Int. J. Anal. Appl. (2022), 20:29

International Journal of Analysis and Applications

# Applications of Spherical Fuzzy Sets in Ternary Semigroups

**Wasitthirawat Krailoet**[1], **Ronnason Chinram**[1,*] **Montakarn Petapirak**[1], **Aiyared Iampan**[2]

[1] *Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand*

[2] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

[*] *Corresponding author:* ronnason.c@psu.ac.th

Abstract. In this paper, we introduce the notions of spherical fuzzy ternary subsemigroups and spherical fuzzy ideals in ternary semigroups by using the concepts of ternary subsemigroups and ideals in ternary semigroups. We investigate their properties. Moreover, we study roughness of spherical fuzzy ideals in ternary semigroups.

## 1. Introduction

The theory of ternary algebraic system was investigated by Lehmer [8] in 1932, but earlier such structures were studied by Kasner [5] who gave the idea of $n$-ary algebras. Furthermore, the ideal theory in ternary semigroups was established by Sioson [12]. In 1965, the notion of fuzzy sets was initiated by Zadeh [14]. The fuzzy set is an extension of classical sets and represented by using a generalization of the indicator of classical sets that is called a membership function. Later, the concept of fuzzy set was applied to study in many algebraic structures. In 1981, Kuroki [6] provided some properties of fuzzy ideals. In 2013, Iampan [4] gave the definition and characterized the properties of ideal extensions in ternary semigroups. After the introduction of ordinary fuzzy sets, the concept of rough sets was given by Pawlak [10] in 1982 which is defined depending on some equivalence relation on a universal finite set. The combination of theories of fuzzy sets and rough sets has been discussed in many research papers through all the years until 1990, when Dubois and Prade [3] proposed the notion of

Received: May 10, 2022.

2010 *Mathematics Subject Classification.* 03E72.

*Key words and phrases.* ideals; fuzzy ideals; rough fuzzy sets; spherical fuzzy sets; rough ideals; rough spherical fuzzy ideals; ternary semigroups.

https://doi.org/10.28924/2291-8639-20-2022-29

ISSN: 2291-8639

rough fuzzy sets. In 2009, Petchkhaew and Chinram [11] studied fuzzy, rough and rough fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) of ternary semigroups. Later, in 2012, Kar and Sarkar [7] focused on studying fuzzy ideals of ternary semigroups and their related properties. In 2016, Wang and Zhan [13] established the rough semigroups and the rough fuzzy semigroups based on fuzzy ideals. In 2019, Ashraf et al. [1] introduced the notion of spherical fuzzy set, which is a generalization of the picture fuzzy sets, intuitionistic fuzzy sets and Pythagorean fuzzy sets when the degree of abstinence is involved, as it provides enlargement of the space of degrees of truthfulness (membership), abstinence (hesitancy) and falseness (non-membership). Recently, in 2020, Chinram and Panityakul [2] introduced rough Pythagorean fuzzy ideals in ternary semigroups and gave some remarkable properties. Our aim of this paper is to study spherical fuzzy ternary subsemigroups and spherical fuzzy ideals in ternary semigroups by using the concepts of ternary subsemigroups and ideals in ternary semigroups. Moreover, we study roughness of spherical fuzzy sets and spherical fuzzy ideals in ternary semigroups.

## 2. Preliminaries

In this section, we shall recall some basic definitions that will be used in this paper.

2.1. **Ternary Semigroups.** A non-empty set $T$ together with a ternary operation, called ternary multiplication, denoted by juxtaposition, is said to be a *ternary semigroup* if

$$(abc)de = a(bcd)e = ab(cde)$$

for all $a, b, c, d, e \in T$. For any three non-empty subsets $A, B$ and $C$ of a ternary semigroup $T$, a *product* $ABC$ is defined by

$$ABC = \{ abc \mid a \in A, b \in B \text{ and } c \in C \}.$$

**Example 2.1.** (1) The following example (Banach's Example) shows that a ternary semigroup does not necessarily reduce an ordinary semigroup. Let $T = \{-i, 0, i\}$ be a ternary semigroup under ternary multiplication over $\mathbb{C}$. We obtain that $T$ is not a semigroup under multiplication over $\mathbb{C}$.

(2) Let $\mathbb{Z}^-$ be the set of all negative integers. Then $\mathbb{Z}^-$ is a ternary semigroup under ternary multiplication over $\mathbb{Z}$. We obtain that $\mathbb{Z}^-$ is not a semigroup under multiplication over $\mathbb{Z}$.

(3) The set of all odd permutation is a ternary semigroup under ternary composition. It is not a semigroup under composition.

A non-empty subset $S$ of a ternary semigroup $T$ is called a *ternary subsemigroup* of $T$ if $S^3 \subseteq S$. Let $I$ be a non-empty subset of a ternary semigroup $T$. Then $I$ is called a *left ideal* of $T$ if $TTI \subseteq I$, a *lateral ideal* of $T$ if $TIT \subseteq I$ and a *right ideal* of $T$ if $ITT \subseteq I$. A non-empty subset $I$ of a ternary semigroup $T$ is called an *ideal* of $T$ if $I$ is a left ideal, a lateral ideal and a right ideal of $T$. An ideal $I$ of a ternary semigroup $T$ is called a *proper ideal* if $I \neq T$.

