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Applications of Standard Methods for Solving the Electric Train Mathematical Model With Proportional Delay

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Abstract. In electric trains, the current is collected via a certain device, called the Pantograph. The governing mathematical model of such physical problem is well-known as the Pantograph delay differential equation (PDDE): $y'(t) = ay(t) + by(ct)$, where c is a proportional delay parameter. In the literature, a special case of the PDDE was analyzed when $c = -1$. The objective of this paper is to determine the general solution of the PDDE for arbitrary c. In addition, it will be shown that the obtained general solution reduces to exact one at $c = -1$. Such exact solution is expressed in terms of several types of functions, e.g., hyberbolic, Mittag-Leffler, and trigonometric functions. Moreover, it is declared that the exact trigonometric solution is periodic with periodicity $\frac{2\pi}{\sqrt{b^2-a^2}}$ which agrees with the corresponding results in the literature. Furthermore, the solution of PDDE is provided at almost all possible cases of the involved parameters a, b, and c. Finally, the solution Ambartsumian delay equation (ADE), which has an application in the theory of surface brightness in the Milky Way, will be recovered as a special case of our results.

1. Introduction

The field of Delay differential equations (DDEs) is a growing area of research due to its wide applications in applied sciences. A specific and famous application of the DDEs is well-known as the Pantograph by which the current can be collected in electric trains. The mathematical model describing the Pantograph technique is called the PDDE [1-5]. Also, other DDEs-models arise in astronomical applications, as an example, the Ambartsumian delay equation (ADE) [6-14] which is

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essential in studying the brightness in the Milky Way. To author's knowledge, there are no standard methods to solve DDEs in contrast to the Ordinary differential equations (ODEs). For examples, the simple ODE $y'(t) = ay(t)$ can be easily solved via applying the separation technique, while the simple DDE $y'(t) = ay(\omega t)$ $(\omega \neq 1)$ or $y'(t) = ay(t-\sigma)$ $(\sigma \in \mathbb{R})$ can't be solved neither by the separation technique nor by the other standard methods that dealing with ODEs. On the other hand, the DDEs can be viewed as a generalization of the ODEs because of the existing of delay parameters. For illustration, the DDEs $y'(t) = ay(\omega t)$ and $y'(t) = ay(t-\sigma)$ reduces to the ordinary version $y'(t) = ay(t)$ when $\omega = 1$ and $\sigma = 0$, respectively. In this paper, an extended version of the above DDEs in the form [15-19]:

$$y'(t) = ay(t) + by(ct), \quad y(0) = \lambda, \tag{1}$$

is to be analyzed for reals a, b, λ, and c. The model (1) is well-known in the literature [15-19] and called the PDDE.

This model reduces to the ADE [6-14]]; $y'(t) = -y(t) + \frac{1}{q}y\left(\frac{t}{q}\right), \quad y(0) = \lambda$ at the particular values $a = -1$ and $b = c = \frac{1}{q}$ $(q > 1)$. This last model and its several generalizations have been addressed by numerous researchers [6-14, 20, 21]. In Ref. [22], the special case of the PDDE, i.e., $y'(t) = ay(t) + by(-t)$ $(c = -1)$ has been solved. The main purpose of this work is to obtain the solution of the PDDE at arbitrary c. Our analysis is based on the standard series method (SSM) and the Maclaurin series expansion (MSE). Our results will also be used to derive previous results in the literature as special cases of ours. The convergence of the obtained series will also be addressed.

