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Interval-Valued Intuitionistic Fuzzy Subalgebras/Ideals of Hilbert Algebras

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Abstract. In this paper, the concept of interval-valued intuitionistic fuzzy sets to subalgebras and ideals of Hilbert algebras is introduced. The inverse image of interval-valued intuitionistic fuzzy subalgebras and interval-valued intuitionistic fuzzy ideals of Hilbert algebras is studied and some related properties are investigated. Equivalence relations on interval-valued intuitionistic fuzzy ideals are discussed.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [15]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 3, 4]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multi-criteria decision-making [6–8]. The concept of Hilbert algebras was introduced in early 50-ties by Henkin and Skolem for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Horn and Diego from algebraic point of view. Diego proved

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(cf. [11]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag (cf. [9], [10]) and Jun (cf. [13]) and some of their filters forming deductive systems were recognized. Dudek (cf. [12]) considered the fuzzification of subalgebras and deductive systems in Hilbert algebras. In this paper, the concept of interval-valued intuitionistic fuzzy sets to subalgebras and ideals of Hilbert algebras is introduced. The inverse image of interval-valued intuitionistic fuzzy subalgebras and interval-valued intuitionistic fuzzy ideals of Hilbert algebras is studied and some related properties are investigated. Equivalence relations on interval-valued intuitionistic fuzzy ideals are discussed.

2. Preliminaries

Before we begin the study, let's review the definition of Hilbert algebras, which was defined by Diego [11] in 1966.

Definition 2.1. *A Hilbert algebra is a triplet $X = (X, \cdot, 1)$, where H is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of X such that the following axioms hold:*

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

The following result was proved in [12].

Lemma 2.1. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.2. *[14] A nonempty subset I of a Hilbert algebra $X = (X, \cdot, 1)$ is called an ideal of X if*

(1) $1 \in I$,
(2) $(\forall x \in X, \forall y \in I)(x \cdot y \in I)$,
(3) $(\forall x \in X, \forall y_1, y_2 \in I)((y_1 \cdot (y_2 \cdot x)) \cdot x \in I)$.

A *fuzzy set* [15] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.3. *[12] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a fuzzy subalgebra of X if the following condition holds:*

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 2.4. *[5] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a fuzzy ideal of X if the following conditions hold:*

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,
(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

Definition 2.5. *[2] Let X be a nonempty set. An intuitionistic fuzzy set A in X is defined to be a structure*

$$A := \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}, \tag{2.1}$$

where $\mu_A : X \to [0, 1]$ is the degree of membership of x to $[0, 1]$ and $\gamma_A : X \to [0, 1]$ is the degree of non-membership of x to $[0, 1]$ such that

$$(\forall x \in X)(0 \leq \mu_A(x) + \gamma_A(x) \leq 1),$$

and the intuitionistic fuzzy set A in (2.1) is simply denoted by $A = (\mu_A, \gamma_A)$.

Let $D[0, 1]$ be the set of all closed subintervals of $[0, 1]$. Consider two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$, then

$$\text{rmin}\{D_1, D_2\} = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$$

and

$$\text{rmax}\{D_1, D_2\} = [\max\{a_1, a_2\}, \max\{b_1, b_2\}].$$

If $D_i = [a_i, b_i] \in D[0, 1]$ for $i = 1, 2, \ldots$, then we define

$$\text{rsup}_i\{D_i\} = [\sup_i\{a_i\}, \sup_i\{b_i\}].$$

Similarly, we define

$$\text{rinf}_i\{D_i\} = [\inf_i\{a_i\}, \inf_i\{b_i\}].$$

Now we call $D_1 \geq D_2$ if and only if $a_1 \geq a_2$ and $b_1 \leq b_2$. Similarly, the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

Definition 2.6. *An interval-valued intuitionistic fuzzy (IVIF) set A in X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$, where $\mu_A : X \to D[0, 1]$ and $\gamma_A : X \to D[0, 1]$. The intervals $\mu_A(x)$ and $\gamma_A(x)$ denote the intervals of the degree of belongingness and non-belongingness of the element* *x to the set* $D[0,1]$*, where* $\mu_A(x) = [\mu_A^l(x), \mu_A^u(x)]$ *and* $\gamma_A(x) = [\gamma_A^l(x), \gamma_A^u(x)]$ *for all* $x \in X$ *with the following condition:*

$$(\forall x \in X)(0 \leq \mu_A^u(x) + \gamma_A^u(x) \leq 1).$$

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ *for the IVIF set* $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$*. Also note that* $\overline{\mu_A}(x) = [1 - \mu_A^u(x), 1 - \mu_A^l(x)]$ *and* $\overline{\gamma_A}(x) = [1 - \gamma_A^u(x), 1 - \gamma_A^l(x)]$ *for all* $x \in X$*, where* $[\overline{\mu_A}(x), \overline{\gamma_A}(x)]$ *represents the complement of* x *in* A*.*

3. IVIF subalgebras of Hilbert algebras

Definition 3.1. *An IVIF set* $A = (\mu_A, \gamma_A)$ *in a Hilbert algebra* X *is called an IVIF subalgebra of* X *if*

$$(\forall x, y \in X)\left(\begin{array}{l} \mu_A(x \cdot y) \geq \text{rmin}\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(x \cdot y) \leq \text{rmax}\{\gamma_A(x), \gamma_A(y)\} \end{array}\right). \tag{3.1}$$

Example 3.1. *Let* $X = \{1, x, y, z, 0\}$ *with the following Cayley table:*

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X *is a Hilbert algebra. We define an IVIF set* $A = (\mu_A, \gamma_A)$ *as follows:*

$$\mu_A(a) = \begin{cases} [0.5, 0.6] & \text{if } a \in \{1, x, y, z\} \\ [0.1, 0.2] & \text{if } a = 0 \end{cases}$$

and

$$\gamma_A(a) = \begin{cases} [0.3, 0.4] & \text{if } a \in \{1, x, y, z\} \\ [0.4, 0.5] & \text{if } a = 0. \end{cases}$$

Then A *is an IVIF subalgebra of* X*.*

Proposition 3.1. *Every IVIF subalgebra* $A = (\mu_A, \gamma_A)$ *of a Hilbert algebra* X *satisfies* $\mu_A(1) \geq \mu_A(x)$ *and* $\gamma_A(1) \leq \gamma_A(x)$ *for all* $x \in X$*, where* $\mu_A(1)$ *and* $\gamma_A(1)$ *are the upper bound and lower bound of* $\mu_A(x)$ *and* $\gamma_A(x)$*, respectively.*

Proof. For any $x \in X$, we have

$$\begin{aligned} \mu_A(1) &= \mu_A(x \cdot x) \\ &\geq \text{rmin}\{\mu_A(x), \mu_A(x)\} \\ &= \text{rmin}\{[\mu_A^l(x), \mu_A^u(x)], [\mu_A^l(x), \mu_A^u(x)]\} \\ &= [\mu_A^l(x), \mu_A^u(x)] \\ &= \mu_A(x) \end{aligned}$$

and

$$\begin{aligned}\gamma_A(1) &= \gamma_A(x\cdot x)\\ &\leq \text{rmax}\{\gamma_A(x),\gamma_A(x)\}\\ &= \text{rmax}\{[\gamma_A^l(x),\gamma_A^u(x)],[\gamma_A^l(x),\gamma_A^u(x)]\}\\ &= [\gamma_A^l(x),\gamma_A^u(x)]\\ &= \gamma_A(x).\end{aligned}$$

□

Proposition 3.2. *If an IVIF set $A=(\mu_A,\gamma_A)$ in a Hilbert algebra X is an IVIF subalgebra, then*

$$(\forall x\in X)\left(\begin{array}{c}\mu_A(1\cdot x)\geq\mu_A(x)\\ \gamma_A(1\cdot x)\leq\gamma_A(x)\end{array}\right). \tag{3.2}$$

Proof. For any $x\in X$, we have

$$\begin{aligned}\mu_A(1\cdot x) &\geq \text{rmin}\{\mu_A(1),\mu_A(x)\}\\ &= \text{rmin}\{\mu_A(x\cdot x),\mu_A(x)\}\\ &\geq \text{rmin}\{\text{rmin}\{\mu_A(x),\mu_A(x)\},\mu_A(x)\}\\ &= \mu_A(x)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(1\cdot x) &\leq \text{rmax}\{\gamma_A(1),\gamma_A(x)\}\\ &= \text{rmax}\{\gamma_A(x\cdot x),\gamma_A(x)\}\\ &\leq \text{rmax}\{\text{rmax}\{\gamma_A(x),\gamma_A(x)\},\gamma_A(x)\}\\ &= \gamma_A(x).\end{aligned}$$

□

Theorem 3.1. *An IVIF set $A=(\mu_A,\gamma_A)=([\mu_A^l,\mu_A^u],[\gamma_A^l,\gamma_A^u])$ in a Hilbert algebra X is an IVIF subalgebra of X if and only if $\mu_A^l,\mu_A^u,\gamma_A^l$, and γ_A^u are fuzzy subalgebras of X.*

Proof. Let μ_A^l and μ_A^u be fuzzy subalgebras of X and $x,y\in X$. Then $\mu_A^l(x\cdot y)\geq\min\{\mu_A^l(x),\mu_A^l(y)\}$ and $\mu_A^u(x\cdot y)\leq\min\{\mu_A^u(x),\mu_A^u(y)\}$. Now,

$$\begin{aligned}\mu_A(x\cdot y) &= [\mu_A^l(x\cdot y),\mu_A^u(x\cdot y)]\\ &\geq [\min\{\mu_A^l(x),\mu_A^l(y)\},\min\{\mu_A^u(x),\mu_A^u(y)\}]\\ &= \text{rmin}\{[\mu_A^l,(x),\mu_A^u(x)],[\mu_A^l(y),\mu_A^u(y)]\}\\ &= \text{rmin}\{\mu_A(x),\mu_A(y)\}.\end{aligned}$$

Again, let γ_A^l and γ_A^u be fuzzy subalgebras of X and $x,y\in X$. Then $\gamma_A^l(x\cdot y)\leq\max\{\gamma_A^l(x),\gamma_A^l(y)\}$ and $\gamma_A^u(x\cdot y)\leq\max\{\gamma_A^u(x),\gamma_A^u(y)\}$. Now,

$$\begin{aligned}\gamma_A(x\cdot y) &= [\gamma_A^l(x\cdot y),\gamma_A^u(x\cdot y)]\\ &\leq [\max\{\gamma_A^l(x),\gamma_A^l(y)\},\max\{\gamma_A^u(x),\gamma_A^u(y)\}]\\ &= \text{rmax}\{[\gamma_A^l,(x),\gamma_A^u(x)],[\gamma_A^l(y),\gamma_A^u(y)]\}\\ &= \text{rmax}\{\gamma_A(x),\gamma_A(y)\}.\end{aligned}$$

Hence, $A = \{[\mu_A^l, \mu_A^u], [\gamma_A^l, \gamma_A^u]\}$ is an IVIF subalgebra of X.