2.2. **Fuzzy sets.** A fuzzy subset of a set $S$ is a function $S \times S \to [0,1]$. Let $f$ and $g$ be any two fuzzy subsets of any set $S$.

(1) $f \subseteq g$ if $f(a) \leq g(a)$ for all $a \in S$.
(2) $(f \cap g)(a) = \min\{f(a), g(a)\}$ for all $a \in S$.
(3) $(f \cup g)(a) = \max\{f(a), g(a)\}$ for all $a \in S$.

A fuzzy subset $f$ of a ternary semigroup $T$ is called a *fuzzy ternary subsemigroup* of $T$ if

$$f(xyz) \geq \min\{f(x), f(y), f(z)\}$$

for all $x, y, z \in T$. A fuzzy subset $f$ of $T$ is called a *fuzzy left ideal* of $T$ if $f(xyz) \geq f(z)$ for all $x, y, z \in T$, a *fuzzy lateral ideal* of $T$ if $f(xyz) \geq f(y)$ for all $x, y, z \in T$ and a *fuzzy right ideal* of $T$ if $f(xyz) \geq f(x)$ for all $x, y, z \in T$. A fuzzy subset $f$ of a ternary semigroup $T$ is called a *fuzzy ideal* of $T$ if it is a fuzzy left ideal, a fuzzy lateral ideal and a fuzzy right ideal of $T$, i.e., $f(xyz) \geq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

For any three fuzzy sets $f_1$, $f_2$ and $f_3$ of a ternary semigroup $T$. The *product* $f_1 \circ f_2 \circ f_3$ of $f_1$, $f_2$ and $f_3$ is defined by

$$(f_1 \circ f_2 \circ f_3)(y) = \begin{cases} \sup_{y = y_1 y_2 y_3} \min\{f_1(y_1), f_2(y_2), f_3(y_3)\} & \text{if } y \in T^3, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the product $f_1 \circ f_2 \circ f_3$ of fuzzy subsets $f_1$, $f_2$ and $f_3$ of a ternary semigroup $T$ is also a fuzzy subset of $T$. Let $\mathcal{F}(T)$ be the set of all fuzzy subsets of a ternary semigroup $T$. Then $\mathcal{F}(T)$ is a ternary semigroup under this product.

2.3. **Spherical Fuzzy Sets.** Let $S$ be a universal set. A *spherical fuzzy set* on $S$

$$\mathcal{S} := \{< x, \mu_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x), \nu_{\mathcal{S}}(x) >| \, x \in S\}$$

where $\mu_{\mathcal{S}} : S \to [0,1]$, $\eta_{\mathcal{S}} : S \to [0,1]$ and $\nu_{\mathcal{S}} : S \to [0,1]$ represent the degree of membership, the degree of hesitancy and the degree of non-membership of $x \in S$ with the condition $0 \leq (\mu_{\mathcal{S}}(x))^2 + (\eta_{\mathcal{S}}(x))^2 + (\nu_{\mathcal{S}}(x))^2 \leq 1$.

We may also denote a spherical fuzzy set $\mathcal{S}$ by $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$.

**Example 2.2.** Let $f$ be any fuzzy subset of a set $S$. Let $\mu_{\mathcal{S}} : S \to [0,1]$, $\eta_{\mathcal{S}} : S \to [0,1]$ and $\nu_{\mathcal{S}} : S \to [0,1]$ be defined by

$$\mu_{\mathcal{S}}(x) = f(x), \eta_{\mathcal{S}}(x) = 0 \text{ and } \nu_{\mathcal{S}}(x) = 1 - f(x).$$

Then $\mathcal{S} := \{< x, \mu_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x), \nu_{\mathcal{S}}(x) >| \, x \in S\}$ is a spherical fuzzy set on $S$.

Let $\mathcal{S}_1 = (\mu_{\mathcal{S}_1}, \eta_{\mathcal{S}_1}, \nu_{\mathcal{S}_1})$ and $\mathcal{S}_2 = (\mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_2})$ be any two spherical fuzzy set of a universal set $S$. We say that $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $\mu_{\mathcal{S}_1}(x) \leq \mu_{\mathcal{S}_2}(x), \eta_{\mathcal{S}_1}(x) \leq \eta_{\mathcal{S}_1}(x)$ and $\nu_{\mathcal{S}_1}(x) \geq \nu_{\mathcal{S}_1}(x)$ for all $x \in S$.

## 3. Main Results

### 3.1. **Spherical fuzzy ideals in ternary semigroups.**

We define spherical fuzzy ternary subsemigroups and spherical fuzzy ideals in ternary semigroups as follows:

**Definition 3.1.** A spherical fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ on a ternary semigroup $T$ is called a *spherical fuzzy ternary subsemigroup* of $T$ if, for all $a, b, c \in T$

(1) $\mu_{\mathcal{S}}(abc) \geq \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\}$,
(2) $\eta_{\mathcal{S}}(abc) \geq \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\}$,
(3) $\nu_{\mathcal{S}}(abc) \leq \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}$.

