjI \subseteq (P:M)$ or $sIK \subseteq P$ or $sjK \subseteq P$. If $sIK \subseteq P$, then we are done and so assume that $sIK \nsubseteq P$ that is $sjI \subseteq (P:M)$ or $sjK \subseteq P$. We claim that $sIJ \subseteq (P:M)$ or $sJK \subseteq P$. Let $j' \in J$ be an arbitrary element. If $0 \neq 4j'IK$, by (2), $sj'I \subseteq (P:M)$ or $sj'K \subseteq P$ since $sIK \nsubseteq P$.Then $j' \in ((P:M):sI)$ or $j' \in (P:sK)$. Hence we get the result.

Now let $4j'IK = 0$. As $0 \neq 4(j+j')IK \subseteq P$, by (2), $s(j+j')I \subseteq (P:M)$ or $s(j+j')K \subseteq P$ since $sIK \nsubseteq P$. Hence we get $s(j+j')I \subseteq (P:M)$ or $s(j+j')K \subseteq P$. Thereby we get the four cases.

Case 1: $sjI \subseteq (P:M)$ and $s(j+j')I \subseteq (P:M)$.

Hence we get $sj'I \subseteq (P:M)$, that is $sIJ \subseteq (P:M)$

Case 2: $sjK \subseteq P$ and $s(j+j')K \subseteq P$

Hence we get $sj'K \subseteq P$, that is $sJK \subseteq P$

Case 3: $sjI \subseteq (P:M)$ and $sjK \nsubseteq P$.

$s(j+j')K \subseteq P$ and $s(j+j')I \nsubseteq (P:M)$.

This can be represented as $j \in ((P:M):sI)$ and $j \notin (P:sK)$, $j+j' \in (P:sK)$ and

$j+j' \notin ((P:M):sI)$. Hence $j+j'+j \notin ((P:M):sI)$ and $j+j'+j \notin (P:sK)$. Now consider $0 \neq 8jIK = 4(j+j'+j)IK$ and by (2), $s(j+j'+j)I \subseteq (P:M)$ or $s(j+j'+j)K \subseteq P$ since $sIK \nsubseteq P$. Hence we get $j+j'+j \in ((P:M):sI)$ or $j+j'+j \in (P:sK)$ and this is not possible. Therefore, since $j \in ((P:M):sI)$ or $j \in (P:sK)$ and $j+j' \in (P:sK)$ or $j+j' \in ((P:M):sI)$, there must be any one of the following holds.

(i) $j \in (P:sK)$ and $j+j' \in (P:sK)$ and $j+j' \notin ((P:M):sI)$, then $j' \in (P:sK)$.

(ii) $j \in ((P:M):sI)$ and $j \notin (P:sK)$ and $j+j' \in ((P:M):sI)$, then $j' \in ((P:M):sI)$.

Case 4: $s(j+j')I \subseteq (P:M)$ and $s(j+j')K \nsubseteq P$

$sjK \subseteq P$ and $sjI \nsubseteq (P:M)$.

Similar to the above case, we have $j' \in ((P:M):sI)$ or $j' \in (P:sK)$. Thus $sIJ \subseteq (P:M)$ or $sJK \subseteq P$.

Type 2: If $0 \neq 8IJP$ and $8IJK = 0$, then $0 \neq 8IJ(K+P) \subseteq P$ and by Type 1, $sIJ \subseteq (P:M)$ or $sJ(K+P) \subseteq P$ or $sI(K+P) \subseteq P$ and so $sIJ \subseteq (P:M)$ or $sJK \subseteq P$ or $sIK \subseteq P$.

Type 3: If $0 \neq 8J(P:M)K$ and $8IJK = 0$, then $0 \neq 8J(P:M)K = 8J(I+(P:M))K$ and so by Type 1, $sJ(I+(P:M)) \subseteq (P:M)$ or $sJK \subseteq P$ or $s(I+(P:M))K \subseteq P$. Hence $sIJ \subseteq (P:M)$ or $sJK \subseteq P$ or $sIK \subseteq P$. Likewise if $0 \neq 8I(P:M)K$, we get the result.