2. Solution by SSM

Based on the SSM, the solution of Eq. (1) is assumed as

$$y(t) = \sum_{n=0}^{\infty} d_n t^n. \tag{2}$$

From Eq. (3) and Eq. (1), it then follows

$$\sum_{n=1}^{\infty} n d_n t^{n-1} = a\sum_{n=0}^{\infty} d_n t^n + b\sum_{n=0}^{\infty} d_n c^n t^n, \tag{3}$$

i.e.,

$$\sum_{n=0}^{\infty}(n+1)d_{n+1}t^{n}=\sum_{n=0}^{\infty}\left(a+bc^{n}\right)d_{n}t^{n}, \tag{4}$$

which yields

$$\sum_{n=0}^{\infty}\left[(n+1)d_{n+1}-\left(a+bc^{n}\right)d_{n}\right]t^{n}=0. \tag{5}$$

Therefore

$$d_{n+1}=\left(\frac{a+bc^{n}}{n+1}\right)d_{n}, \qquad n\geq 0. \tag{6}$$

Accordingly,

$$\begin{aligned}
d_1 &= \frac{1}{1!}\left(a+b\right)d_0,\\
d_2 &= \frac{1}{2}\left(a+bc\right)d_1=\frac{1}{2}\left(a+b\right)\left(a+bc\right)d_0=\frac{1}{2!}\prod_{k=0}^{1}\left(a+bc^{k}\right)d_0,\\
d_3 &= \frac{1}{3}\left(a+bc^{2}\right)d_2=\frac{1}{6}\left(a+b\right)\left(a+bc\right)\left(a+bc^{2}\right)d_0=\frac{1}{3!}\prod_{k=0}^{2}\left(a+bc^{k}\right)d_0,\\
d_4 &= \frac{1}{4}\left(a+bc^{3}\right)d_3=\frac{1}{24}\left(a+b\right)\left(a+bc\right)\left(a+bc^{2}\right)\left(a+bc^{3}\right)d_0=\frac{1}{4!}\prod_{k=0}^{3}\left(a+bc^{k}\right)d_0,\\
&\vdots\\
d_n &= \frac{1}{n!}\left(a+b\right)\left(a+bc\right)\ldots\left(a+bc^{n-2}\right)\left(a+bc^{n-1}\right)d_0=\frac{1}{n!}\prod_{k=0}^{n-1}\left(a+bc^{k}\right)d_0.
\end{aligned} \tag{7}$$

Hence

$$\begin{aligned}
y(t) &= d_0+\sum_{n=1}^{\infty}d_n\ t^{n},\\
&= d_0+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=0}^{n-1}\left(a+bc^{k}\right)d_0\ t^{n},\\
&= d_0\left[1+\sum_{n=1}^{\infty}\prod_{k=0}^{n-1}\left(a+bc^{k}\right)\frac{t^{n}}{n!}\right].
\end{aligned} \tag{8}$$

Applying the initial condition $y(0)=\lambda$, yields

$$y(t)=\lambda\left[1+\sum_{n=1}^{\infty}\frac{t^{n}}{n!}\prod_{k=0}^{n-1}\left(a+bc^{k}\right)\right]. \tag{9}$$

3. Solution by the MSE

According to the MSE, we have

$$y(t)=\sum_{n=0}^{\infty}\frac{y^{(n)}(0)}{n!}t^{n}. \tag{10}$$

At $t=0$, we get from Eq. (1) that

$$y^{(1)}(0)=ay(0)+by\left(0\right)=\lambda\left(a+b\right). \tag{11}$$

Differentiating Eq. (11) w.r.t. to t, then

$$y^{(2)}(t)=ay^{(1)}(t)+bcy^{(1)}\left(ct\right), \tag{12}$$

and hence,

$$y^{(2)}(0)=\left(a+bc\right)y^{(1)}(0)=\lambda\left(a+b\right)\left(a+bc\right). \tag{13}$$

Similarly, we have from Eq. (13) that

$$y^{(3)}(t)=ay^{(2)}(t)+bc^{2}y^{(2)}\left(ct\right). \tag{14}$$

Accordingly,

$$y^{(3)}(0)=\left(a+bc^{2}\right)y^{(2)}\left(0\right)=\lambda\left(a+b\right)\left(a+bc\right)\left(a+bc^{2}\right). \tag{15}$$