**Definition 3.2.** A spherical fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ on a ternary semigroup $T$ is called

(1) a *spherical fuzzy left ideal* of $T$ if for all $a, b, c \in T$,

$$\mu_{\mathcal{S}}(abc) \geq \mu_{\mathcal{S}}(c), \ \eta_{\mathcal{S}}(abc) \geq \eta_{\mathcal{S}}(c) \quad \text{and} \quad \nu_{\mathcal{S}}(abc) \leq \nu_{\mathcal{S}}(c),$$

(2) a *spherical fuzzy lateral ideal* of $T$ if for all $a, b, c \in T$,

$$\mu_{\mathcal{S}}(abc) \geq \mu_{\mathcal{S}}(b), \ \eta_{\mathcal{S}}(abc) \geq \eta_{\mathcal{S}}(b) \quad \text{and} \quad \nu_{\mathcal{S}}(abc) \leq \nu_{\mathcal{S}}(b),$$

(3) a *spherical fuzzy right ideal* of $T$ if for all $a, b, c \in T$,

$$\mu_{\mathcal{S}}(abc) \geq \mu_{\mathcal{S}}(a), \ \eta_{\mathcal{S}}(abc) \geq \eta_{\mathcal{S}}(a) \quad \text{and} \quad \nu_{\mathcal{S}}(abc) \leq \nu_{\mathcal{S}}(a),$$

(4) a *spherical fuzzy ideal* of $T$ if for all $a, b, c \in T$,

$$\mu_{\mathcal{S}}(abc) \geq \max\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\},$$

$$\eta_{\mathcal{S}}(abc) \geq \max\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\}$$

and

$$\nu_{\mathcal{S}}(abc) \leq \min\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}.$$

Next, we define the product of three spherical fuzzy sets.

**Definition 3.3.** Let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be any three spherical fuzzy sets on a ternary semigroup $T$. The *product* $\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_3$ of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ is defined by

$$\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_3 = ((\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3}), (\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3}), (\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3}))$$

where

$$(\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(x) = \begin{cases} \sup\limits_{x=abc} \min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\}, & \text{if } x \in T^3; \\ 0, & \text{otherwise,} \end{cases}$$

$$(\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(x) = \begin{cases} \sup\limits_{x=abc} \min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\}, & \text{if } x \in T^3; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(x) = \begin{cases} \inf\limits_{x=abc} \max\{\nu_{\mathcal{S}_1}(a), \nu_{\mathcal{S}_2}(b), \nu_{\mathcal{S}_3}(c)\}, & \text{if } x \in T^3; \\ 1, & \text{otherwise.} \end{cases}$$

**Theorem 3.1.** *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be a spherical fuzzy set on a ternary semigroup $T$. Then $\mathcal{S}$ is a spherical fuzzy ternary subsemigroup of $T$ if and only if $\mathcal{S} \circ \mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S}$.*

*Proof.* Assume that $\mathcal{S}$ is a spherical fuzzy ternary subsemigroup of $T$. Let $x \in T$. If $x \notin T^3$, we obtain that

$$(\mu_{\mathcal{S}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{S}})(x) = 0 \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{S}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{S}})(x) = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{S}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{S}})(x) = 1 \geq \nu_{\mathcal{S}}(x).$$

Now, assume that $x \in T^3$, we obtain that

$$(\mu_{\mathcal{S}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{S}})(x) = \sup_{x=abc} \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\} \leq \sup_{x=abc} \mu_{\mathcal{S}}(abc) = \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{S}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{S}})(x) = \sup_{x=abc} \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\} \leq \sup_{x=abc} \eta_{\mathcal{S}}(abc) = \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{S}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{S}})(x) = \inf_{x=abc} \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\} \geq \inf_{x=abc} \nu_{\mathcal{S}}(abc) = \nu_{\mathcal{S}}(x).$$

Hence $\mathcal{S} \circ \mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S}$.

Conversely, let $a, b, c \in T$.

$$\begin{aligned} \mu_{\mathcal{S}}(abc) &\geq (\mu_{\mathcal{S}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{S}})(abc) \\ &= \sup_{abc=x_1x_2x_3} \min\{\mu_{\mathcal{S}}(x_1), \mu_{\mathcal{S}}(x_2), \mu_{\mathcal{S}}(x_3)\} \\ &\geq \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\}, \end{aligned}$$

$$\begin{aligned} \eta_{\mathcal{S}}(abc) &\geq (\eta_{\mathcal{S}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{S}})(abc) \\ &= \sup_{abc=x_1x_2x_3} \min\{\eta_{\mathcal{S}}(x_1), \eta_{\mathcal{S}}(x_2), \eta_{\mathcal{S}}(x_3)\} \\ &\geq \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\} \end{aligned}$$

and

$$\begin{aligned} \nu_{\mathcal{S}}(abc) &\leq (\nu_{\mathcal{S}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{S}})(abc) \\ &= \inf_{abc=x_1x_2x_3} \max\{\nu_{\mathcal{S}}(x_1), \nu_{\mathcal{S}}(x_2), \nu_{\mathcal{S}}(x_3)\} \\ &\leq \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}. \end{aligned}$$

This implies that $\mathcal{S}$ is a spherical fuzzy ternary subsemigroup of $T$. □

Let $\mathcal{T} := (\mu_{\mathcal{T}}, \eta_{\mathcal{T}}, \nu_{\mathcal{T}})$ be a spherical fuzzy set on a ternary semigroup $T$ defined by $\mu_{\mathcal{T}}(x) = 1$ and $\eta_{\mathcal{T}}(x) = \nu_{\mathcal{T}}(x) = 0$ for all $x \in T$. The following theorem holds.