Type 4: If $0 \neq 8J(P:M)P$ and $8IJK = 8IJP = 8J(P:M)K = 8I(P:M)K = 0$. Then $0 \neq 8J(P:M)P = 8J(I+(P:M))(K+P)$ and by Type 1, $sJ(I+(P:M)) \subseteq (P:M)$ or $sJ(K+P) \subseteq P$ or $s(I+(P:M))(K+P) \subseteq P$. Hence $sIJ \subseteq (P:M)$ or $sJK \subseteq P$ or $sIK \subseteq P$. Likewise if $0 \neq 8I(P:M)P$, we have the result.

To prove the particular case, let $(M,+)$ be a group having no subgroups of order 2. We have to show that $0 \neq 8IJK$. If this happens, We get the result by Type 1. Suppose $8IJK = 0$. Let $0 \neq a \in IJK$. As $8a = 0$, so the group $(M,+)$ has a subgroup of order 2, 4 or 8, which is a contradiction.

Corollary 2.2. *Let S be a multiplicatively closed subset of R and I be a weakly S-2-absorbing ideal of R.*

(1) There exists $s \in S$ such that for any a, $b \in R$ and for any ideal A of R, if $abA \subseteq I$ and $0 \neq 2abA$, then $sab \in I$ or $saA \subseteq I$ or $sbA \subseteq I$.

(2) There exists $s \in S$ such that for any $a \in R$, ideals A, B of R, if $aAB \subseteq I$ and $0 \neq 4aAB$, then $saA \subseteq I$ or $saB \subseteq I$ or $sAB \subseteq I$.

(3) There exists $s \in S$ such that for any ideals A, B, C of R, if $0 \neq ABC \subseteq I$ and $0 \neq 8(AB(C+I)+AC(B+I)+BC(A+I)+AI(B+C)+BI(A+C)+CI(A+B)+I^2(A+B+C))$, then $sAB \subseteq I$ or $sBC \subseteq I$ or $sAC \subseteq I$. In particular, this happens when the group $(R,+)$ has no elements of order 2.

Proposition 2.8. *Let ϕ: $M_1 \to M_2$ be a module homomorphism where M_1 and M_2 are R-modules and S be a multiplicatively closed subset of R. Then the following holds.*

1. If ϕ is a monomorphism and K is a weakly S-2-absorbing submodule of M_2 with $(\phi^{-1}(K):$

$M_1) \cap S = \emptyset$, then $\phi^{-1}(K)$ is a weakly S-2-absorbing submodule of M_1.

2. If ϕ is an epimorphism and P is a weakly S-2-absorbing submodule of M_1 containing $\ker\phi$, then $\phi(P)$ is a weakly S-2-absorbing submodule of M_2.

Proof. 1. Let $a, b \in R$ and $m_1 \in M_1$ be such that $0 \neq abm_1 \in \phi^{-1}(K)$. Then $0 \neq \phi(abm_1) = ab\phi(m_1) \in K$ as ϕ is a monomorphism. since K is weakly S-2-absorbing, there exists $s \in S$ such that $sab \in (K : M_2)$ or $sa\phi(m_1) \in K$ or $sb\phi(m_1) \in K$. If $sab \in (K : M_2)$, then $sab \in (\phi^{-1}(K) : M_1)$ since $(K : M_2) \subseteq (\phi^{-1}(K) : M_1)$ and if $sa\phi(m_1) \in K$ or $sb\phi(m_1) \in K$, we have $\phi(sam_1) \in K$ implies $sam_1 \in \phi^{-1}(K)$ or $\phi(sbm_1) \in K$ implies $sbm_1 \in \phi^{-1}(K)$. Hence $\phi^{-1}(K)$ is a weakly S-2-absorbing submodule of M_1.