Repeating this procedure n-times, we get

$$\begin{aligned} y^{(n)}(0)&=\lambda\left(a+b\right)\left(a+bc\right)\left(a+bc^{2}\right)\ldots\ldots\left(a+bc^{n-2}\right)\left(a+bc^{n-1}\right),\\ &=\lambda\prod_{k=0}^{n-1}\left(a+bc^{k}\right),\qquad n\geq 1. \end{aligned} \tag{16}$$

Therefore

$$\begin{aligned} y(t)&=y(0)+\sum_{n=1}^{\infty}y^{(n)}(0)\frac{t^{n}}{n!},\\ &=\lambda+\lambda\sum_{n=1}^{\infty}\prod_{k=0}^{n-1}\left(a+bc^{k}\right)\frac{t^{n}}{n!},\\ &=\lambda\left[1+\sum_{n=1}^{\infty}\frac{t^{n}}{n!}\prod_{k=0}^{n-1}\left(a+bc^{k}\right)\right], \end{aligned} \tag{17}$$

which is also the same solution obtained in the previous section.

4. Convergence analysis

Theorem 1. *For $a, b \in \mathbb{R}$, the closed-form series solution:*

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=0}^{n-1} \left(a + bc^k \right) \right], \tag{18}$$

has infinite radius of convergence $\forall\ c \in [-1, 1]$ and hence the series is uniformly convergent on any compact interval on $\mathbb{R}$.

Proof: Eq. (18) can be expressed as

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} C_n(t) \right], \tag{19}$$

where C_n is defined by

$$C_n(t) = \frac{t^n}{n!} \prod_{k=0}^{n-1} \left(a + bc^k \right), \quad n \geq 1. \tag{20}$$

Let ρ is the radius of convergence, by ratio test we can write

$$\begin{aligned} \frac{1}{\rho} &= \lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{t^{n+1}}{(n+1)!} \prod_{k=0}^{n} \left(a + bc^k \right)}{\frac{t^n}{n!} \prod_{k=0}^{n-1} \left(a + bc^k \right)} \right|, \\ &= \lim_{n\to\infty} \left| \frac{(a + bc^n) \prod_{k=0}^{n-1} \left(a + bc^k \right)}{(n+1) \prod_{k=0}^{n-1} \left(a + bc^k \right)} \right| |t|, \\ &= \lim_{n\to\infty} \left| \frac{a + bc^n}{n+1} \right| |t|. \end{aligned} \tag{21}$$

For $c \in (-1, 1)$, we have $\lim_{n\to\infty} c^n = 0$, then

$$\frac{1}{\rho} = \lim_{n\to\infty} \left| \frac{a + bc^n}{n+1} \right| |t| = 0, \quad \forall\ c \in (-1, 1),\ t \in \mathbb{R}. \tag{22}$$

At $c = 1$ we have $c^n = 1$ ($\forall n \in \mathbb{N}$), then the limit in Eq. (22) reduces to

$$\frac{1}{\rho} = \lim_{n\to\infty} \left| \frac{a + b}{n+1} \right| |t| = 0, \quad \text{where } c = 1,\ t \in \mathbb{R}. \tag{23}$$

At $c = -1$ we have $c^n = -1$ (if n is odd) and $c^n = 1$ (if n is even), Accordingly, the limit (22) becomes

$$\frac{1}{\rho} = \lim_{n\to\infty} \left| \frac{a \pm b}{n+1} \right| |t| = 0, \quad \text{where } c = -1,\ t \in \mathbb{R}, \tag{24}$$

which completes the proof. □

5. Special cases & Comparisons

It was shown in the previous section that the solution of the PDDE is given by the closed-form:

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=0}^{n-1} \left(a + bc^k \right) \right]. \tag{25}$$

In this section, the solutions of several special cases are determined and some of them are compared with those in the relevant literature.

5.1. $\lambda = 1$.