**Theorem 3.2.** *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be a spherical fuzzy set on a ternary semigroup $T$. If $\mathcal{S}$ is a spherical fuzzy left ideal of $T$, then $\mathcal{T} \circ \mathcal{T} \circ \mathcal{S} \subseteq \mathcal{S}$.*

*Proof.* Assume that $\mathcal{S}$ is a spherical fuzzy left ideal of $T$. If $x \notin T^3$, we obtain that

$$(\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{S}})(x) = 0 \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{T}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{S}})(x) = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{T}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{S}})(x) = 1 \geq \nu_{\mathcal{S}}(x).$$

Now, assume that $x \in T^3$, we obtain that

$$(\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{S}})(x) = \sup_{x=abc} \min\{\mu_{\mathcal{T}}(a), \mu_{\mathcal{T}}(b), \mu_{\mathcal{S}}(c)\} = \sup_{x=abc} \mu_{\mathcal{S}}(c) \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{T}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{S}})(x) = \sup_{x=abc} \min\{\eta_{\mathcal{T}}(a), \eta_{\mathcal{T}}(b), \eta_{\mathcal{S}}(c)\} = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{T}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{S}})(x) = \inf_{x=abc} \max\{\nu_{\mathcal{T}}(a), \nu_{\mathcal{T}}(b), \nu_{\mathcal{S}}(c)\} = \inf_{x=abc} \nu_{\mathcal{S}}(c) \geq \nu_{\mathcal{S}}(x).$$

Hence $\mathcal{T} \circ \mathcal{T} \circ \mathcal{S} \subseteq \mathcal{S}$. □

**Theorem 3.3.** *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be a spherical fuzzy set on a ternary semigroup $T$. If $\mathcal{S}$ is a spherical fuzzy lateral ideal of $T$, then $\mathcal{T} \circ \mathcal{S} \circ \mathcal{T} \subseteq \mathcal{S}$.*

*Proof.* The proof is similar to that of Theorem 3.2. □

**Theorem 3.4.** *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be a spherical fuzzy set on a ternary semigroup $T$. If $\mathcal{S}$ is a spherical fuzzy right ideal of $T$, then $\mathcal{S} \circ \mathcal{T} \circ \mathcal{T} \subseteq \mathcal{S}$.*

*Proof.* The proof is similar to that of Theorem 3.2. □

3.2. **Rough Spherical Fuzzy Sets in Ternary Semigroups.** The aims of this subsection is to connect rough set theory and spherical fuzzy sets of ternary semigroups.

**Definition 3.4.** An equivalence relation $\rho$ on a ternary semigroup $T$ is called a *congruence* if for all $x_1, x_2, x_3, y_1, y_2, y_3 \in T$

$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \rho \Rightarrow (x_1x_2x_3, y_1y_2y_3) \in \rho.$$

The congruence class of $x \in T$ is denoted by $[x]_\rho$. A congruence $\rho$ on $T$ is called *complete* if $[y_1]_\rho[y_2]_\rho[y_3]_\rho = [y_1y_2y_3]_\rho$ for all $y_1, y_2, y_3 \in T$.

**Definition 3.5.** Let $\rho$ be a congruence on a ternary semigroup $T$ and $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be the spherical fuzzy set on a ternary semigroup $T$.

(1) The *lower approximation* is defined as

$$\underline{App}(\mathcal{S}) = \{< y, \underline{\mu_{\mathcal{S}}}(y), \underline{\eta_{\mathcal{S}}}(y), \underline{\nu_{\mathcal{S}}}(y) >| \ y \in T\},$$

where $\underline{\mu_{\mathcal{S}}}(y) = \inf\limits_{y' \in [y]_\rho} \mu_{\mathcal{S}}(y')$, $\underline{\eta_{\mathcal{S}}}(y) = \inf\limits_{y' \in [y]_\rho} \eta_{\mathcal{S}}(y')$ and $\underline{\nu_{\mathcal{S}}}(y) = \sup\limits_{y' \in [y]_\rho} \nu_{\mathcal{S}}(y')$ with the condition that

$$0 \leq (\underline{\mu_{\mathcal{S}}}(y))^2 + (\underline{\eta_{\mathcal{S}}}(y))^2 + (\underline{\nu_{\mathcal{S}}}(y))^2 \leq 1.$$