2. First observe that $(\phi(P) : M_2) \cap S = \emptyset$. Indeed, assume that $s' \in (\phi(P) : M_2) \cap S$. Then $\phi(s'M_1) = s'\phi(M_1) = s'M_2 \subseteq \phi(P)$ and so $s'M_1 \subseteq P$ as $\ker\phi \subseteq P$. This shows that $s' \in (P : M_1)$ and so $(P : M_1) \cap S \neq \emptyset$, a contradiction occurs since P is a weakly S-2-absorbing submodule of M_1. Now, let $a, b \in R$ and $m_2 \in M_2$ be such that $0 \neq abm_2 \in \phi(P)$. As we can write $m_2 = \phi(m_1)$ for some $m_1 \in M_1$ and so $0 \neq abm_2 = ab(\phi(m_1)) = \phi(abm_1) \in \phi(P)$. Since $\ker\phi \subseteq P$, we have $0 \neq abm_1 \in P$. Then there exists $s \in S$ such that $sab \in (P : M_1)$ or $sam_1 \in P$ or $sbm_1 \in P$. Consequently we get $sab \in (\phi(P) : M_2)$ or $\phi(sam_1) = sa\phi(m_1) = sam_2 \in \phi(P)$ or $\phi(sbm_1) = sb\phi(m_1) = sbm_2 \in \phi(P)$. Hence $\phi(P)$ is weakly S-2-absorbing submodule of M_2.

Corollary 2.3. *Let S be a multiplicatively closed subset of R. P_1 and P_2 are two submodules of M with $P_2 \subseteq P_1$.*

1. If K is a weakly S-2-absorbing submodule of M with $(K : P_1) \cap S = \emptyset$, then $K \cap P_1$ is a weakly S-2-absorbing submodule of P_1.

2. If P_1 is a weakly S-2-absorbing submodule of M, then P_1/P_2 is a weakly S-2-absorbing submodule of M/P_2.

3. If P_1/P_2 is a weakly S-2-absorbing submodule of M/P_2 and P_2 is a weakly S-2-absorbing submodule of M, then P_1 is a weakly S-2-absorbing submodule of M.

Proof. 1. Consider the injection $i\colon P_1 \to M$ defined by $i(p_1) = p_1$ for all $p_1 \in P_1$. We have to show that $(i^{-1}(K) : P_1) \cap S = \emptyset$. Indeed, if $s \in (i^{-1}(K) : P_1) \cap S$, then $sP_1 \subseteq i^{-1}(K)$. As $i^{-1}(K) = K \cap P_1$, we have $sP_1 \subseteq K \cap P_1 \subseteq K$ and so $s \in (K : P_1) \cap S$, a contradiction as K is weakly S-2-absorbing. Thus by Proposition 2.8(1), we conclude the result.

2. Consider the canonical epimorphism $\pi\colon M \to M/P_2$ defined by $\pi(m) = m + P_2$. Then $\pi(P_1) = P_1/P_2$ is a weakly S-2-absorbing submodule of M/P_2 by Proposition 2.8(2).

3. Let $a, b \in R$ and $m \in M$ be such that $0 \neq abm \in P_1$. Then $ab(m + P_2) = abm + P_2 \in P_1/P_2$. If $ab(m + P_2) \neq P_2$, then there exists $s_1 \in S$ of P_1/P_2 implies $s_1ab \in (P_1/P_2 : M/P_2)$ or $s_1a(m + P_2) \in P_1/P_2$ or $s_1b(m + P_2) \in P_1/P_2$. Hence $s_1ab \in (P_1 : M)$ or $s_1am \in P_1$ or $s_1bm \in P_1$. If $abm \in P_2$, then by assumption, there exists $s_2 \in S$ of P_2 such that $s_2ab \in (P_2 : M) \subseteq (P_1 : M)$ or

$s_2am \in P_2 \subseteq P_1$ or $s_2bm \in P_2 \subseteq P_1$. It follows that P_1 is a weakly S-2-absorbing submodule of M associated with $s = s_1s_2 \in S$.

We need to recall the following Lemma for the next result.

Lemma 2.2. *[2] For an ideal $I$of a ring R and a submodule N of a finitely generated faithful multiplication R-module M, the following hold.*

1. *$(IN :_R M) = I(N :_R M)$.*
2. *If I is finitely generated faithful multiplication, then*
   (a) $(IN :_M I) = N$.
   (b) Whenever $N \subseteq IM$, then $(JN :_M I) = J(N :_M I)$ for any ideal J of R.