At $\lambda = 1$, the PDDE (1) becomes

$$y'(t) = ay(t) + by\left(ct\right), \quad y(0) = 1, \tag{26}$$

which has been solved by Fox et. al [19]. The solution of Eqs. (26) can be directly obtained from (18) by substituting $\lambda = 1$, and this gives

$$y(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=0}^{n-1} \left(a + bc^k \right), \tag{27}$$

which is the corresponding solution in [19].

5.2. $b = 0$.

At $b = 0$, the present PDDE reduces to

$$y'(t) = ay(t), \quad y(0) = \lambda, \tag{28}$$

which has the exact solution:

$$y(t) = \lambda e^{at}. \tag{29}$$

Such exact solution can also be derived as a special case of our solution (18). To do that, we substitute $b = 0$ into (18) to obtain

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=0}^{n-1} a \right], \tag{30}$$

i.e.,

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!} \right], \tag{31}$$

where

$$\prod_{k=0}^{n-1} a = a^n, \quad \forall\ n \geq 1. \tag{32}$$

Eq. (32) can be written as

$$y(t) = \lambda \sum_{n=0}^{\infty} \frac{(at)^n}{n!} = \lambda e^{at}, \tag{33}$$

which agrees with the exact solution of given in Eq. (29).

5.3. $a = 0$. At $a = 0$, the PDDE yields

$$y'(t) = by(ct), \quad y(0) = \lambda. \tag{34}$$

The solution of such case can be deduced from (18) as follows. Inserting $a = 0$ into (18) leads to

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=0}^{n-1} \left(bc^k\right)\right] = \lambda \left[1 + \sum_{n=1}^{\infty} c^{\frac{1}{2}n(n-1)} \frac{(bt)^n}{n!}\right], \tag{35}$$

where

$$\prod_{k=0}^{n-1} \left(bc^k\right) = b^n c^{\frac{1}{2}n(n-1)}, \quad \forall\ n \geq 1. \tag{36}$$

The solution (36) is equivalent to the form:

$$y(t) = \lambda \sum_{n=0}^{\infty} c^{\frac{1}{2}n(n-1)} \frac{(bt)^n}{n!}. \tag{37}$$

5.4. $c = 1$. At $c = 1$, we have PDDE becomes the ODE:

$$y'(t) = (a + b)y(t), \quad y(0) = \lambda, \tag{38}$$

and its solution is known as

$$y(t) = \lambda e^{(a+b)t}. \tag{39}$$

This solution can also be obtained by substituting $c = 1$ into (18) which gives

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=0}^{n-1} (a + b)\right], \tag{40}$$

i.e.,

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{(a + b)^n t^n}{n!}\right], \tag{41}$$

where

$$\prod_{k=0}^{n-1} (a + b) = (a + b)^n, \quad \forall\ n \geq 1. \tag{42}$$

Eq. (42) can be written as

$$y(t) = \lambda \sum_{n=0}^{\infty} \frac{((a + b)t)^n}{n!} = \lambda e^{(a+b)t}, \tag{43}$$

which is the same exact solution of the Eqs. (39).

5.5. $c = -1$, $a \neq \pm b$. At $c = -1$, the PDDE becomes

$$y'(t) = ay(t) + by(-t), \quad y(0) = \lambda. \tag{44}$$

Indeed, the model (44) is interesting and its exact solution is obtained here in terms of hyperbolic and trigonometric functions. In addition, such exact solution will be expressed in an equivalent form via Mittag-Leffler functions. Substituting $c = -1$ into (18), we obtain

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!}\psi_n\right], \tag{45}$$

where ψ_n is defined by

$$\psi_n = \prod_{k=0}^{n-1} \left(a + b(-1)^k\right), \quad n \geq 1. \tag{46}$$

Using the products rules, it can be shown that

$$\psi_n = \prod_{k=0}^{n-1} \left(a + b(-1)^k\right) = \begin{cases} (a^2 - b^2)^{\frac{n}{2}}, & \text{if } n \text{ even}, \\ \\ (a+b)^{\frac{n+1}{2}}(a-b)^{\frac{n-1}{2}}, & \text{if } n \text{ odd}. \end{cases}. \tag{47}$$