(2) The *upper approximation* is defined as

$$\overline{App}(\mathcal{S}) = \{< y, \overline{\mu_{\mathcal{S}}}(y), \overline{\eta_{\mathcal{S}}}(y), \overline{\nu_{\mathcal{S}}}(y) >| \ y \in T\},$$

where $\overline{\mu_{\mathcal{S}}}(y) = \sup\limits_{y' \in [y]_\rho} \mu_{\mathcal{S}}(y')$, $\overline{\eta_{\mathcal{S}}}(y) = \sup\limits_{y' \in [y]_\rho} \eta_{\mathcal{S}}(y')$ and $\overline{\nu_{\mathcal{S}}}(y) = \inf\limits_{y' \in [y]_\rho} \nu_{\mathcal{S}}(y')$ with the condition that

$$0 \leq (\overline{\mu_{\mathcal{S}}}(y))^2 + (\overline{\eta_{\mathcal{S}}}(y))^2 + (\overline{\nu_{\mathcal{S}}}(y))^2 \leq 1.$$

(3) The *rough spherical fuzzy set* of $T$ is defined by

$$App(\mathcal{S}) = (\underline{App}(\mathcal{S}), \overline{App}(\mathcal{S})).$$

**Theorem 3.5.** *Let $\rho$ be a congruence on a ternary semigroup $T$ and $\mathcal{S}_1 = (\mu_{\mathcal{S}_1}, \eta_{\mathcal{S}_1}, \nu_{\mathcal{S}_1})$ and $\mathcal{S}_2 = (\mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_2})$ be any two spherical fuzzy sets on $T$. The following statements hold.*

(1) *If $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\overline{App}(\mathcal{S}_1) \subseteq \overline{App}(\mathcal{S}_2)$ and $\underline{App}(\mathcal{S}_1) \subseteq \underline{App}(\mathcal{S}_2)$.*
(2) *$\overline{App}(\mathcal{S}_1 \cap \mathcal{S}_2) \subseteq \overline{App}(\mathcal{S}_1) \cap \overline{App}(\mathcal{S}_2)$.*
(3) *$\overline{App}(\mathcal{S}_1 \cup \mathcal{S}_2) = \overline{App}(\mathcal{S}_1) \cup \overline{App}(\mathcal{S}_2)$.*
(4) *$\underline{App}(\mathcal{S}_1 \cap \mathcal{S}_2) = \underline{App}(\mathcal{S}_1) \cap \underline{App}(\mathcal{S}_2)$.*
(5) *$\underline{App}(\mathcal{S}_1) \cup \underline{App}(\mathcal{S}_2) \subseteq \underline{App}(\mathcal{S}_1 \cup \mathcal{S}_2)$.*

*Proof.* (1) Assume that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then $\mu_{\mathcal{S}_1} \leq \mu_{\mathcal{S}_2}$, $\eta_{\mathcal{S}_2} \leq \eta_{\mathcal{S}_1}$ and $\nu_{\mathcal{S}_2} \geq \nu_{\mathcal{S}_1}$. Thus for all $y \in T$, we have

$$\overline{\mu_{\mathcal{S}_1}}(y) = \sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_1}(y') \leq \sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_2}(y') = \overline{\mu_{\mathcal{S}_2}}(y),$$

$$\overline{\eta_{\mathcal{S}_1}}(y) = \sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_1}(y') \leq \sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_2}(y') = \overline{\eta_{\mathcal{S}_2}}(y)$$

and

$$\overline{\nu_{\mathcal{S}_1}}(y) = \inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_1}(y') \geq \inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_2}(y') = \overline{\nu_{\mathcal{S}_2}}(y).$$

This implies that $\overline{App}(\mathcal{S}_1) \subseteq \overline{App}(\mathcal{S}_2)$. Similarly, we have $\underline{App}(\mathcal{S}_1) \subseteq \underline{App}(\mathcal{S}_2)$.

(2) Since $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq \mathcal{S}_1$ and $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq \mathcal{S}_2$, $\overline{App}(\mathcal{S}_1 \cap \mathcal{S}_2) \subseteq \overline{App}(\mathcal{S}_1) \cap \overline{App}(\mathcal{S}_2)$ by (1).

(3) Note that

$$\overline{App}(\mathcal{S}_1) \cup \overline{App}(\mathcal{S}_2) = (\overline{\mu_{\mathcal{S}_1}} \cup \overline{\mu_{\mathcal{S}_2}}, \overline{\eta_{\mathcal{S}_1}} \cap \overline{\eta_{\mathcal{S}_2}}, \overline{\nu_{\mathcal{S}_1}} \cap \overline{\nu_{\mathcal{S}_2}})$$

and

$$\overline{App}(\mathcal{S}_1 \cup \mathcal{S}_2) = (\overline{\mu_{\mathcal{S}_1 \cup \mathcal{S}_2}}, \overline{\eta_{\mathcal{S}_1 \cup \mathcal{S}_2}}, \overline{\nu_{\mathcal{S}_1 \cup \mathcal{S}_2}}).$$