Conversely, assume that A is an IVIF subalgebra of X. For any $x, y \in X$,

$$\begin{aligned}[\mu_A^l(x\cdot y), \mu_A^u(x\cdot y)] &= \mu_A(x\cdot y)\\ &\geq \text{rmin}\{\mu_A(x), \mu_A(y)\}\\ &= \text{rmin}\{[\mu_A^l(x), \mu_A^u(x)], [\mu_A^l(y), \mu_A^u(y)]\}\\ &= [\min\{\mu_A^l(x), \mu_A^l(y)\}, \min\{\mu_A^u(x), \mu_A^u(y)\}]\end{aligned}$$

and

$$\begin{aligned}[\gamma_A^l(x\cdot y), \gamma_A^u(x\cdot y)] &= \gamma_A(x\cdot y)\\ &\leq \text{rmax}\{\gamma_A(x), \gamma_A(y)\}\\ &= \text{rmax}\{[\gamma_A^l(x), \gamma_A^u(x)], [\gamma_A^l(y), \gamma_A^u(y)]\}\\ &= [\max\{\gamma_A^l(x), \gamma_A^l(y)\}, \max\{\gamma_A^u(x), \gamma_A^u(y)\}].\end{aligned}$$

Thus $\mu_A^l(x\cdot y) \geq \min\{\mu_A^l(x), \mu_A^l(y)\}$, $\mu_A^u(x\cdot y) \geq \min\{\mu_A^u(x), \mu_A^u(y)\}$, $\gamma_A^l(x\cdot y) \leq \max\{\gamma_A^l(x), \gamma_A^l(y)\}$, and $\gamma_A^u(x\cdot y) \leq \max\{\gamma_A^u(x), \gamma_A^u(y)\}$. Therefore, $\mu_A^l, \mu_A^u, \gamma_A^l$, and γ_A^u are fuzzy subalgebras of X. □

Theorem 3.2. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two IVIF subalgebras of a Hilbert algebra X, then $A \cap B = (\mu_{A\cap B}, \gamma_{A\cup B})$ is an IVIF subalgebra of X.*

Proof. Let $x, y \in X$. Since A and B are IVIF subalgebras of X, by Theorem 3.1, we have

$$\begin{aligned}\mu_{A\cap B}(x\cdot y) &= [\mu_{A\cap B}^l(x\cdot y), \mu_{A\cap B}^u(x\cdot y)]\\ &= [\min\{\mu_A^l(x\cdot y), \mu_B^l(x\cdot y)\}, \min\{\mu_A^u(x\cdot y), \mu_B^u(x\cdot y)\}]\\ &\geq [\min\{\mu_{A\cap B}^l(x), \mu_{A\cap B}^l(y)\}, \min\{\mu_{A\cap B}^u(x), \mu_{A\cap B}^u(y)\}]\\ &= \text{rmin}\{\mu_{A\cap B}(x), \mu_{A\cap B}(y)\}\end{aligned}$$

and

$$\begin{aligned}\gamma_{A\cup B}(x\cdot y) &= [\gamma_{A\cup B}^l(x\cdot y), \gamma_{A\cup B}^u(x\cdot y)]\\ &= [\max\{\gamma_A^l(x\cdot y), \gamma_B^l(x\cdot y)\}, \max\{\gamma_A^u(x\cdot y), \gamma_B^u(x\cdot y)\}]\\ &\leq [\max\{\gamma_{A\cup B}^l(x), \gamma_{A\cup B}^l(y)\}, \max\{\gamma_{A\cup B}^u(x), \gamma_{A\cup B}^u(y)\}]\\ &= \text{rmax}\{\gamma_{A\cup B}(x), \gamma_{A\cup B}(y)\}.\end{aligned}$$

Hence, $A \cap B = (\mu_{A\cap B}, \gamma_{A\cup B})$ is an IVIF subalgebra of X. □

Corollary 3.1. *Let $\{A_i \mid i = 1, 2, 3, \cdots\}$ be a family of IVIF subalgebras of a Hilbert algebra X. Then $\bigcap_{i=1}^{\infty} A_i$ is also an IVIF subalgebra of X, where $\bigcap_{i=1}^{\infty} A_i = \{(x, \text{rmin}\mu_{A_i}(x), \text{rmax}\gamma_{A_i}(x)) \mid x \in X\}$.*

For any elements x and y of a Hilbert algebra X, let $\prod^{n} x\cdot y$ denotes the expression $x\cdot(\cdots(x\cdot(x\cdot y)))$, where x occurred n times.

Theorem 3.3. *Let $A = (\mu_A, \gamma_A)$ be an IVIF subalgebra of a Hilbert algebra X and let $n \in \mathbb{N}$. Then*

(1) $\mu_A\left(\prod^{n} x\cdot x\right) \geq \mu_A(x)$ for any odd number n,

(2) $\gamma_A\left(\prod^{n} x\cdot x\right) \leq \gamma_A(x)$ for any odd number n,

(3) $\mu_A\left(\prod^{n} x \cdot x\right) = \mu_A(x)$ *for any even number n,*

(4) $\gamma_A\left(\prod^{n} x \cdot x\right) = \gamma_A(x)$ *for any even number n.*

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2p - 1$ for some positive integer p. We prove the theorem by induction. Now, $\mu_A(x \cdot x) = \mu_A(1) \geq \mu_A(x)$ and $\gamma_A(x \cdot x) = \gamma_A(1) \leq \gamma_A(x)$. Suppose that $\mu_A\left(\prod^{2p-1} x \cdot x\right) \geq \mu_A(x)$ and $\gamma_A\left(\prod^{2p-1} x \cdot x\right) \leq \gamma_A(x)$. Then by assumption,

$$\begin{aligned} \mu_A\left(\prod^{2(p+1)-1} x \cdot x\right) &= \mu_A\left(\prod^{2p+1} x \cdot x\right) \\ &= \mu_A\left(\prod^{2p-1} x \cdot (x \cdot (x \cdot x))\right) \\ &= \mu_A\left(\prod^{2p-1} x \cdot x\right) \\ &\geq \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} \gamma_A\left(\prod^{2(p+1)-1} x \cdot x\right) &= \gamma_A\left(\prod^{2p+1} x \cdot x\right) \\ &= \gamma_A\left(\prod^{2p-1} x \cdot (x \cdot (x \cdot x))\right) \\ &= \gamma_A\left(\prod^{2p-1} x \cdot x\right) \\ &\leq \gamma_A(x), \end{aligned}$$

which proves (1) and (2). Proofs are similar for the cases (3) and (4). □

Definition 3.2. *Let* $A = (\mu_A, \gamma_A)$ *be an IVIF set defined in a Hilbert algebra* X*. The IVIF sets* $\oplus A$ *and* $\otimes A$ *are defined as* $\oplus A = \{(x, \mu_A(x), \overline{\mu_A}(x)) \mid x \in X\}$ *and* $\otimes A = \{(x, \overline{\gamma_A}(x), \gamma_A(x)) \mid x \in X\}$.

Theorem 3.4. *If* $A = (\mu_A, \gamma_A)$ *is an IVIF subalgebra of a Hilbert algebra* X*, then* $\oplus A$ *and* $\otimes A$ *both are IVIF subalgebras.*

Proof. Let $x, y \in X$. Then $\mu_A(x \cdot y) = [1, 1] - \mu_A(x \cdot y) \leq [1, 1] - \text{rmin}\{\mu_A(x), \mu_A(y)\} = \text{rmax}\{1 - \mu_A(x), 1 - \mu_A(y)\} = \text{rmax}\{\mu_A(x), \mu_A(y)\}$. Hence, $\oplus A$ is an IVIF subalgebra of X. Let $x, y \in X$. Then $\gamma_A(x \cdot y) = [1, 1] - \gamma_A(x \cdot y) \geq [1, 1] - \text{rmax}\{\gamma_A(x), \gamma_A(y)\} = \text{rmin}\{1 - \gamma_A(x), 1 - \gamma_A(y)\} = \text{rmin}\{\gamma_A(x), \gamma_A(y)\}$. Hence, $\otimes A$ is also an IVIF subalgebra of X. □

The sets $\{x \in X \mid \mu_A(x) = \mu_A(1)\}$ and $\{x \in X \mid \gamma_A(x) = \gamma_A(1)\}$ are denoted by μ_A^1 and γ_A^1, respectively. These two sets are also subalgebra of a Hilbert algebra X.

Theorem 3.5. *Let* $A = (\mu_A, \gamma_A)$ *be an IVIF subalgebra of a Hilbert algebra* X*, then the sets* μ_A^1 *and* γ_A^1 *are subalgebras of* X.

Proof. Let $x, y \in \mu_A^1$. Then $\mu_A(x) = \mu_A(1) = \mu_A(y)$ and so

$$\mu_A(x \cdot y) \leq \text{rmin}\{\mu_A(x), \mu_A(y)\} = \mu_A(1).$$

By using Proposition 3.1, we have $\mu_A(x \cdot y) = \mu_A(1)$; hence, $x \cdot y \in \mu_A^1$. Again, let $x, y \in \gamma_A^1$. Then $\gamma_A(x) = \gamma_A(1) = \gamma_A(y)$ and so, $\gamma_A(x \cdot y) \leq \text{rmax}\{\gamma_A(x), \gamma_A(y)\} = \gamma_A(1)$. Again, by Proposition 3.1, we have $\gamma_A(x \cdot y) = \gamma_A(1)$; hence, $x \cdot y \in \gamma_A^1$. Therefore, the sets μ_A^1 and γ_A^1 are subalgebras of X. □

Theorem 3.6. *Let B be a nonempty subset of a Hilbert algebra X and $A = (\mu_A, \gamma_A)$ be an IVIF set in X defined by* $\mu_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in B \\ [\beta_1, \beta_2] & \text{otherwise} \end{cases}$ *and* $\gamma_A(x) = \begin{cases} [\theta_1, \theta_2] & \text{if } x \in B \\ [\delta_1, \delta_2] & \text{otherwise} \end{cases}$ *for all* $[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\theta_1, \theta_2], [\delta_1, \delta_2] \in D[0, 1]$ *with* $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$ *and* $[\theta_1, \theta_2] \leq [\delta_1, \delta_2]$ *and* $\alpha_2 + \theta_2 \leq 1$ *and* $\beta_2 + \delta_2 \leq 1$. *Then A is an IVIF subalgebra of X if and only if B is a subalgebra of X. Moreover,* $\mu_A^1 = B = \gamma_A^1$.

Proof. Let A be an IVIF subalgebra of X. Let $x, y \in B$. Then

$$\mu_A(x \cdot y) \geq \text{rmin}\{\mu_A(x), \mu_A(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$$

and

$$\gamma_A(x \cdot y) \leq \text{rmax}\{\gamma_A(x), \gamma_A(y)\} = \text{rmax}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2].$$

So $x \cdot y \in B$. Hence, B is a subalgebra of X.

Conversely, suppose that B is a subalgebra of X. Let $x, y \in X$. Consider two cases:

Case (i): If $x, y \in B$, then $x \cdot y \in B$. Thus

$$\mu_A(x \cdot y) = [\alpha_1, \alpha_2] = \text{rmin}\{\mu_A(x), \mu_A(y)\}$$

and

$$\gamma_A(x \cdot y) = [\theta_1, \theta_2] = \text{rmax}\{\gamma_A(x), \gamma_A(y)\}.$$

Case (ii): If $x \notin B$ or $y \notin B$, then $\mu_A(x \cdot y) \geq [\beta_1, \beta_2] = \text{rmin}\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(x \cdot y) \leq [\theta_1, \theta_2] = \text{rmax}\{\gamma_A(x), \gamma_A(y)\}$. Hence, A is an IVIF subalgebra of X. Now, $\mu_A^1 = \{x \in X \mid \mu_A(x) = \mu_A(1)\} = \{x \in X \mid \mu_A(x) = [\alpha_1, \alpha_2]\} = B$ and $\gamma_A^1 = \{x \in X \mid \gamma_A(x) = \gamma_A(1)\} = \{x \in X \mid \gamma_A(x) = [\theta_1, \theta_2]\} = B$. □

Definition 3.3. *Let $A = (\mu_A, \gamma_A)$ be an IVIF subalgebra of a Hilbert algebra X. For $[s_1, s_2], [t_1, t_2] \in D[0, 1]$, the sets $U(\mu_A : [s_1, s_2]) = \{x \in X \mid \mu_A(x) \geq [s_1, s_2]\}$ is called an upper $[s_1, s_2]$-level of A and $L(\gamma_A : [t_1, t_2]) = \{x \in X \mid \gamma_A(x) \leq [t_1, t_2]\}$ is called a lower $[t_1, t_2]$-level of A.*

Theorem 3.7. *If $A = (\mu_A, \gamma_A)$ is an IVIF subalgebra of a Hilbert algebra X, then the upper $[s_1, s_2]$-level and lower $[t_1, t_2]$-level of A are subalgebras of X.*