The series solution (45) can be written as

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!}\psi_{2n-1} + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\psi_{2n}\right], \tag{48}$$

where ψ_{2n-1} and ψ_{2n} are obtained from (47) by

$$\psi_{2n-1} = (a+b)^n(a-b)^{n-1}, \qquad \psi_{2n} = (a^2 - b^2)^n. \tag{49}$$

Accordingly, Eq. (48) gives

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!}(a+b)^n(a-b)^{n-1} + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}(a^2 - b^2)^n\right], \tag{50}$$

i.e.,

$$y(t) = \lambda \left[1 + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}(a+b)^{n+1}(a-b)^n + \sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!}(a^2 - b^2)^{n+1}\right]. \tag{51}$$

5.5.1. *Hyperbolic functions ($a > b$).* Eq. (51) gives

$$
\begin{aligned}
y(t) &= \lambda \left[1 + (a+b)\sum_{n=0}^{\infty} \frac{\left(\sqrt{a^2-b^2}\right)^{2n} t^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{\left(\sqrt{a^2-b^2}\right)^{2n+2} t^{2n+2}}{(2n+2)!}\right], \\
&= \lambda \left[1 + \frac{a+b}{\sqrt{a^2-b^2}}\sum_{n=0}^{\infty} \frac{\left(\sqrt{a^2-b^2}\right)^{2n+1} t^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{\left(\sqrt{a^2-b^2}\ t\right)^{2n+2}}{(2n+2)!}\right], \\
&= \lambda \left[1 + \sqrt{\frac{a+b}{a-b}}\sum_{n=0}^{\infty} \frac{\left(\sqrt{a^2-b^2}\ t\right)^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{\left(\sqrt{a^2-b^2}\ t\right)^{2n+2}}{(2n+2)!}\right], \\
&= \lambda \left[1 + \sqrt{\frac{a+b}{a-b}}\sinh\left(\sqrt{a^2-b^2}\ t\right) + \sum_{n=1}^{\infty} \frac{\left(\sqrt{a^2-b^2}\ t\right)^{2n}}{(2n)!}\right], \\
&= \lambda \left[\sqrt{\frac{a+b}{a-b}}\sinh\left(\sqrt{a^2-b^2}\ t\right) + \cosh\left(\sqrt{a^2-b^2}\ t\right)\right], \quad a > b. \qquad (52)
\end{aligned}
$$

The curves of the hyperbolic solution in Eq. (52) are depicted in Fig. 1 at three different sets for the values of a and b.

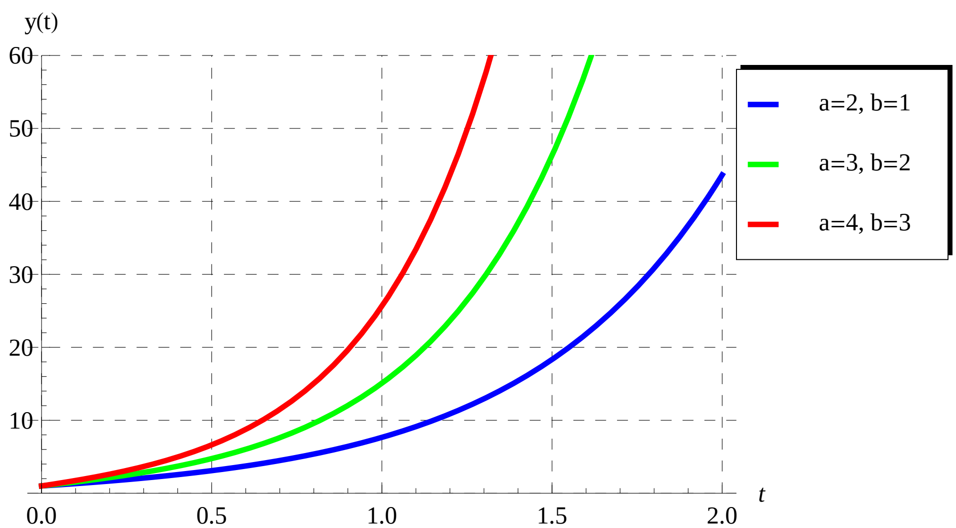

Figure 1. Plots of the hyperbolic solution in Eq. (52) at $a = 2, b = 1$, $a = 3, b = 2$, and $a = 4, b = 3$.