Let $y \in T$. Then

$$\begin{aligned}
(\overline{\mu_{\mathcal{S}_1}} \cup \overline{\mu_{\mathcal{S}_2}})(y) &= \max\{\overline{\mu_{\mathcal{S}_1}}(y), \overline{\mu_{\mathcal{S}_2}}(y)\} \\
&= \max\{\sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_1}(y'), \sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \max\{\mu_{\mathcal{S}_1}(y'), \mu_{\mathcal{S}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_1 \cup \mathcal{S}_2}(y') \\
&= \overline{\mu_{\mathcal{S}_1 \cup \mathcal{S}_2}}(y),
\end{aligned}$$

$$\begin{aligned}
(\overline{\eta_{\mathcal{S}_1}} \cup \overline{\eta_{\mathcal{S}_2}})(y) &= \max\{\overline{\eta_{\mathcal{S}_1}}(y), \overline{\eta_{\mathcal{S}_2}}(y)\} \\
&= \max\{\sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_1}(y'), \sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \max\{\eta_{\mathcal{S}_1}(y'), \eta_{\mathcal{S}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_1 \cup \mathcal{S}_2}(y') \\
&= \overline{\eta_{\mathcal{S}_1 \cup \mathcal{S}_2}}(y)
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\nu_{\mathcal{S}_1}} \cap \overline{\nu_{\mathcal{S}_2}})(y) &= \min\{\overline{\nu_{\mathcal{S}_1}}(y), \overline{\nu_{\mathcal{S}_2}}(y)\} \\
&= \min\{\inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_1}(y'), \inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \min\{\nu_{\mathcal{S}_1}(y'), \nu_{\mathcal{S}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_1 \cup \mathcal{S}_2}(y') \\
&= \overline{\nu_{\mathcal{S}_1 \cup \mathcal{S}_2}}(y).
\end{aligned}$$

(4) Note that

$$\underline{App}(\mathcal{S}_1) \cap \underline{App}(\mathcal{S}_2) = (\underline{\mu_{\mathcal{S}_1}} \cap \underline{\mu_{\mathcal{S}_2}}, \underline{\eta_{\mathcal{S}_1}} \cap \underline{\eta_{\mathcal{S}_2}}, \underline{\nu_{\mathcal{S}_1}} \cup \underline{\nu_{\mathcal{S}_2}})$$

and

$$\underline{App}(\mathcal{S}_1 \cap \mathcal{S}_2) = (\underline{\mu_{\mathcal{S}_1 \cap \mathcal{S}_2}}, \underline{\eta_{\mathcal{S}_1 \cap \mathcal{S}_2}}, \underline{\nu_{\mathcal{S}_1 \cap \mathcal{S}_2}}).$$

Let $y \in T$. Then

$$\begin{aligned}(\underline{\mu_{\mathcal{S}_1}} \cap \underline{\mu_{\mathcal{S}_2}})(y) &= \min\{\underline{\mu_{\mathcal{S}_1}}(y), \underline{\mu_{\mathcal{S}_2}}(y)\} \\ &= \min\{\inf_{y' \in [y]_\rho} \mu_{\mathcal{S}_1}(y'), \inf_{y' \in [y]_\rho} \mu_{\mathcal{S}_2}(y')\} \\ &= \inf_{y' \in [y]_\rho} \min\{\mu_{\mathcal{S}_1}(y'), \mu_{\mathcal{S}_2}(y')\} \\ &= \inf_{y' \in [y]_\rho} \mu_{\mathcal{S}_1 \cap \mathcal{S}_2}(y') \\ &= \underline{\mu_{\mathcal{S}_1 \cap \mathcal{S}_2}}(y),\end{aligned}$$

$$\begin{aligned}(\underline{\eta_{\mathcal{S}_1}} \cap \underline{\eta_{\mathcal{S}_2}})(y) &= \min\{\underline{\eta_{\mathcal{S}_1}}(y), \underline{\eta_{\mathcal{S}_2}}(y)\} \\ &= \min\{\inf_{y' \in [y]_\rho} \eta_{\mathcal{S}_1}(y'), \inf_{y' \in [y]_\rho} \eta_{\mathcal{S}_2}(y')\} \\ &= \inf_{y' \in [y]_\rho} \min\{\eta_{\mathcal{S}_1}(y'), \eta_{\mathcal{S}_2}(y')\} \\ &= \inf_{y' \in [y]_\rho} \eta_{\mathcal{S}_1 \cap \mathcal{S}_2}(y') \\ &= \underline{\eta_{\mathcal{S}_1 \cap \mathcal{S}_2}}(y)\end{aligned}$$

and

$$\begin{aligned}(\underline{\nu_{\mathcal{S}_1}} \cup \underline{\nu_{\mathcal{S}_2}})(y) &= \max\{\underline{\nu_{\mathcal{S}_1}}(y), \underline{\nu_{\mathcal{S}_2}}(y)\} \\ &= \max\{\sup_{y' \in [y]_\rho} \nu_{\mathcal{S}_1}(y'), \sup_{y' \in [y]_\rho} \nu_{\mathcal{S}_2}(y')\} \\ &= \sup_{y' \in [y]_\rho} \max\{\nu_{\mathcal{S}_1}(y'), \nu_{\mathcal{S}_2}(y')\} \\ &= \sup_{y' \in [y]_\rho} \nu_{\mathcal{S}_1 \cap \mathcal{S}_2}(y') \\ &= \underline{\nu_{\mathcal{S}_1 \cap \mathcal{S}_2}}(y).\end{aligned}$$