Proof. Let $x, y \in U(\mu_A : [s_1, s_2])$. Then $\mu_A(x) \leq [s_1, s_2]$ and $\mu_A(y) \leq [s_1, s_2]$. It follows that $\mu_A(x \cdot y) \leq \text{rmin}\{\mu_A(x), \mu_A(y)\} \leq [s_1, s_2]$ so that $x \cdot y \in U(\mu_A : [s_1, s_2])$. Hence, $U(\mu_A : [s_1, s_2])$ is a subalgebra of X. Let $x, y \in L(\gamma_A : [t_1, t_2])$. Then $\gamma_A(x) \leq [t_1, t_2]$ and $\gamma_A(y) \leq [t_1, t_2]$. Thus $\gamma_A(x \cdot y) \leq \text{rmax}\{\gamma_A(x), \gamma_A(y)\} \leq [t_1, t_2]$ so that $x \cdot y \in L(\gamma_A : [t_1, t_2])$. Hence, $L(\gamma_A : [t_1, t_2])$ is a subalgebra of X. □

Theorem 3.8. *Let $A = (\mu_A, \gamma_A)$ be an IVIF set in a Hilbert algebra X such that the sets $U(\mu_A : [s_1, s_2])$ and $L(\gamma_A : [t_1, t_2])$ are subalgebras of X for every $[s_1, s_2], [t_1, t_2] \in D[0, 1]$. Then A is an IVIF subalgebra of X.*

Proof. Let $[s_1, s_2], [t_1, t_2] \in D[0, 1]$ be such that $U(\mu_A : [s_1, s_2])$ and $L(\gamma_A : [t_1, t_2])$ are subalgebras of X. In contrary, let $x_0, y_0 \in X$ be such that $\mu_A(x_0 \cdot y_0) < \text{rmin}\{\mu_A(x_0), \mu_A(y_0)\}$. Let $\mu_A(x_0) = [\theta_1, \theta_2], \mu_A(y_0) = [\theta_3, \theta_4]$, and $\mu_A(x_0 \cdot y_0) = [s_1, s_2]$. Then $[s_1, s_2] < \text{rmin}\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]$. So, $s_1 < \min\{\theta_1, \theta_3\}$ and $s_2 < \min\{\theta_2, \theta_4\}$. Consider,

$$\begin{aligned} [\rho_1, \rho_2] &= \tfrac{1}{2}[\mu_A(x_0 \cdot y_0) + \text{rmin}\{\mu_A(x_0), \mu_A(y_0)\}] \\ &= \tfrac{1}{2}[[s_1, s_2] + [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]] \\ &= [\tfrac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}), \tfrac{1}{2}(s_2 + \min\{\theta_2, \theta_4\})]. \end{aligned}$$

Therefore, $\min\{\theta_1, \theta_3\} > \rho_1 = \frac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}) > s_1$ and $\min\{\theta_2, \theta_4\} > \rho_2 = \frac{1}{2}(s_2 + \min\{\theta_2, \theta_4\}) > s_2$. Hence, $[\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2] > [s_1, s_2]$, so that $x_0 \cdot y_0 \notin U(\mu_A : [s_1, s_2])$, which is a contradiction because

$$\mu_A(x_0) = [\theta_1, \theta_2] \geq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$$

and

$$\mu_A(y_0) = [\theta_3, \theta_4] \geq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2].$$

This implies that $x_0 \cdot y_0 \in U(\mu_A : [s_1, s_2])$. Thus $\mu_A(x \cdot y) \leq \text{rmin}\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$. Again, in contrary, let $x_0, y_0 \in X$ be such that $\gamma_A(x_0 \cdot y_0) > \text{rmax}\{\gamma_A(x_0), \gamma_A(y_0)\}$. Let $\gamma_A(x_0) = [\eta_1, \eta_2], \gamma_A(y_0) = [\eta_3, \eta_4]$, and $\gamma_A(x_0 \cdot y_0) = [t_1, t_2]$. Then $[t_1, t_2] > \text{rmax}\{[\eta_1, \eta_2], [\eta_3, \eta_4]\} = [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}]$. So $t_1 > \max\{\eta_1, \eta_3\}$ and $t_2 > \max\{\eta_2, \eta_4\}$. Let us consider,

$$\begin{aligned} [\beta_1, \beta_2] &= \tfrac{1}{2}[\gamma_A(x_0 \cdot y_0) + \text{rmax}\{\gamma_A(x_0), \gamma_A(y_0)\}] \\ &= \tfrac{1}{2}[[t_1, t_2] + [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] \\ &= [\tfrac{1}{2}(t_1 + \max\{\eta_1, \eta_3\}), \tfrac{1}{2}(t_2 + \max\{\eta_2, \eta_4\})]. \end{aligned}$$

Therefore, $\max\{\eta_1, \eta_3\} < \beta_1 = \frac{1}{2}[(t_1 + \max\{\eta_1, \eta_3\})] < t_1$ and $\max\{\eta_2, \eta_4\} < \beta_2 = \frac{1}{2}[(t_2 + \max\{\eta_2, \eta_4\})] < t_2$. Hence, $[\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] < [\beta_1, \beta_2] < [t_1, t_2]$ so that $x_0 \cdot y_0 \notin L(\gamma_A : [t_1, t_2])$, which is a contradiction because

$$\gamma_A(x_0) = [\eta_1, \eta_2] \leq [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] > [\beta_1, \beta_2]$$

and

$$\gamma_A(y_0) = [\eta_3, \eta_4] \geq [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] > [\beta_1, \beta_2].$$

This implies that $x_0 \cdot y_0 \in L(\gamma_A : [t_1, t_2])$. Thus $\gamma_A(x \cdot y) \geq \text{rmax}\{\gamma_A(x), \gamma_A(y)\}$ for all $x, y \in X$. Therefore, A is an IVIF subalgebra of X. □

Theorem 3.9. *Any subalgebra of a Hilbert algebra X can be realized as both the upper $[s_1, s_2]$-level and lower $[t_1, t_2]$-level of some IVIF subalgebra of X.*

Proof. Let B be an IVIF subalgebra of X and A be an IVIF set on X defined by $\mu_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in B \\ [0, 0] & \text{otherwise} \end{cases}$ and $\gamma_A(x) = \begin{cases} [\beta_1, \beta_2] & \text{if } x \in B \\ [1, 1] & \text{otherwise} \end{cases}$ for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0, 1]$ and $\alpha_2 + \beta_2 \leq 1$. We consider the following cases:

Case (i) If $x, y \in B$, then $\mu_A(x) = [\alpha_1, \alpha_2], \gamma_A(x) = [\beta_1, \beta_2]$, and $\mu_A(y) = [\alpha_1, \alpha_2]$, $\gamma_A(y) = [\beta_1, \beta_2]$. Thus

$$\mu_A(x \cdot y) = [\alpha_1, \alpha_2] = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = \text{rmin}\{\mu_A(x), \mu_A(y)\}$$

and

$$\gamma_A(x \cdot y) = [\beta_1, \beta_2] = \text{rmax}\{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = \text{rmax}\{\gamma_A(x), \gamma_A(y)\}.$$

Case (ii) If $x \in B$ and $y \notin B$, then $\mu_A(x) = [\alpha_1, \alpha_2], \gamma_A(x) = [\beta_1, \beta_2]$, and $\mu_A(y) = [0, 0]$, $\gamma_A(y) = [1, 1]$. Thus

$$\mu_A(x \cdot y) \geq [0, 0] = \text{rmin}\{[\alpha_1, \alpha_2], [0, 0]\} = \text{rmin}\{\mu_A(x), \mu_A(y)\}$$

and

$$\gamma_A(x \cdot y) \geq [1, 1] = \text{rmax}\{[\beta_1, \beta_2], [1, 1]\} = \text{rmax}\{\gamma_A(x), \gamma_A(y)\}.$$

Case (iii) If $x \notin B$ and $y \in B$, then $\mu_A(x) = [0, 0], \gamma_A(x) = [1, 1], \mu_A(y) = [\alpha_1, \alpha_2]$, and $\gamma_A(y) = [\beta_1, \beta_2]$. Thus

$$\mu_A(x \cdot y) \geq [0, 0] = \text{rmin}\{[0, 0], [\alpha_1, \alpha_2]\} = \text{rmin}\{\mu_A(x), \mu_A(y)\}$$

and

$$\gamma_A(x \cdot y) \leq [1, 1] = \text{rmax}\{[1, 1], [\beta_1, \beta_2]\} = \text{rmax}\{\gamma_A(x), \gamma_A(y)\}.$$

Case (iv) If $x \notin B$ and $y \notin B$, then $\mu_A(x) = [0, 0], \gamma_A(x) = [1, 1], \mu_A(y) = [0, 0]$, and $\gamma_A(y) = [1, 1]$. Thus

$$\mu_A(x \cdot y) \leq [0, 0] = \text{rmin}\{[0, 0], [0, 0]\} = \text{rmin}\{\mu_A(x), \mu_A(y)\}$$

and

$$\gamma_A(x \cdot y) \geq [1, 1] = \text{rmax}\{[1, 1], [1, 1]\} = \text{rmax}\{\gamma_A(x), \gamma_A(y)\}.$$

Therefore, A is an IVIF subalgebra of X. □

Theorem 3.10. *Let B be a subset of a Hilbert algebra X and A be an IVIF set in X defined by* $\mu_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in B \\ [0,0] & \text{otherwise} \end{cases}$ *and* $\gamma_A(x) = \begin{cases} [\beta_1, \beta_2] & \text{if } x \in B \\ [1,1] & \text{otherwise} \end{cases}$ *for all* $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0,1]$ *and* $\alpha_2 + \beta_2 \leq 1$. *If A is realized as a lower level subalgebra and an upper level subalgebra of some IVIF subalgebra of X, then B is a subalgebra of X.*

Proof. Let A be an IVIF subalgebra of X and $x, y \in B$. Then $\mu_A(x) = [\alpha_1, \alpha_2] = \mu_A(y)$ and $\gamma_A(x) = [\beta_1, \beta_2] = \gamma_A(y)$. Thus

$$\mu_A(x \cdot y) \leq \text{rmin}\{\mu_A(x), \mu_A(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$$

and

$$\gamma_A(x \cdot y) \geq \text{rmax}\{\gamma_A(x), \gamma_A(y)\} = \text{rmax}\{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = [\beta_1, \beta_2],$$

which imply that $x \cdot y \in B$. Hence, B is a subalgebra of X. □

4. IVIF ideals of Hilbert algebras

Definition 4.1. *An IVIF set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra X is said to be an IVIF ideal of X if the following conditions are hold:*

$$(\forall x \in X) \begin{pmatrix} \mu_A(1) \geq \mu_A(x) \\ \gamma_A(1) \leq \gamma_A(x) \end{pmatrix}, \tag{4.1}$$

$$(\forall x, y \in X) \begin{pmatrix} \mu_A(x \cdot y) \geq \mu_A(y) \\ \gamma_A(x \cdot y) \leq \gamma_A(y) \end{pmatrix}, \tag{4.2}$$

$$(\forall x, y_1, y_2 \in X) \begin{pmatrix} \mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \text{rmin}\{\mu_A(y_1), \mu_A(y_2)\} \\ \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \text{rmax}\{\gamma_A(y_1), \gamma_A(y_2)\} \end{pmatrix}. \tag{4.3}$$

Example 4.1. *Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:*

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define an IVIF set $A = (\mu_A, \gamma_A)$ as follows:

$$\mu_A(x) = \begin{cases} [0.5, 0.6] & \text{if } x \in \{1, x, y, z\} \\ [0.1, 0.2] & \text{if } x = 0 \end{cases}$$

and

$$\gamma_A(x) = \begin{cases} [0.3, 0.4] & \text{if } x \in \{1, x, y, z\} \\ [0.4, 0.5] & \text{if } x = 0. \end{cases}$$

Then A is an IVIF ideal of X.