5.5.2. *Mittag-Leffler functions.* Here, it is noted that Eq. (51) can be written using the Gamma function in the form:

$$
y(t) = \lambda \left[1 + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{\Gamma(2n+2)}(a+b)^{n+1}(a-b)^{n} + \sum_{n=0}^{\infty} \frac{t^{2n+2}}{\Gamma(2n+3)}(a^2-b^2)^{n+1}\right], \qquad (53)
$$

or

$$y(t)=\lambda\left[1+(a+b)t\sum_{n=0}^{\infty}\frac{\left((a^2-b^2)t^2\right)^n}{\Gamma(2n+2)}+(a^2-b^2)t^2\sum_{n=0}^{\infty}\frac{\left((a^2-b^2)t^2\right)^n}{\Gamma(2n+3)}\right]. \tag{54}$$

Using the definition of the two-parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{55}$$

then Eq. (54) takes the following final form:

$$y(t)=\lambda\left[1+(a+b)tE_{2,2}\left((a^2-b^2)t^2\right)+(a^2-b^2)t^2E_{2,3}\left((a^2-b^2)t^2\right)\right]. \tag{56}$$

This last form can also be used to establish the solution in terms of trigonometric functions via some properties of the Mittag-Leffler functions as shown below.

5.5.3. *Trigonometric functions ($b > a$).* Suppose that $b > a$, then we can rewrite (56) as

$$y(t)=\lambda\left[1+(a+b)tE_{2,2}\left(-(\sqrt{b^2-a^2}\ t)^2\right)-(b^2-a^2)t^2E_{2,3}\left(-(\sqrt{b^2-a^2}\ t)^2\right)\right]. \tag{57}$$

Applying the following properties [21]:

$$E_{2,2}(-z^2)=\frac{\sin(z)}{z}, \qquad E_{2,3}(-z^2)=\frac{1-\cos(z)}{z^2}, \tag{58}$$

for $z=\sqrt{b^2-a^2}\ t$ we have

$$E_{2,2}\left(-(\sqrt{b^2-a^2}\ t)^2\right)=\frac{\sin\left(\sqrt{b^2-a^2}\ t\right)}{\sqrt{b^2-a^2}\ t}, \tag{59}$$

$$E_{2,3}\left(-(\sqrt{b^2-a^2}\ t)^2\right)=\frac{1-\cos\left(\sqrt{b^2-a^2}\ t\right)}{(b^2-a^2)t^2}. \tag{60}$$

Substituting (59) and (60) into (57) and simplifying, we obtain

$$y(t)=\lambda\left[\sqrt{\frac{b+a}{b-a}}\sin\left(\sqrt{b^2-a^2}\ t\right)+\cos\left(\sqrt{b^2-a^2}\ t\right)\right], \quad b>a. \tag{61}$$

This also the same solution reported by Ebaid and Al-Jeaid [22]. It can be easily seen that the solution is periodic with periodicity $P=\frac{2\pi}{\sqrt{b^2-a^2}}$. Fig. 2 shows three different periodic solutions with $P=2\pi,\pi,$ and $\pi/2$.