(5) Since $\mathcal{S}_1 \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_2 \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$, $\underline{App}(\mathcal{S}_1) \cup \underline{App}(\mathcal{S}_2) \subseteq \underline{App}(\mathcal{S}_1 \cup \mathcal{S}_2)$ by (1). □

**Theorem 3.6.** *Let $\rho$ be a congruence relation on a ternary semigroup $T$ and $\mathcal{S}$ be a spherical fuzzy set on $T$. Then $\underline{App}(\mathcal{S})$ is also a spherical fuzzy set on $T$.*

*Proof.* Let $y \in T$. Then

$$\begin{aligned}
&(\underline{\mu_{\mathcal{S}}}(y))^2 + (\underline{\eta_{\mathcal{S}}}(y))^2 + (\underline{\nu_{\mathcal{S}}}(y))^2 \\
&= (\inf_{y' \in [y]_\rho} \mu_{\mathcal{S}}(y'))^2 + (\inf_{y' \in [y]_\rho} \eta_{\mathcal{S}}(y'))^2 + (\sup_{y' \in [y]_\rho} \nu_{\mathcal{S}}(y'))^2 \\
&= \inf_{y' \in [y]_\rho} (\mu_{\mathcal{S}}(y'))^2 + \inf_{y' \in [y]_\rho} (\eta_{\mathcal{S}}(y'))^2 + \sup_{y' \in [y]_\rho} (\nu_{\mathcal{S}}(y'))^2 \\
&\leq \inf_{y' \in [y]_\rho} (\mu_{\mathcal{S}}(y'))^2 + \inf_{y' \in [y]_\rho} (\eta_{\mathcal{S}}(y'))^2 + \sup_{y' \in [y]_\rho} (1 - (\mu_{\mathcal{S}}(y'))^2 - (\eta_{\mathcal{S}}(y'))^2) \\
&\leq \inf_{y' \in [y]_\rho} (\mu_{\mathcal{S}}(y'))^2 + \inf_{y' \in [y]_\rho} (\eta_{\mathcal{S}}(y'))^2 + 1 - \inf_{y' \in [y]_\rho} (\mu_{\mathcal{S}}(y'))^2 - \inf_{y' \in [y]_\rho} (\eta_{\mathcal{S}}(y'))^2 = 1.
\end{aligned}$$

This implies that $0 \leq (\underline{\mu_{\mathcal{S}}}(y))^2 + (\underline{\eta_{\mathcal{S}}}(y))^2 + (\underline{\nu_{\mathcal{S}}}(y))^2 \leq 1$. Therefore, $\underline{App}(\mathcal{S})$ is a spherical fuzzy set on $T$. □

Let $\mathcal{S}$ be a spherical fuzzy set on a ternary semigroup $T$. Note that $\overline{App}(\mathcal{S})$ need not be a spherical fuzzy set on $T$, as can be seen in the following example.

**Example 3.1.** Let $T = \{i, -i\}$ be the ternary semigroup under the ternary multiplication, $\rho = T \times T$ and $\mathcal{S}$ be a spherical fuzzy set on $T$ defined by

$$\mu_{\mathcal{S}}(i) = 1, \eta_{\mathcal{S}}(i) = 0, \nu_{\mathcal{S}}(i) = 0 \text{ and } \mu_{\mathcal{S}}(-i) = 0, \eta_{\mathcal{S}}(-i) = 1, \nu_{\mathcal{S}}(-i) = 0.$$

Then

$$\overline{\mu_{\mathcal{S}}}(i) = \overline{\mu_{\mathcal{S}}}(-i) = 1, \overline{\eta_{\mathcal{S}}}(i) = \overline{\eta_{\mathcal{S}}}(-i) = 1, \overline{\nu_{\mathcal{S}}}(i) = \overline{\nu_{\mathcal{S}}}(-i) = 0.$$

In this example, we have that $\overline{App}(\mathcal{S})$ is not a spherical fuzzy set on $T$.

3.3. **Rough Spherical Fuzzy Ideals in Ternary Semigroups.** The aims of this subsection is to connect rough set theory and spherical fuzzy ideals of ternary semigroups.