Proposition 4.1. *If $A = (\mu_A, \gamma_A)$ is an IVIF ideal of a Hilbert algebra X, then*

$$(\forall x, y \in X) \left(\begin{array}{l} \mu_A((y \cdot x) \cdot x) \geq \mu_A(y) \\ \gamma_A((y \cdot x) \cdot x) \leq \gamma_A(y) \end{array} \right). \tag{4.4}$$

Proof. Putting $y_1 = y$ and $y_2 = 1$ in (4.3), we have

$$\mu_A((y \cdot x) \cdot x) \geq \mathrm{rmin}\{\mu_A(y), \mu_A(1)\} = \mu_A(y)$$

and

$$\gamma_A((y \cdot x) \cdot x) \leq \mathrm{rmax}\{\gamma_A(y), \gamma_A(1)\} = \gamma_A(y).$$

□

Lemma 4.1. *If $A = (\mu_A, \gamma_A)$ is an IVIF ideal of a Hilbert algebra X, then*

$$(\forall x, y \in X) \left(x \leq y \ \Rightarrow \begin{cases} \mu_A(x) \leq \mu_A(y) \\ \gamma_A(x) \geq \gamma_A(y) \end{cases} \right). \tag{4.5}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{aligned} \mu_A(y) &= \mu_A(1 \cdot y) \\ &= \mu_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\geq \mathrm{rmin}\{\mu_A(x \cdot y), \mu_A(x)\} \\ &\geq \mathrm{rmin}\{\mu_A(1), \mu_A(x)\} \\ &= \mu_A(x). \end{aligned}$$

Thus

$$\begin{aligned} \gamma_A(y) &= \gamma_A(1 \cdot y) \\ &= \gamma_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\leq \mathrm{rmax}\{\gamma_A(x \cdot y), \gamma_A(x)\} \\ &\leq \mathrm{rmax}\{\gamma_A(1), \gamma_A(x)\} \\ &= \gamma_A(x). \end{aligned}$$

□

Theorem 4.1. *Every IVIF ideal of a Hilbert algebra X is an IVIF subalgebra of X.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an IVIF ideal of X. Since $y \leq x \cdot y$ for all $x, y \in X$, it follows from Lemma 4.1 that

$$\mu_A(y) \geq \mu_A(x \cdot y), \gamma_A(y) \leq \gamma_A(x \cdot y).$$

It follows from (4.2) that

$$\begin{aligned} \mu_A(x \cdot y) &\geq \mu_A(y) \\ &\geq \mathrm{rmin}\{\mu_A(x \cdot y), \mu_A(x)\} \\ &\geq \mathrm{rmin}\{\mu_A(x), \mu_A(y)\} \end{aligned}$$

and

$$\begin{aligned}\gamma_A(x\cdot y) &\leq \gamma_A(y)\\ &\leq \operatorname{rmax}\{\gamma_A(x\cdot y),\gamma_A(x)\}\\ &\leq \operatorname{rmax}\{\gamma_A(x),\gamma_A(y)\}.\end{aligned}$$

Hence, A is an IVIF subalgebra of X. □

Theorem 4.2. *An IVIF set $A=(\mu_A,\gamma_A)=\{[\mu_A^l,\mu_A^u],[\gamma_A^l,\gamma_A^u]\}$ in a Hilbert algebra X is an IVIF ideal of X if and only if $\mu_A^l,\mu_A^u,\gamma_A^l$, and γ_A^u are fuzzy ideals of X.*

Proof. Since $\mu_A^l(1)\geq\mu_A^l(x),\mu_A^u(1)\geq\mu_A^u(x),\gamma_A^l(1)\leq\gamma_A^l(x)$, and $\gamma_A^u(1)\leq\gamma_A^u(x)$, we have $\mu_A(1)\geq\mu_A(x)$ and $\gamma_A(1)\leq\gamma_A(x)$. Let $x,y\in X$. Then

$$\mu_A(x\cdot y)=[\mu_A^l(x\cdot y),\mu_A^u(x\cdot y)]\geq[\mu_A^l(y),\mu_A^u(y)]=\mu_A(y)$$

and

$$\gamma_A(x\cdot y)=[\gamma_A^l(x\cdot y),\gamma_A^u(x\cdot y)]\leq[\gamma_A^l(y),\gamma_A^u(y)]=\gamma_A(y).$$

Let $x,y_1,y_2\in X$. Then

$$\begin{aligned}\mu_A((y_1\cdot(y_2\cdot x))\cdot x) &= [\mu_A^l((y_1\cdot(y_2\cdot x))\cdot x),\mu_A^u((y_1\cdot(y_2\cdot x))\cdot x)]\\ &\geq [\min\{\mu_A^l(y_1),\mu_A^l(y_2)\},\min\{\mu_A^u(y_1),\mu_A^u(y_2)\}]\\ &= \operatorname{rmin}\{[\mu_A^l(y_1),\mu_A^u(y_1)],[\mu_A^l(y_2),\mu_A^u(y_2)]\}\\ &= \operatorname{rmin}\{\mu_A(y_1),\mu_A(y_2)\}\end{aligned}$$

and

$$\begin{aligned}\gamma_A((y_1\cdot(y_2\cdot x))\cdot x) &= [\gamma_A^l((y_1\cdot(y_2\cdot x))\cdot x),\gamma_A^u((y_1\cdot(y_2\cdot x))\cdot x)]\\ &\leq [\max\{\gamma_A^l(y_1),\gamma_A^l(y_2)\},\max\{\gamma_A^u(y_1),\gamma_A^u(y_2)\}]\\ &= \operatorname{rmax}\{[\gamma_A^l(y_1),\gamma_A^u(y_1)],[\gamma_A^l(y_2),\gamma_A^u(y_2)]\}\\ &= \operatorname{rmax}\{\gamma_A(y_1),\gamma_A(y_2)\}.\end{aligned}$$

Hence, $A=\{[\mu_A^l,\mu_A^u],[\gamma_A^l,\gamma_A^u]\}$ is an IVIF ideal of X.

Conversely, assume that A is an IVIF ideal of X. Let $x\in X$. Then $[\mu_A^l(1),\mu_A^u(1)]=\mu_A(1)\geq\mu_A(x)=[\mu_A^l(x),\mu_A^u(x)]$; hence, $\mu_A^l(1)\geq\mu_A^l(x)$ and $\gamma_A^l(1)\leq\gamma_A^l(x)$. Let $x,y\in X$. Then $[\mu_A^l(x\cdot y),\mu_A^u(x\cdot y)]=\mu_A(x\cdot y)\geq\mu_A(y)=[\mu_A^l(y),\mu_A^u(y)]$; hence, $\mu_A^l(x\cdot y)\geq\mu_A^l(y)$ and $\mu_A^u(x\cdot y)\geq\mu_A^u(y)$. Also, $[\gamma_A^l(x\cdot y),\gamma_A^u(x\cdot y)]=\gamma_A(x\cdot y)\leq\gamma_A(y)=[\gamma_A^l(y),\gamma_A^u(y)]$; hence, $\gamma_A^l(x\cdot y)\leq\gamma_A^l(y)$ and $\gamma_A^u(x\cdot y)\leq\gamma_A^u(y)$. Let $x,y_1,y_2\in X$. Then

$$\begin{aligned}&\mu_A^l((y_1\cdot(y_2\cdot x))\cdot x),\mu_A^u((y_1\cdot(y_2\cdot x))\cdot x)]\\ &=\mu_A((y_1\cdot(y_2\cdot x))\cdot x)\\ &\geq\operatorname{rmin}\{\mu_A(y_1),\mu_A(y_2)\}\\ &=\operatorname{rmin}\{[\mu_A^l(y_1),\mu_A^u(y_1)],[\mu_A^l(y_2),\mu_A^u(y_2)]\}\\ &=[\min\{\mu_A^l(y_1),\mu_A^l(y_2)\},\min\{\mu_A^u(y_1),\mu_A^u(y_2)\}].\end{aligned}$$

Hence, $\mu_A^l((y_1\cdot(y_2\cdot x))\cdot x)\geq\min\{\mu_A^l(y_1),\mu_A^l(y_2)\}$ and $\mu_A^u((y_1\cdot(y_2\cdot x))\cdot x)\geq\min\{\mu_A^u(y_1),\mu_A^u(y_2)\}$. Also,

$$\begin{aligned}&[\gamma_A^l((y_1\cdot(y_2\cdot x))\cdot x),\gamma_A^u((y_1\cdot(y_2\cdot x))\cdot x)]\\&=\gamma_A((y_1\cdot(y_2\cdot x))\cdot x)\\&\leq \mathrm{rmax}\{\gamma_A(y_1),\gamma_A(y_2)\}\\&=\mathrm{rmax}\{[\gamma_A^l(y_1),\gamma_A^u(y_1)],[\gamma_A^l(y_2),\gamma_A^u(y_2)]\}\\&=[\max\{\gamma_A^l(y_1),\gamma_A^l(y_2)\},\max\{\gamma_A^u(y_1),\gamma_A^u(y_2)\}].\end{aligned}$$

Hence, $\gamma_A^l((y_1\cdot(y_2\cdot x))\cdot x)\leq\max\{\gamma_A^l(y_1),\gamma_A^l(y_2)\}$ and $\gamma_A^u((y_1\cdot(y_2\cdot x))\cdot x)\leq\max\{\gamma_A^u(y_1),\gamma_A^u(y_2)\}$. Therefore, $\mu_A^l,\mu_A^u,\gamma_A^l$, and γ_A^u are fuzzy ideals of X. □

Proposition 4.2. *If $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ are IVIF ideals of a Hilbert algebra X, then $A\cap B$ is an IVIF ideal of X.*

Proof. Let $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ be IVIF ideals of X. Let $x\in X$. Then

$$\begin{aligned}\mu_{A\cap B}(1)&=[\mu_{A\cap B}^l(1),\mu_{A\cap B}^u(1)]\\&=[\min\{\mu_A^l(1),\mu_B^l(1)\},\min\{\mu_A^u(1),\mu_B^u(1)\}]\\&\geq[\min\{\mu_A^l(x),\mu_B^l(x)\},\min\{\mu_A^u(x),\mu_B^u(x)\}]\\&=[\mu_{A\cap B}^l(x),\mu_{A\cap B}^u(x)]\\&=\mu_{A\cap B}(x)\end{aligned}$$

and

$$\begin{aligned}\gamma_{A\cup B}(1)&=[\gamma_{A\cup B}^l(1),\gamma_{A\cup B}^u(1)]\\&=[\max\{\gamma_A^l(1),\gamma_B^l(1)\},\max\{\gamma_A^u(1),\gamma_B^u(1)\}]\\&\leq[\max\{\gamma_A^l(x),\gamma_B^l(x)\},\max\{\gamma_A^u(x),\gamma_B^u(x)\}]\\&=[\gamma_{A\cup B}^l(x),\gamma_{A\cup B}^u(x)]\\&=\gamma_{A\cup B}(x).\end{aligned}$$