Proposition 2.9. *Let I be a finitely generated faithful multiplication ideal of a ring R, S be a multiplicatively closed subset of R and P be a submodule of a finitely generated faithful multiplication cyclic R-module M.*

1. If IP is a weakly S-2-absorbing submodule of M and $(P : M) \cap S = \emptyset$, then either I is a weakly S-2-absorbing ideal of R or P is a weakly S-2-absorbing submodule of M.

2. P is a weakly S-2-absorbing submodule of IM if and only if $(P :_M I)$ is a weakly S-2-absorbing submodule of M.

Proof. (1) Suppose $P = M$, we get $I = I(P :_R M) = (IP :_R M)$ by Lemma 2.2. Since IP is a weakly S-2-absorbing submodule of M, by Proposition 2.5, I is a weakly S-2-absorbing ideal of R. Now, suppose P is a proper submodule of M. By Lemma 2.2, $(IP :_M I) = P$ and so $(P : M) = ((IP :_M I) :_R M) = (I(P :_R M) :_M I)$. Let $a, b \in R$ and $m \in M$ be such that $0 \neq abm \in P$. Since I is faithful, then $(0 :_M I) = Ann_R(I)M = 0$ [2], and so $0 \neq abIm \subseteq IP$. By Proposition 2.7, there exists $s \in S$ of IP such that $sab \in (IP : M)$ or $saIm \subseteq IP$ or $sbIm \subseteq IP$. If $sab \in (IP : M)$, then $sab \in (P : M)$. If $saIm \subseteq IP$, then $sam \in (IP : I) = P$. Likewise if $sbIm \subseteq IP$, then $sbm \in P$. Hence P is a weakly S-2-absorbing submodule of M.

(2) Suppose P is a weakly S-2-absorbing submodule of IM. Then $(P :_R IM) \cap S = ((P :_M I) :_R M) \cap S = \emptyset$. Let $a, b \in R$ and $m \in M$ be such that $0 \neq abm \in (P :_M I)$. If $abIm = 0$, then $abm \in (0 :_M I) = Ann_R(I)M = 0$, a contradiction. Hence $0 \neq abIm \subseteq P$. By Proposition 2.7, there exists $s \in S$ of P such that $sab \in (P : IM)$ or $saIm \subseteq P$ or $sbIm \subseteq P$. If $sab \in (P :_R IM)$, then $sab \in ((P :_M I) :_R M)$. If $saIm \subseteq P$, then $sam \in (P : I)$ and similarly if $sbIm \subseteq P$, we get $sbm \in (P : I)$ as required.

Conversely, suppose $(P :_M I)$ is a weakly S-2-absorbing submodule of M. Then clearly $((P :_M I) :_R M) \cap S = (P :_R IM) \cap S = \emptyset$. Let $a, b \in R$ and $x \in IM$ be such that $0 \neq abx \in P$. Clearly $ab < x > \subseteq P$. Since $x \in IM$, by Lemma 2.2, $ab(< x >:_M I) = (ab < x >:_M I) \subseteq (P :_M I)$. If $ab(< x >:_M I) = 0$, then since $abx \in (abIx :_M I)$ and $Ix \subseteq IM$, by Lemma 2.2, we have $abx \in ab(Ix :_M I) \subseteq ab(< x >:_M I) = 0$, a contradiction. So we have $0 \neq ab(< x >:_M I) \subseteq$

$(P :_M I)$. By Proposition 2.7, there exists $s' \in S$ of $(P :_M I)$ such that $s'ab \in ((P :_M I) :_R M)$ or $s'a(< x >:_M I) \subseteq (P :_M I)$ or $s'b(< x >:_M I) \subseteq (P :_M I)$. If $s'ab \in ((P :_M I) :_R M)$, then $s'ab \in (P :_R IM)$. If $s'a(< x >:_M I) \subseteq (P :_M I)$, then $Is'a(< x >:_M I) \subseteq P$. Since $s'ax \in s'a < x >= s'a(I < x >:_M I) = s'aI(< x >:_M I) \subseteq P$ by Lemma 2.2. Likewise if $s'b(< x >:_M I) \subseteq (P :_M I)$, then $s'bx \in P$. Hence P is a weakly S-2-absorbing submodule of IM.

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