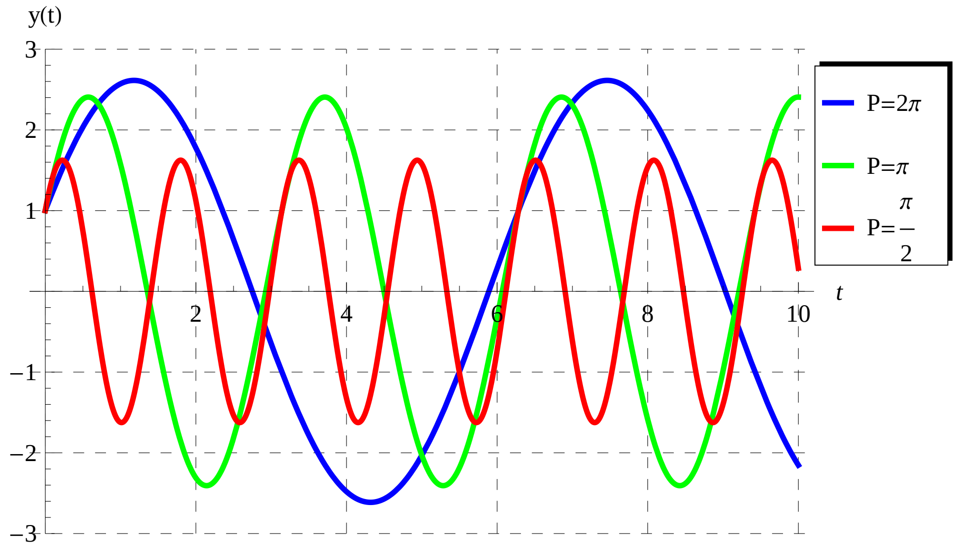

Figure 2. Plots of the periodic solution in Eq. (61) at $a = 1, b = \sqrt{2}$ $(P = 2\pi)$, $a = \sqrt{3}, b = \sqrt{7}$ $(P = \pi)$, and $a = 1, b = \sqrt{17}$ $(P = \frac{\pi}{2})$.

6. The ADE $(a = -1, b = c = \frac{1}{q}, q > 1)$

At $a = -1$ and $b = c = \frac{1}{q}$ $(q > 1)$, the PDDE (1) becomes the ADE [10-12]:

$$y'(t) = -y(t) + \frac{1}{q} y\left(\frac{t}{q}\right), \quad y(0) = \lambda, \tag{62}$$

Substituting the above values into (18) implies that

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=0}^{n-1} \left(q^{-(k+1)} - 1\right)\right], \tag{63}$$

i.e.,

$$y(t) = \lambda \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=1}^{n} \left(q^{-k} - 1\right)\right]. \tag{64}$$

which is in full agreement with the corresponding result in [10].

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] H.I. Andrews, Third Paper: Calculating the Behaviour of an Overhead Catenary System for Railway Electrification, Proc. Inst. Mech. Eng. 179 (1964), 809–846. https://doi.org/10.1243/pime_proc_1964_179_050_02.

[2] M.R. Abbott, Numerical Method for Calculating the Dynamic Behaviour of a Trolley Wire Overhead Contact System for Electric Railways, Computer J. 13 (1970), 363–368. https://doi.org/10.1093/comjnl/13.4.363.

[3] G. Gilbert, H.E.H. Davies, Pantograph Motion on a Nearly Uniform Railway Overhead Line, Proc. Inst. Electr. Eng. 113 (1966), 485-492. https://doi.org/10.1049/piee.1966.0078.

[4] P.M. Caine, P.R. Scott, Single-Wire Railway Overhead System, Proc. Inst. Electr. Eng. 116 (1969), 1217-1221. https://doi.org/10.1049/piee.1969.0226.

[5] J.R. Ockendon, A.B. Tayler, The Dynamics of a Current Collection System for an Electric Locomotive, Proc. R. Soc. Lond. A. 322 (1971), 447–468. https://doi.org/10.1098/rspa.1971.0078.

[6] V.A. Ambartsumian, On the Fluctuation of the Brightness of the Milky Way, Dokl. Akad. Nauk USSR, 44 (1994), 223-226.