Let $x,y\in X$. Then

$$\begin{aligned}\mu_{A\cap B}(x\cdot y)&=[\mu_{A\cap B}^l(x\cdot y),\mu_{A\cap B}^u(x\cdot y)]\\&=[\min\{\mu_A^l(x\cdot y),\mu_B^l(x\cdot y)\},\min\{\mu_A^u(x\cdot y),\mu_B^u(x\cdot y)\}]\\&\geq[\min\{\mu_A^l(y),\mu_B^l(y)\},\min\{\mu_A^u(y),\mu_B^u(y)\}]\\&=[\mu_{A\cap B}^l(y),\mu_{A\cap B}^u(y)]\\&=\mu_{A\cap B}(y)\end{aligned}$$

and

$$\begin{aligned}\gamma_{A\cup B}(x\cdot y)&=[\gamma_{A\cup B}^l(x\cdot y),\gamma_{A\cup B}^u(x\cdot y)]\\&=[\max\{\gamma_A^l(x\cdot y),\gamma_B^l(x\cdot y)\},\max\{\gamma_A^u(x\cdot y),\gamma_B^u(x\cdot y)\}]\\&\leq[\max\{\gamma_A^l(y),\gamma_B^l(y)\},\max\{\gamma_A^u(y),\gamma_B^u(y)\}]\\&=[\gamma_{A\cup B}^l(y),\gamma_{A\cup B}^u(y)]\\&=\gamma_{A\cup B}(y).\end{aligned}$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}
&\mu_{A\cap B}((y_1 \cdot (y_2 \cdot x)) \cdot x) \\
&= [\mu^l_{A\cap B}((y_1 \cdot (y_2 \cdot x)) \cdot x), \mu^u_{A\cap B}((y_1 \cdot (y_2 \cdot x)) \cdot x)] \\
&= \left[\begin{array}{l} \min\{\mu^l_A((y_1 \cdot (y_2 \cdot x)) \cdot x), \mu^l_B((y_1 \cdot (y_2 \cdot x)) \cdot x)\}, \\ \min\{\mu^u_A((y_1 \cdot (y_2 \cdot x)) \cdot x), \mu^u_B((y_1 \cdot (y_2 \cdot x)) \cdot x)\} \end{array}\right] \\
&\geq \left[\begin{array}{l} \min\{\min\{\mu^l_A(y_1), \mu^l_A(y_2)\}, \min\{\mu^l_B(y_1), \mu^l_B(y_2)\}\}, \\ \min\{\min\{\mu^u_A(y_1), \mu^u_A(y_2)\}, \min\{\mu^u_B(y_1), \mu^u_B(y_2)\}\} \end{array}\right] \\
&= \left[\begin{array}{l} \min\{\min\{\mu^l_A(y_1), \mu^l_B(y_1)\}, \min\{\mu^l_A(y_2), \mu^l_B(y_2)\}\}, \\ \min\{\min\{\mu^u_A(y_1), \mu^u_B(y_2)\}, \min\{\mu^u_A(y_2), \mu^u_B(y_2)\}\} \end{array}\right] \\
&= [\min\{\mu^l_{A\cap B}(y_1), \mu^l_{A\cap B}(y_2)\}, \min\{\mu^l_{A\cap B}(y_1), \mu^l_{A\cap B}(y_2)\}] \\
&= \mathrm{rmin}\{\mu_{A\cap B}(y_1), \mu_{A\cap B}(y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{A\cup B}((y_1 \cdot (y_2 \cdot x)) \cdot x) \\
&= [\gamma^l_{A\cup B}((y_1 \cdot (y_2 \cdot x)) \cdot x), \gamma^u_{A\cup B}((y_1 \cdot (y_2 \cdot x)) \cdot x)] \\
&= \left[\begin{array}{l} \max\{\gamma^l_A((y_1 \cdot (y_2 \cdot x)) \cdot x), \gamma^l_B((y_1 \cdot (y_2 \cdot x)) \cdot x)\}, \\ \max\{\gamma^u_A((y_1 \cdot (y_2 \cdot x)) \cdot x), \gamma^u_B((y_1 \cdot (y_2 \cdot x)) \cdot x)\} \end{array}\right] \\
&\leq \left[\begin{array}{l} \max\{\min\{\gamma^l_A(y_1), \gamma^l_A(y_2)\}, \min\{\gamma^l_B(y_1), \gamma^l_B(y_2)\}\}, \\ \max\{\min\{\gamma^u_A(y_1), \gamma^u_A(y_2)\}, \min\{\gamma^u_B(y_1), \gamma^u_B(y_2)\}\} \end{array}\right] \\
&= \left[\begin{array}{l} \max\{\min\{\gamma^l_A(y_1), \gamma^l_B(y_1)\}, \min\{\gamma^l_A(y_2), \gamma^l_B(y_2)\}\}, \\ \max\{\min\{\gamma^u_A(y_1), \gamma^u_B(y_2)\}, \min\{\gamma^u_A(y_2), \gamma^u_B(y_2)\}\} \end{array}\right] \\
&= [\max\{\gamma^l_{A\cap B}(y_1), \gamma^l_{A\cap B}(y_2)\}, \max\{\gamma^l_{A\cap B}(y_1), \gamma^l_{A\cap B}(y_2)\}] \\
&= \mathrm{rmax}\{\gamma_{A\cap B}(y_1), \gamma_{A\cap B}(y_2)\}.
\end{aligned}$$

Hence, $A \cap B$ is an IVIF ideal of X. □

Corollary 4.1. *If $\{A_i = (\mu_{A_i}, \gamma_{A_i}) \mid i \in \Delta\}$ is a family of IVIF ideals of a Hilbert algebra X, then $\bigcap\limits_{i\in\Delta} A_i$ is an IVIF ideal of X.*

Corollary 4.2. *If $A = (\mu_A, \gamma_A)$ is an IVIF ideal of a Hilbert algebra X, then $\overline{A}$ is also an IVIF ideal of X.*

Theorem 4.3. *If $A = (\mu_A, \gamma_A)$ is an IVIF ideal of a Hilbert algebra X, then $\oplus A$ and $\otimes A$ are both IVIF ideals.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an IVIF ideal of X. Let $x \in X$. Then $\overline{\mu_A}(1) = 1 - \mu_A(1) \leq 1 - \mu_A(x) \leq \overline{\mu_A}(x)$. Let $x, y \in X$. Then $\overline{\mu_A}(x \cdot y) = 1 - \mu_A(x \cdot y) \leq 1 - \mu_A(y) \leq \overline{\mu_A}(y)$. Let

$x, y_1, y_2 \in X$. Then

$$\begin{aligned}\overline{\mu_A}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= 1 - \mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \\ &\leq 1 - \text{rmin}\{\mu_A(y_1), \mu_A(y_2)\} \\ &= \text{rmax}\{1 - \mu_A(y_1), 1 - \mu_A(y_2)\} \\ &= \text{rmax}\{\overline{\mu_A}(y_1), \overline{\mu_A}(y_2)\}.\end{aligned}$$

Hence, $\oplus A$ is an IVIF ideal of X.

Let $x \in X$. Then $\overline{\gamma_A}(1) = 1 - \gamma_A(1) \geq 1 - \gamma_A(x) \geq \overline{\gamma_A}(x)$. Let $x, y \in X$. Then $\overline{\gamma_A}(x \cdot y) = 1 - \gamma_A(x \cdot y) \geq 1 - \gamma_A(y) \geq \overline{\gamma_A}(y)$. Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\overline{\gamma_A}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= 1 - \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \\ &\geq 1 - \text{rmax}\{\gamma_A(y_1), \gamma_A(y_2)\} \\ &= \text{rmin}\{1 - \gamma_A(y_1), 1 - \gamma_A(y_2)\} \\ &= \text{rmin}\{\overline{\gamma_A}(y_1), \overline{\gamma_A}(y_2)\}.\end{aligned}$$

Hence, $\otimes A$ is an IVIF ideal of X. □

Theorem 4.4. *An IVIF set $A = (\mu_A, \gamma_A)$ is an IVIF ideal of a Hilbert algebra X if and only if for every $[s_1, s_2], [t_1, t_2] \in D[0, 1]$, the sets $U(\mu_A : [t_1, t_2])$ and $L(g, [s_1, s_2])$ are either empty or ideals of X.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an IVIF ideal of X and let $[s_1, s_2], [t_1, t_2] \in D[0, 1]$ be such that $U(\mu_A : [t_1, t_2])$ and $L(\gamma_A : [s_1, s_2])$ are nonempty sets of X. It is clear that $1 \in U(\mu_A : [t_1, t_2]) \cap L(\gamma_A : [s_1, s_2])$ since $\mu_A(1) \geq [t_1, t_2]$ and $\gamma_A(1) \leq [s_1, s_2]$. Let $x \in X$ and $y \in U(\mu_A : [t_1, t_2])$. Then $\mu_A(y) \geq [t_1, t_2]$. It follows that $\mu_A(x \cdot y) \geq \mu_A(y) \geq [t_1, t_2]$ so that $x \cdot y \in U(\mu_A : [t_1, t_2])$. Let $x \in X$ and $y_1, y_2 \in U(\mu_A : [t_1, t_2])$. Then $\mu_A(y_1) \geq [t_1, t_2]$ and $\mu_A(y_2) \geq [t_1, t_2]$. Hence, $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu_A(y_1), \mu_A(y_2)\} \geq [t_1, t_2]$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\mu_A : [t_1, t_2])$. Hence, $U(\mu_A : [t_1, t_2])$ is an ideal of X. Let $x \in X$ and $y \in L(\gamma_A : [s_1, s_2])$. Then $\gamma_A(y) \leq [s_1, s_2]$. It follows that $\gamma_A(x \cdot y) \leq \gamma_A(y) \leq [s_1, s_2]$ so that $x \cdot y \in L(\gamma_A : [s_1, s_2])$. Let $x \in X$ and $y_1, y_2 \in L(\gamma_A : [s_1, s_2])$. Then $\gamma_A(y_1) \leq [s_1, s_2]$ and $\gamma_A(y_2) \leq [s_1, s_2]$. Hence, $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{\gamma_A(y_1), \gamma_A(y_2)\} \leq [s_1, s_2]$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(\gamma_A : [s_1, s_2])$. Hence, $L(\gamma_A : [s_1, s_2])$ is an ideal of X.

Assume now that every nonempty sets $U(\mu_A : [t_1, t_2])$ and $L(\gamma_A : [s_1, s_2])$ are ideals of X. If $\mu_A(1) \geq \mu_A(x)$ is not true for all $x \in X$, then there exists $x_0 \in X$ such that $\mu_A(1) < \mu_A(x_0)$. But in this case for $[s_1, s_2] = \frac{1}{2}(\mu_A(1) + \mu_A(x_0))$. Then $x_0 \in U(\mu_A : [s_1, s_2])$, that is $U(\mu_A : [s_1, s_2]) \neq \emptyset$. Since by the assumption, $U(\mu_A : [s_1, s_2])$ is an ideal of X, then $\mu_A(1) \geq [s_1, s_2]$, which is impossible. Hence, $\mu_A(1) \geq \mu_A(x)$. If $\gamma_A(1) \leq \gamma_A(x)$ is not true, then there exists $y_0 \in X$ such that $\gamma_A(1) < \gamma_A(y_0)$. But in this case for $[s_0', s_0''] = \frac{1}{2}(\gamma_A(1) + \gamma_A(y_0))$. Then $y_0 \in L(\gamma_A : [s_0', s_0''])$, that is $L(\gamma_A : [s_0', s_0'']) \neq \emptyset$. Since by the assumption, $L(\gamma_A : [s_0', s_0''])$ is an ideal of X, then $\gamma_A(1) \leq [s_0', s_0'']$, which is impossible. Hence, $\gamma_A(1) \leq \gamma_A(x)$. If $\mu_A(x \cdot y) \geq \mu_A(y)$ is not true for all $x, y \in X$, then there exist $x_0, y_0 \in X$ such that $\mu_A(x_0 \cdot y_0) < \mu_A(y_0)$. Let $[t_1, t_2] = \frac{1}{2}(\mu_A(x_0 \cdot y_0) + \mu_A(y_0))$. Then $t \in [0, 1]$ and $\mu_A(x_0 \cdot y_0) < t < \mu_A(y_0)$, which prove that $y_0 \in U(\mu_A : t)$. Since $U(\mu_A : t)$ is an ideal of X,

$x_0 \cdot y_0 \in U(\mu_A : t)$. Hence, $\mu_A(x_0 \cdot y_0) \geq t$, a contradiction. Thus $\mu_A(x \cdot y) \geq \mu_A(y)$ is true for all $x, y \in X$. If $\gamma_A(x \cdot y) \leq \gamma_A(y)$ is not true for all $x, y \in X$, then there exist $x_0, y_0 \in X$ such that $\gamma_A(x_0 \cdot y_0) > \gamma_A(y_0)$. Let $[t_0', t_0''] = \frac{1}{2}(\gamma_A(x_0 \cdot y_0) + \gamma_A(y_0))$. Then $[t_0', t_0''] \in D[0,1]$ and $\gamma_A(x_0 \cdot y_0) > [t_0', t_0''] > \gamma_A(y_0)$, which prove that $y_0 \in L(\gamma_A : [t_0', t_0''])$. Since $L(\gamma_A : [t_0', t_0''])$ is an ideal of X, $x_0 \cdot y_0 \in L(\gamma_A : [t_0', t_0''])$. Hence, $\gamma_A(x_0 \cdot y_0) \leq [t_0', t_0'']$, a contradiction. Thus $\gamma_A(x \cdot y) \leq \gamma_A(y)$ is true for all $x, y \in X$. Suppose that $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \mathrm{rmin}\{\mu_A(y_1), \mu_A(y_2)\}$ is not true for all $x, y_1, y_2 \in X$. Then there exist $u_0, v_0, x_0 \in X$ such that $\mu_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) < \mathrm{rmin}\{\mu_A(u_0), \mu_A(v_0)\}$. Taking $[p', p''] = \frac{1}{2}(\mu_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) + \mathrm{rmin}\{\mu_A(u_0), \mu_A(v_0)\})$. Then $\mu_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) < [p', p''] < \mathrm{rmin}\{\mu_A(u_0), \mu_A(v_0)\}$, which prove that $u_0, v_0 \in U(\mu_A : [p', p''])$. Since $U(\mu_A : p[p', p''])$ is an ideal of X, $(u_0 \cdot (v_0 \cdot x_0))x_0 \in U(\mu_A : [p', p''])$, a contradiction. Thus $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \mathrm{rmin}\{\mu_A(y_1), \mu_A(y_2)\}$ is true for all $x, y_1, y_2 \in X$. Suppose that $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \mathrm{rmax}\{\gamma_A(y_1), \gamma_A(y_2)\}$ is not true for all $x, y_1, y_2 \in X$. Then there exist $u_0, v_0, x_0 \in X$ such that $\gamma_A((u_0 \cdot (v_0 \cdot x_0) \cdot x_0) > \mathrm{rmax}\{\gamma_A(u_0), \gamma_A(v_0)\}$. Taking $[p_0', p_0''] = \frac{1}{2}(\gamma_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) + \mathrm{rmax}\{\gamma_A(u_0), \gamma_A(v_0)\})$. Then $\gamma_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) > [p_0', p_0''] > \mathrm{rmax}\{\gamma_A(u_0), \gamma_A(v_0)\}$, which prove that $u_0, v_0 \in L(\gamma_A : [p_0', p_0''])$. Since $L(\gamma_A : [p_0', p_0''])$ is an ideal of X, $(u_0 \cdot (v_0 \cdot x_0)) \cdot x_0 \in L(\gamma_A : [p_0', p_0''])$, a contradiction. Thus $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \mathrm{rmax}\{\gamma_A(y_1), \gamma_A(y_2)\}$ is true for all $x, y_1, y_2 \in X$. Hence, A is an IVIF ideal of X. □

5. Product of IVIF subalgebras/ideals in Hilbert algebras

Definition 5.1. *Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IVIF sets in Hilbert algebras X and Y, respectively. The cartesian product $A \times B = \{((x, y), (\mu_A \times \mu_B)(x, y), (\gamma_A \times \gamma_B)(x, y)) \mid x \in X, y \in Y\}$ defined by*

$$(\mu_A \times \mu_B)(x, y) = \mathrm{rmin}\{\mu_A(x), \mu_B(y)\}$$

and

$$(\gamma_A \times \gamma_B)(x, y) = \mathrm{rmax}\{\gamma_A(x), \gamma_B(y)\},$$

where $\mu_A \times \mu_B : X \times Y \to D[0,1]$ and $\gamma_A \times \gamma_B : X \times Y \to D[0,1]$ for all $x \in X$ and $y \in Y$.

Remark 5.1. *Let X and Y be Hilbert algebras. We define the binary operation $\cdot$ on $X \times Y$ by $(x, y) \cdot (u, v) = (x \cdot u, y \cdot v)$ for every $(x, y), (u, v) \in X \times Y$, then clearly $(X \times Y, \cdot, (1, 1))$ is a Hilbert algebra.*

Proposition 5.1. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IVIF subalgebras of Hilbert algebras X and Y, respectively, then the cartesian product $A \times B$ is also an IVIF subalgebra of $X \times Y$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
&(\mu_A \times \mu_B)((x_1, y_1) \cdot (x_2, y_2)) \\
&= (\mu_A \times \mu_B)((x_1 \cdot x_2), (y_1 \cdot y_2)) \\
&= \mathrm{rmin}\{\mu_A(x_1 \cdot x_2), \mu_B(y_1 \cdot y_2)\} \\
&\geq \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(x_1), \mu_A(x_2)\}, \mathrm{rmin}\{\mu_B(y_1), \mu_B(y_2)\}\} \\
&= \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(x_1), \mu_B(y_1)\}, \mathrm{rmin}\{\mu_A(x_2), \mu_B(y_2)\}\} \\
&= \mathrm{rmin}\{(\mu_A \times \mu_B)(x_1, y_1), (\mu_A \times \mu_B)(x_2, y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
&(\gamma_A \times \gamma_B)((x_1, y_1) \cdot (x_2, y_2)) \\
&= (\gamma_A \times \gamma_B)((x_1 \cdot x_2), (y_1 \cdot y_2)) \\
&= \mathrm{rmin}\{\gamma_A(x_1 \cdot x_2), \gamma_B(y_1 \cdot y_2)\} \\
&\leq \mathrm{rmin}\{\mathrm{rmax}\{\gamma_A(x_1), \gamma_A(x_2)\}, \mathrm{rmax}\{\gamma_B(y_1), \gamma_B(y_2)\}\} \\
&= \mathrm{rmax}\{\mathrm{rmin}\{\gamma_A(x_1), \gamma_B(y_1)\}, \mathrm{rmin}\{\gamma_A(x_2), \gamma_B(y_2)\}\} \\
&= \mathrm{rmax}\{(\mu_A \times \mu_B)(x_1, y_1), (\mu_A \times \mu_B)(x_2, y_2)\}.
\end{aligned}$$

Hence, $A \times B$ is an IVIF subalgebra of $X \times Y$. □

Lemma 5.1. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two IVIF subalgebras of Hilbert algebras X and Y, respectively, then $\oplus(A \times B) = (\mu_A \times \mu_B, \overline{\mu_A} \times \overline{\mu_B})$ is an IVIF subalgebra of $X \times Y$.*

Proof. It is sufficient to prove only the part of $\overline{\mu_A} \times \overline{\mu_B}$. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
&(\overline{\mu_A} \times \overline{\mu_B})((x_1, y_1) \cdot (x_2, y_2)) \\
&= (\overline{\mu_A} \times \overline{\mu_B})((x_1 \cdot x_2), (y_1 \cdot y_2)) \\
&= \mathrm{rmax}\{\overline{\mu_A}(x_1 \cdot x_2), \overline{\mu_B}(y_1 \cdot y_2)\} \\
&= \mathrm{rmax}\{1 - \mu_A(x_1 \cdot x_2), 1 - \mu_B(y_1 \cdot y_2)\} \\
&\leq \mathrm{rmax}\{1 - \mathrm{rmin}\{\mu_A(x_1), \mu_A(x_2)\}, 1 - \mathrm{rmin}\{\mu_B(y_1), \mu_B(y_2)\}\} \\
&= \mathrm{rmax}\{\mathrm{rmax}\{1 - \mu_A(x_1), 1 - \mu_B(y_1)\}, \mathrm{rmax}\{1 - \mu_A(x_2), 1 - \mu_B(y_2)\}\} \\
&= \mathrm{rmax}\{\mathrm{rmax}\{\overline{\mu_A}(x_1), \overline{\mu_B}(y_1)\}, \mathrm{rmax}\{\overline{\mu_A}(x_2), \overline{\mu_B}(y_2)\}\} \\
&= \mathrm{rmax}\{(\overline{\mu_A} \times \overline{\mu_B})(x_1, y_1), (\overline{\mu_A} \times \overline{\mu_B})(x_2, y_2)\}.
\end{aligned}$$

Hence, $\oplus(A \times B)$ is an IVIF subalgebra of $X \times Y$. □

Lemma 5.2. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two IVIF subalgebras of Hilbert algebras X and Y, respectively, then $\otimes(A \times B) = (\overline{\gamma_A} \times \overline{\gamma_B}, \gamma_A \times \gamma_B)$ is an IVIF subalgebra of $X \times Y$.*

Proof. It is sufficient to prove only the part of $\overline{\gamma_A} \times \overline{\gamma_B}$. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$
\begin{aligned}
&(\overline{\gamma_A} \times \overline{\gamma_B})((x_1, y_1) \cdot (x_2, y_2)) \\
&= (\overline{\gamma_A} \times \overline{\gamma_B})((x_1 \cdot x_2), (y_1 \cdot y_2)) \\
&= \text{rmin}\{\overline{\gamma_A}(x_1 \cdot x_2), \overline{\gamma_B}(y_1 \cdot y_2)\} \\
&= \text{rmin}\{1 - \gamma_A(x_1 \cdot x_2), 1 - \gamma_B(y_1 \cdot y_2)\} \\
&\geq \text{rmin}\{1 - \text{rmax}\{\gamma_A(x_1), \gamma_A(x_2)\}, 1 - \text{rmax}\{\gamma_B(y_1), \gamma_B(y_2)\}\} \\
&= \text{rmin}\{\text{rmin}\{1 - \gamma_A(x_1), 1 - \gamma_B(y_1)\}, \text{rmin}\{1 - \gamma_A(x_2), 1 - \gamma_B(y_2)\}\} \\
&= \text{rmin}\{\text{rmin}\{\overline{\gamma_A}(x_1), \overline{\gamma_B}(y_1)\}, \text{rmin}\{\overline{\gamma_A}(x_2), \overline{\gamma_B}(y_2)\}\} \\
&= \text{rmin}\{(\overline{\gamma_A} \times \overline{\gamma_B})(x_1, y_1), (\overline{\gamma_A} \times \overline{\gamma_B})(x_2, y_2)\}.
\end{aligned}
$$

Hence, $\otimes(A \times B)$ is an IVIF subalgebra of $X \times Y$. □

Theorem 5.1. *The IVIF sets $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IVIF subalgebras of Hilbert algebras X and Y, respectively if and only if $\oplus(A \times B)$ and $\otimes(A \times B)$ are IVIF subalgebras of $X \times Y$.*

Proof. It follows from Lemmas 5.1 and 5.2. □

Proposition 5.2. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two IVIF ideals of Hilbert algebras X and Y, respectively, then the cartesian product $A \times B$ is also an IVIF ideal of $X \times Y$.*

Proof. Let $(x, y) \in X \times Y$. Then

$$
\begin{aligned}
(\mu_A \times \mu_B)(1, 1) &= \text{rmin}\{\mu_A(1), \mu_B(1)\} \\
&\geq \text{rmin}\{\mu_A(x), \mu_B(y)\} \\
&= (\mu_A \times \mu_B)(x, y)
\end{aligned}
$$

and

$$
\begin{aligned}
(\gamma_A \times \gamma_B)(1, 1) &= \text{rmax}\{\gamma_A(1), \gamma_B(1)\} \\
&\leq \text{rmax}\{\gamma_A(x), \gamma_B(y)\}\} \\
&= (\gamma_A \times \gamma_B)(x, y).
\end{aligned}
$$

Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$
\begin{aligned}
(\mu_A \times \mu_B)((x_1, x_2) \cdot (y_1, y_2)) &= (\mu_A \times \mu_B)((x_1 \cdot y_1), (x_2 \cdot y_2)) \\
&= \text{rmin}\{\mu_A(x_1 \cdot y_1), \mu_B(x_2 \cdot y_2)\} \\
&\geq \text{rmin}\{\mu_A(y_1), \mu_B(y_2)\} \\
&= (\mu_A \times \mu_B)(y_1, y_2)
\end{aligned}
$$

and

$$
\begin{aligned}
(\gamma_A \times \gamma_B)((x_1, x_2) \cdot (y_1, y_2)) &= (\gamma_A \times \gamma_B)((x_1 \cdot y_1), (x_2 \cdot y_2)) \\
&= \text{rmax}\{\gamma_A(x_1 \cdot y_1), \gamma_B(x_2 \cdot y_2)\} \\
&\leq \text{rmax}\{\gamma_A(y_1), \gamma_B(y_2)\} \\
&= (\gamma_A \times \gamma_B)(y_1, y_2).
\end{aligned}
$$