[7] J. Patade, S. Bhalekar, On Analytical Solution of Ambartsumian Equation, Natl. Acad. Sci. Lett. 40 (2017), 291–293. https://doi.org/10.1007/s40009-017-0565-2.

[8] F.M. Alharbi, A. Ebaid, New Analytic Solution for Ambartsumian Equation, J. Math. Syst. Sci. 8 (2018), 182-186. https://doi.org/10.17265/2159-5291/2018.07.002.

[9] H. Bakodah, A. Ebaid, Exact Solution of Ambartsumian Delay Differential Equation and Comparison with Daftardar-Gejji and Jafari Approximate Method, Mathematics. 6 (2018), 331. https://doi.org/10.3390/math6120331.

[10] N.O. Alatawi, A. Ebaid, Solving a Delay Differential Equation by Two Direct Approaches, J. Math. Syst. Sci. 9 (2019), 54-56. https://doi.org/10.17265/2159-5291/2019.02.003.

[11] A. Ebaid, A. Al-Enazi, B.Z. Albalawi, M.D. Aljoufi, Accurate Approximate Solution of Ambartsumian Delay Differential Equation via Decomposition Method, Math. Comput. Appl. 24 (2019), 7. https://doi.org/10.3390/mca24010007.

[12] A.A. Alatawi, M. Aljoufi, F.M. Alharbi, A. Ebaid, Investigation of the Surface Brightness Model in the Milky Way via Homotopy Perturbation Method, J. Appl. Math. Phys. 8 (2020), 434–442. https://doi.org/10.4236/jamp.2020.83033.

[13] E. A. Algehyne, E. R. El-Zahar, F. M. Alharbi, A. Ebaid, Development of Analytical Solution for a Generalized Ambartsumian Equation, AIMS Math. 5 (2020), 249–258. https://doi.org/10.3934/math.2020016.

[14] S.M. Khaled, E.R. El-Zahar, A. Ebaid, Solution of Ambartsumian Delay Differential Equation with Conformable Derivative, Mathematics. 7 (2019), 425. https://doi.org/10.3390/math7050425.

[15] T. Kato, J.B. McLeod, The Functional-Differential Equation $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc. 77 (1971), 891-935.

[16] A. Iserles, On the Generalized Pantograph Functional-Differential Equation, Eur. J. Appl. Math. 4 (1993), 1–38. https://doi.org/10.1017/s0956792500000966.

[17] G. Derfel, A. Iserles, The Pantograph Equation in the Complex Plane, J. Math. Anal. Appl. 213 (1997), 117–132. https://doi.org/10.1006/jmaa.1997.5483.

[18] J. Patade, S. Bhalekar, Erratum to: Analytical Solution of Pantograph Equation with Incommensurate Delay, Phys. Sci. Rev. 3 (2018), 20165103. https://doi.org/10.1515/psr-2016-9103.

[19] L. Fox, D.F. Mayers, J.R. Ockendon, A.B. Tayler, On a Functional Differential Equation, IMA J. Appl. Math. 8 (1971), 271–307. https://doi.org/10.1093/imamat/8.3.271.

[20] D. Kumar, J. Singh, D. Baleanu, S. Rathore, Analysis of a Fractional Model of the Ambartsumian Equation, Eur. Phys. J. Plus. 133 (2018), 259. https://doi.org/10.1140/epjp/i2018-12081-3.

[21] A. Ebaid, C. Cattani, A.S. Al Juhani, E.R. El-Zahar, A Novel Exact Solution for the Fractional Ambartsumian Equation, Adv. Differ. Equ. 2021 (2021), 88. https://doi.org/10.1186/s13662-021-03235-w.

[22] A. Ebaid, H.K. Al-Jeaid, On the Exact Solution of the Functional Differential Equation $y'(t) = ay(t) + by(-t)$, Adv. Differ. Equ. Control Processes, 26 (2022), 39-49. https://doi.org/10.17654/0974324322003.