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$. Then

$$\begin{aligned}
&(\mu_A \times \mu_B)(((x_2, y_2) \cdot ((x_3, y_3) \cdot (x_1, y_1))) \cdot (x_1, y_1))\\
&= (\mu_A \times \mu_B)(((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1)), (y_2 \cdot (y_3 \cdot y_1)) \cdot y_1))\\
&= \mathrm{rmin}\{\mu_A((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1), \mu_B((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1)\}\\
&\geq \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(x_2), \mu_A(x_3)\}, \mathrm{rmin}\{\mu_B(y_2), \mu_B(y_3)\}\}\\
&= \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(x_2), \mu_B(y_2)\}, \mathrm{rmin}\{\mu_A(x_3), \mu_B(y_3)\}\}\\
&= \mathrm{rmin}\{(\mu_A \times \mu_B)(x_2, y_2), (\mu_A \times \mu_B)(x_3, y_3)\}
\end{aligned}$$

and

$$\begin{aligned}
&(\gamma_A \times \gamma_B)(((x_2, y_2) \cdot ((x_3, y_3) \cdot (x_1, y_1))) \cdot (x_1, y_1))\\
&= (\gamma_A \times \gamma_B)(((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1)), (y_2 \cdot (y_3 \cdot y_1)) \cdot y_1))\\
&= \mathrm{rmax}\{\gamma_A((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1), \gamma_B((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1)\}\\
&\leq \mathrm{rmax}\{\mathrm{rmax}\{\gamma_A(x_2), \gamma_A(x_3)\}, \mathrm{rmax}\{\gamma_B(y_2), \gamma_B(y_3)\}\}\\
&= \mathrm{rmax}\{\mathrm{rmax}\{\gamma_A(x_2), \gamma_B(y_2)\}, \mathrm{rmax}\{\gamma_A(x_3), \gamma_B(y_3)\}\}\\
&= \mathrm{rmax}\{(\gamma_A \times \gamma_B)(x_2, y_2), (\gamma_A \times \gamma_B)(x_3, y_3)\}.
\end{aligned}$$

Hence, $A \times B$ is an IVIF ideal of $X \times Y$. □

Lemma 5.3. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two IVIF ideals of Hilbert algebras X and Y, respectively, then $\oplus(A \times B) = (\mu_A \times \mu_B, \overline{\mu_A} \times \overline{\mu_B})$ is an IVIF ideal of $X \times Y$.*

Proof. It is sufficient to prove only the part of $\overline{\mu_A} \times \overline{\mu_B}$. Let $(x, y) \in X \times Y$. Then

$$\begin{aligned}
(\overline{\mu_A} \times \overline{\mu_B})(1, 1) &= \mathrm{rmax}\{\overline{\mu_A}(1), \overline{\mu_B}(1)\}\\
&= \mathrm{rmax}\{1 - \mu_A(1), 1 - \mu_B(1)\}\\
&\leq \mathrm{rmax}\{1 - \mu_A(x), 1 - \mu_B(y)\}\\
&= (\overline{\mu_A} \times \overline{\mu_B})(x, y).
\end{aligned}$$

Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$\begin{aligned}
(\overline{\mu_A} \times \overline{\mu_B})((x_1, x_2) \cdot (y_1, y_2)) &= (\overline{\mu_A} \times \overline{\mu_B})((x_1 \cdot y_1), (x_2 \cdot y_2))\\
&= \mathrm{rmax}\{\overline{\mu_A}(x_1 \cdot y_1), \overline{\mu_B}(x_2 \cdot y_2)\}\\
&= \mathrm{rmax}\{1 - \mu_A(x_1 \cdot y_1), 1 - \mu_B(x_2 \cdot y_2)\}\\
&\leq \mathrm{rmax}\{1 - \mu_A(y_1), 1 - \mu_B(y_2)\}\\
&= \mathrm{rmax}\{\overline{\mu_A}(y_1), \overline{\mu_B}(y_2)\}\\
&= (\overline{\mu_A} \times \overline{\mu_B})(y_1, y_2).
\end{aligned}$$

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$. Then

$$\begin{aligned}
&(\overline{\mu_A} \times \overline{\mu_B})(((x_2, y_2) \cdot ((x_3, y_3) \cdot (x_1, y_1))) \cdot (x_1, y_1)) \\
&= (\overline{\mu_A} \times \overline{\mu_B})(((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1)), (y_2 \cdot (y_3 \cdot y_1)) \cdot y_1)) \\
&= \text{rmax}\{\overline{\mu_A}((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1), \overline{\mu_A}((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1)\} \\
&= \text{rmax}\{1 - \mu_A((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1), 1 - \mu_A((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1)\} \\
&\leq \text{rmax}\{1 - \text{rmin}\{\mu_A(x_2), \mu_A(x_3)\}, 1 - \text{rmin}\{\mu_B(y_2), \mu_B(y_3)\}\} \\
&= \text{rmax}\{\text{rmax}\{1 - \mu_A(x_2), 1 - \mu_B(y_2)\}, \text{rmax}\{1 - \mu_A(x_3), 1 - \mu_B(y_3)\}\} \\
&= \text{rmax}\{\text{rmax}\{\overline{\mu_A}(x_2), \overline{\mu_B}(y_2)\}, \text{rmax}\{\overline{\mu_A}(x_3), \overline{\mu_B}(y_3)\}\} \\
&= \text{rmax}\{(\overline{\mu_A} \times \overline{\mu_B})(x_2, y_2), (\overline{\mu_A} \times \overline{\mu_B})(x_3, y_3)\}.
\end{aligned}$$

Hence, $\oplus(A \times B)$ is an IVIF ideal of $X \times Y$. □

Lemma 5.4. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two IVIF ideals of Hilbert algebras X and Y, respectively, then $\otimes(A \times B) = (\overline{\gamma_A} \times \overline{\gamma_B}, \gamma_A \times \gamma_B)$ is an IVIF ideal of $X \times Y$.*

Proof. It is sufficient to prove only the part of $\overline{\gamma_A} \times \overline{\gamma_B}$. Let $(x, y) \in X \times Y$. Then

$$\begin{aligned}
(\overline{\gamma_A} \times \overline{\gamma_B})(1, 1) &= \text{rmin}\{\overline{\gamma_A}(1), \overline{\gamma_B}(1)\} \\
&= \text{rmin}\{1 - \gamma_A(1), 1 - \gamma_B(1)\} \\
&\geq \text{rmin}\{1 - \gamma_A(x), 1 - \gamma_B(y)\} \\
&= (\overline{\gamma_A} \times \overline{\gamma_B})(x, y).
\end{aligned}$$

Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$\begin{aligned}
(\overline{\gamma_A} \times \overline{\gamma_B})((x_1, x_2) \cdot (y_1, y_2)) &= (\overline{\gamma_A} \times \overline{\gamma_B})((x_1 \cdot y_1), (x_2 \cdot y_2)) \\
&= \text{rmin}\{\overline{\gamma_A}(x_1 \cdot y_1), \overline{\gamma_B}(x_2 \cdot y_2)\} \\
&= \text{rmin}\{1 - \gamma_A(x_1 \cdot y_1), 1 - \gamma_B(x_2 \cdot y_2)\} \\
&\geq \text{rmin}\{1 - \gamma_A(y_1), 1 - \gamma_B(y_2)\} \\
&= \text{rmin}\{\overline{\gamma_A}(y_1), \overline{\gamma_B}(y_2)\} \\
&= (\overline{\gamma_A} \times \overline{\gamma_B})(y_1, y_2).
\end{aligned}$$

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$. Then

$$\begin{aligned}
&(\overline{\gamma_A} \times \overline{\gamma_B})(((x_2, y_2) \cdot ((x_3, y_3) \cdot (x_1, y_1))) \cdot (x_1, y_1)) \\
&= (\overline{\gamma_A} \times \overline{\gamma_B})(((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1), ((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1)) \\
&= \text{rmin}\{\overline{\gamma_A}((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1), \overline{\gamma_A}((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1)\} \\
&= \text{rmin}\{1 - \gamma_A((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1), 1 - \gamma_A((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1)\} \\
&\geq \text{rmin}\{1 - \text{rmax}\{\gamma_A(x_2), \gamma_A(x_3)\}, 1 - \text{rmax}\{\gamma_B(y_2), \gamma_B(y_3)\}\} \\
&= \text{rmin}\{\text{rmin}\{1 - \gamma_A(x_2), 1 - \gamma_B(y_2)\}, \text{rmin}\{1 - \gamma_A(x_3), 1 - \gamma_B(y_3)\}\} \\
&= \text{rmin}\{\text{rmin}\{\overline{\gamma_A}(x_2), \overline{\gamma_B}(y_2)\}, \text{rmin}\{\overline{\gamma_A}(x_3), \overline{\gamma_B}(y_3)\}\} \\
&= \text{rmin}\{(\overline{\gamma_A} \times \overline{\gamma_B})(x_2, y_2), (\overline{\gamma_A} \times \overline{\gamma_B})(x_3, y_3)\}.
\end{aligned}$$

Hence, $\otimes(A \times B)$ is an IVIF subalgebra of $X \times Y$. □

Theorem 5.2. *The IVIF sets $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IVIF ideals of Hilbert algebras X and Y, respectively if and only if $\oplus(A \times B)$ and $\otimes(A \times B)$ are IVIF ideals of $X \times Y$.*

Proof. It follows from Lemmas 5.3 and 5.4. □

Theorem 5.3. *Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any two IVIF sets in Hilbert algebras X and Y, respectively. If $A \times B$ is an IVIF subalgebra of $X \times Y$, then nonempty upper $[s_1, s_2]$-level cut $U(\mu_A \times \mu_B : [s_1, s_2])$ and nonempty lower $[t_1, t_2]$-level cut $L(\gamma_A \times \gamma_B : [t_1, t_2])$ are subalgebras of $X \times Y$ for all $[s_1, s_2], [t_1, t_2] \in D[0, 1]$.*

Proof. It follows from Theorem 3.7. □

Theorem 5.4. *Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any two IVIF sets in Hilbert algebras X and Y, respectively. If $A \times B$ is an IVIF ideal of $X \times Y$, then nonempty upper $[s_1, s_2]$-level cut $U(\mu_A \times \mu_B : [s_1, s_2])$ and nonempty lower $[t_1, t_2]$-level cut $L(\gamma_A \times \gamma_B : [t_1, t_2])$ are ideals of $X \times Y$ for all $[s_1, s_2], [t_1, t_2] \in D[0, 1]$.*

Proof. It follows from Theorem 4.4. □

A mapping $f : X \to Y$ of Hilbert algebras is called a homomorphism if $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1) = 1$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras. For any IVIF set $A = (\mu_A, \gamma_A)$ in Y, we define an IVIF set $f^{-1}(A) = (\mu_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$ in X by

$$\mu_{f^{-1}(A)}(x) = \mu_A(f(x)) \text{ and } \gamma_{f^{-1}(A)}(x) = \gamma_A(f(x)) \ \forall x \in X.$$

Proposition 5.3. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y and $A = (\mu_A, \gamma_A)$ an IVIF subalgebra of Y. Then the inverse image $f^{-1}(A)$ of A is an IVIF subalgebra of X.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \mu_{f^{-1}(A)}(x \cdot y) &= \mu_A(f(x \cdot y)) \\ &= \mu_A(f(x) \cdot f(y)) \\ &\geq \text{rmin}\{\mu_A(f(x)), \mu_A(f(y))\} \\ &= \text{rmin}\{\mu_{f^{-1}(A)}(x), \mu_{f^{-1}(A)}(y)\} \end{aligned}$$

and

$$\begin{aligned} \gamma_{f^{-1}(A)}(x \cdot y) &= \gamma_A(f(x \cdot y)) \\ &= \gamma_A(f(x) \cdot f(y)) \\ &\leq \text{rmax}\{\gamma_A(f(x)), \gamma_A(f(y))\} \\ &= \text{rmax}\{\gamma_{f^{-1}(A)}(x), \gamma_{f^{-1}(A)}(y)\}. \end{aligned}$$

Hence, $f^{-1}(A)$ of A is an IVIF subalgebra of X. □

Theorem 5.5. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y and $A = (\mu_A, \gamma_A)$ be an IVIF ideal of Y. Then the inverse image $f^{-1}(A) = (\mu_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$ is an IVIF ideal of X.*

Proof. Since f is a homomorphism of X into Y, then $f(1) = 1 \in Y$ and, by the assumption, $\mu_A(f(1)) = \mu_A(1) \geq \mu_A(y)$ for every $y \in Y$. In particular, $\mu_A(f(1)) \geq \mu_A(f(x))$ for all $x \in X$. Hence, $\mu_{f^{-1}(A)}(1) \geq \mu_{f^{-1}(A)}(x)$ for all $x \in X$. Also, $\gamma_A(f(1)) = \gamma_A(1) \leq \gamma_A(y)$ for every $y \in Y$. In particular, $\gamma_B(f(1)) \leq \gamma_B(f(x))$ for all $x \in X$. Hence, $\gamma_{f^{-1}(A)}(1) \leq \gamma_{f^{-1}(A)}(x)$ for all $x \in X$, which proves (4.1). Now let $x, y \in X$. Then, by the assumption,

$$\mu_{f^{-1}(A)}(x \cdot y) = \mu_A(f(x \cdot y)) = \mu_A(f(x) \cdot f(y)) \geq \mu_A(f(y)) = \mu_{f^{-1}(A)}(y)$$

and

$$\gamma_{f^{-1}(A)}(x \cdot y) = \gamma_A(f(x \cdot y)) = \gamma_A(f(x) \cdot f(y)) \leq \gamma_A(f(y)) = \gamma_{f^{-1}(A)}(y),$$

which proves (4.2). Let $x, y_1, y_2 \in X$. Then by assumption,

$$\begin{aligned}
\mu_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \mu_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\
&= \mu_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\
&= \mu_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\
&= \mu_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x) \\
&\geq \text{rmin}\{\mu_A(f(y_1)), \mu_A(f(y_2))\} \\
&= \text{rmin}\{\mu_{f^{-1}(A)}(y_1), \mu_{f^{-1}(A)}(y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \gamma_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\
&= \gamma_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\
&= \gamma_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\
&= \gamma_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\
&\leq \text{rmax}\{\gamma_A(f(y_1)), \gamma_A(f(y_2))\} \\
&= \text{rmax}\{\gamma_{f^{-1}(A)}(y_1), \gamma_{f^{-1}(A)}(y_2)\},
\end{aligned}$$

which proves (4.3). Hence, $f^{-1}(A)$ is an IVIF ideal of X. □

6. Equivalence relations on IVIF subalgebras/ideals

Let $\mathscr{I}(H)$ be the family of all IVIF ideals of a Hilbert algebra X and let $t = [t_1, t_2] \in D[0, 1]$. Define binary relations U^t and L^t on $\mathscr{I}(H)$ as follows:

$$(A, B) \in U^t \;\Leftrightarrow\; U(\mu_A : t) = U(\mu_B : t), \;(A, B) \in L^t \;\Leftrightarrow\; L(\gamma_A : t) = L(\gamma_B : t),$$

respectively, for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ in $\mathscr{I}(H)$. Then clearly U^t and L^t are equivalence relations on $\mathscr{I}(H)$. For any $A = (\mu_A, \gamma_A) \in \mathscr{I}(H)$, let $[A]_{U^t}$ (resp., $[A]_{L^t}$) denote the equivalence class of A modulo U^t (resp., L^t), and denote by $\mathscr{I}(H)/U^t$ (resp., $\mathscr{I}(H)/L^t$) the system of all equivalence classes modulo U^t (resp., L^t); so

$$\mathscr{I}(H)/U^t = \{[A]_{U^t} \mid A = (\mu_A, \gamma_A) \in \mathscr{I}(H)\},$$

respectively,

$$\mathscr{I}(H)/L^t = \{[A]_{L^t} \mid A = (\mu_A, \gamma_A) \in \mathscr{I}(H)\}.$$

Now let $I(H)$ denote the family of all ideals of X and let $t = [t_1, t_2] \in D[0,1]$. Define maps f_t and g_t from $\mathscr{I}(H)$ to $I(H) \cup \{\emptyset\}$ by $f_t(A) = U(\mu_A : t)$ and $g_t(A) = L(\gamma_A : t)$, respectively, for all $A = (\mu_A, \gamma_A) \in \mathscr{I}(H)$. Then f_t and g_t are clearly well defined.

Theorem 6.1. *For any $t = [t_1, t_2] \in D[0,1]$, the maps f_t and g_t are surjective from $\mathscr{I}(H)$ to $I(H) \cup \{\emptyset\}$.*

Proof. Let $t = [t_1, t_2] \in D[0,1]$. Note that $\overline{\mathbf{0}} = (\mathbf{0}, \mathbf{1})$ is in $\mathscr{I}(H)$, where $\mathbf{0}$ and $\mathbf{1}$ are IVIF sets in X defined by $\mathbf{0}(x) = [0,0]$ and $\mathbf{1}(x) = [1,1]$ for all $x \in X$. Obviously, $f_t(\overline{\mathbf{0}}) = U(\mathbf{0}, t) = U([0,0] : [t_1, t_2]) = \emptyset = L([1,1] : [t_1, t_2]) = L(\mathbf{1} : t) = g_t(\overline{\mathbf{0}})$. Let $G(\neq \emptyset) \in I(H)$. For $\overline{G} = (\chi_G, \overline{\chi_G}) \in \mathscr{I}(H)$, we have $f_t(\overline{G}) = U(\chi_G : t) = G$ and $g_t(\overline{G}) = L(\overline{\chi_G}; t) = G$. Hence, f_t and g_t are surjective. □

Theorem 6.2. *The quotient sets $\mathscr{I}(H)/U^t$ and $\mathscr{I}(H)/L^t$ are equipotent to $I(H) \cup \{\emptyset\}$ for every $t = [t_1, t_2] \in D[0,1]$.*

Proof. For $t = [t_1, t_2] \in D[0,1]$, let f_t^* (resp., g_t^*) be a map from $\mathscr{I}(H)/U^t$ (resp., $\mathscr{I}(H)/L^t$) to $I(H) \cup \{\emptyset\}$ defined by $f_t^*([A]_{U^t}) = f_t(A)$ (resp., $g_t^*([A]_{L^t}) = g_t(A)$) for all $A = (\mu_A, \gamma_A) \in \mathscr{I}(H)$. If $U(\mu_A : t) = U(\mu_B : t)$ and $L(\gamma_A : t) = L(\gamma_B : t)$ for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B) \in \mathscr{I}(H)$, then $(A, B) \in U^t$ and $(A, B) \in L^t$; hence, $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. Therefore, the maps f_t^* and g_t^* are injective. Now let $G(\neq \emptyset) \in I(H)$. For $\overline{G} = (\chi_G, \overline{\chi_G}) \in \mathscr{I}(H)$, we have

$$f_t^*([\overline{G}]_{U^t} = f_t(\overline{G}) = U(\chi_G : t) = G,$$

$$g_t^*([\overline{G}]_{L^t} = g_t(\overline{G}) = L(\overline{\chi_G}, t) = G.$$

Finally, for $\overline{\mathbf{0}} = (\mathbf{0}, \mathbf{1}) \in \mathscr{I}(H)$, we get

$$f_t^*([\overline{\mathbf{0}}]_{U^t} = f_t(\overline{\mathbf{0}}) = U(\mathbf{0}, t) = \emptyset,$$

$$g_t^*([\overline{\mathbf{0}}]_{L^t} = g_t(\overline{\mathbf{0}}) = L(\mathbf{1}, t) = \emptyset.$$

This shows that f_t^* and g_t^* are surjective. □

For any $t = [t_1, t_2] \in D[0,1]$, we define another relation R^t on $\mathscr{I}(H)$ as follows:

$$(A, B) \in R^t \;\Leftrightarrow\; U(\mu_A : t) \cap L(\gamma_A : t) = U(\mu_B : t) \cap L(\gamma_B : t)$$

for any $A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B) \in \mathscr{I}(H)$. Then the relation R^t is also an equivalence relation on $\mathscr{I}(H)$.

Theorem 6.3. *For any $t = [t_1, t_2] \in D[0, 1]$, the map $\varphi_t : \mathscr{I}(H) \to I(H) \cup \{\emptyset\}$ is defined by $\varphi_t(A) = f_t(A) \cap g_t(A)$ for each $A = (\mu_A, \gamma_A) \in \mathscr{I}(H)$ as surjective.*

Proof. Let $t = [t_1, t_2] \in D[0, 1]$. For $\overline{\mathbf{0}} = (\mathbf{0}, \mathbf{1}) \in \mathscr{I}(H)$,

$$\varphi_t(\overline{\mathbf{0}}) = f_t(\overline{\mathbf{0}}) \cap g_t(\overline{\mathbf{0}}) = U(\mathbf{0}, t) \cap L(\mathbf{0}, t) = \emptyset.$$

For any $H \in \mathscr{I}(H)$, there exists $\overline{H} = (\chi_H, \overline{\chi_H}) \in \mathscr{I}(H)$ such that

$$\varphi_t(\overline{H}) = f_t(\overline{H}) \cap g_t(\overline{H}) = U(\chi_H : t) \cap L(\overline{\chi_H}, t) = H.$$

Hence, φ_t is surjective. □

Theorem 6.4. *For any $t = [t_1, t_2] \in D[0, 1]$, the quotient set $\mathscr{I}(H)/R^t$ is equipotent to $I(H) \cup \{\emptyset\}$.*

Proof. Let $t = [t_1, t_2] \in D[0, 1]$ and let $\varphi_t^* : \mathscr{I}(H)/R^t \to I(H) \cup \{\emptyset\}$ be a map defined by $\varphi_t^*([A]_{R^t}) = \varphi_t(A)$ for all $[A]_{R^t} \in \mathscr{I}(H)/R^t$. If $\varphi_t^*([A]_{R^t}) = \varphi_t^*([B]_{R^t})$ for any $[A]_{R^t}, [B]_{R^t} \in \mathscr{I}(H)/R^t$, then $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, that is, $U(\mu_A : t) \cap L(\gamma_A : t) = U(\mu_B : t) \cap L(\gamma_B : t)$; hence, $(A, B) \in R^t$. It follows that $[A]_{R^t} = [B]_{R^t}$ so that φ_t^* is injective. For $\overline{\mathbf{0}} = (\mathbf{0}, \mathbf{1}) \in \mathscr{I}(H)$,

$$\varphi_t^*([\overline{\mathbf{0}}]_{R^t}) = \varphi_t(\overline{\mathbf{0}}) = f_t(\overline{\mathbf{0}}) \cap g_t(\overline{\mathbf{0}}) = U(\mathbf{0}, t) \cap L(\mathbf{1}, t) = \emptyset.$$

If $H \in \mathscr{I}(H)$, then for $\overline{H} = (\chi_H, \overline{\chi_H}) \in \mathscr{I}(H)$, we have

$$\varphi_t^*([\overline{H}]_{R^t}) = \varphi_t(\overline{H}) = f_t(\overline{H}) \cap g_t(\overline{H}) = U(\chi_H : t) \cap L(\overline{\chi_H}, t) = H.$$

Hence, φ_t^* is surjective, this completes the proof. □

The same type of results are also true for IVIF subalgebras.

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