**Theorem 3.7.** *Let $\rho$ be a complete congruence relation on a ternary semigroup $T$. If $\mathcal{S}$ is a spherical fuzzy left ideal [spherical fuzzy lateral ideal, spherical fuzzy right ideal] of $T$, then $\underline{App}(\mathcal{S})$ is a spherical fuzzy left ideal [spherical fuzzy lateral ideal, spherical fuzzy right ideal] of $T$.*

*Proof.* Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}
\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y \in [y_1y_2y_3]_\rho} \mu_{\mathcal{S}}(y) \\
&= \inf_{y \in [y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(y) = \inf_{abc \in [y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(abc) \\
&\geq \inf_{abc \in [y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(c) = \inf_{c \in [y_3]} \mu_{\mathcal{S}}(c) = \underline{\mu_{\mathcal{S}}}(y_3),
\end{aligned}$$

$$\begin{aligned}\underline{\eta_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \eta_{\mathcal{S}}(y)\\ &= \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(abc)\\ &\geq \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(c) = \inf_{c\in[y_3]} \eta_{\mathcal{S}}(c) = \underline{\eta_{\mathcal{S}}}(y_3)\end{aligned}$$

and

$$\begin{aligned}\underline{\nu_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{S}}(y)\\ &= \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(abc)\\ &\leq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(c) = \sup_{c\in[y_3]} \nu_{\mathcal{S}}(c) = \underline{\nu_{\mathcal{S}}}(y_3).\end{aligned}$$

This implies that $\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \underline{\mu_{\mathcal{S}}}(y_3)$, $\underline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \underline{\eta_{\mathcal{S}}}(y_3)$ and $\underline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \underline{\nu_{\mathcal{S}}}(y_3)$. Then $\underline{App}(\mathcal{S})$ is a spherical fuzzy left ideal of $T$.

The proofs of other cases are similar. □

**Corollary 3.1.** *Let $\rho$ be a complete congruence relation on a ternary semigroup $T$. If $\mathcal{S}$ is a spherical fuzzy ideal of $T$, then $\underline{App}(\mathcal{S})$ is a spherical fuzzy ideal of $T$.*

*Proof.* This follows from Theorem 3.7. □

**Theorem 3.8.** *Let $\rho$ be a congruence relation on a ternary semigroup $T$. If $\mathcal{S}$ is a spherical fuzzy left ideal [spherical fuzzy lateral ideal, spherical fuzzy right ideal] of $T$ and $\overline{App}(\mathcal{S})$ is a spherical fuzzy set of $T$, then $\overline{App}(\mathcal{S})$ is a spherical fuzzy left ideal [spherical fuzzy lateral ideal, spherical fuzzy right ideal] of $T$.*

*Proof.* Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{S}}(y)\\ &\geq \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(abc)\\ &\geq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(c) = \sup_{c\in[y_3]_\rho} \mu_{\mathcal{S}}(c) = \overline{\mu_{\mathcal{S}}}(y_3),\end{aligned}$$

$$\begin{aligned}\overline{\eta_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \eta_{\mathcal{S}}(y)\\ &\geq \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(abc)\\ &\geq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(c) = \sup_{c\in[y_3]_\rho} \eta_{\mathcal{S}}(c) = \overline{\eta_{\mathcal{S}}}(y_3)\end{aligned}$$

and

$$\begin{aligned}\overline{\nu_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{S}}(y)\\ &\leq \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(abc)\\ &\leq \inf_{c\in[y_3]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(c) = \inf_{c\in[y_3]_\rho} \nu_{\mathcal{S}}(c) = \overline{\nu_{\mathcal{S}}}(y_3).\end{aligned}$$

This implies that $\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \overline{\mu_{\mathcal{S}}}(y_3)$, $\overline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \overline{\eta_{\mathcal{S}}}(y_3)$ and $\overline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \overline{\nu_{\mathcal{S}}}(y_3)$. Then $\overline{App}(\mathcal{S})$ is a spherical fuzzy left ideal of $T$.

The proofs of other cases are similar. □

**Corollary 3.2.** *Let $\rho$ be a congruence relation on a ternary semigroup $T$. If $\mathcal{S}$ is a spherical fuzzy ideal of $T$ and $\overline{App}(\mathcal{S})$ is a spherical fuzzy set of $T$, then $\overline{App}(\mathcal{S})$ is a spherical fuzzy ideal of $T$.*

*Proof.* This follows from Theorem 3.8. □

**Conflicts of Interest:** The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Ashraf, S. Abdullah, T. Mahmood, F. Gahni, T. Mahmood, Spherical Fuzzy Sets and Their Applications in Multi-Attribute Decision Making Problems, J. Intell. Fuzzy Syst. 36 (2019), 2829–2844. https://doi.org/10.3233/JIFS-172009

[2] R. Chinram, T. Panityakul, Rough Pythagorean Fuzzy Ideals in Ternary Semigroups, J. Math. Computer Sci. 20 (2020), 302–312. https://doi.org/10.22436/jmcs.020.04.04

[3] D. Dubois, H. Prade, Rough Fuzzy Sets and Fuzzy Rough Sets, Int. J. Gen. Syst. 17 (1990), 191–209. https://doi.org/10.1080/03081079008935107.

[4] A. Iampan, Some Properties of Ideal Extensions in Ternary Semigroups, Iran. J. Math. Sci. Inform. 8 (2013), 67–74. https://doi.org/10.7508/ijmsi.2013.01.007.

[5] E. Kasner, An Extension of the Group Concept, Bull. Amer. Math. Soc. 10 (1904), 290-291

[6] N. Kuroki, on Fuzzy Ideals and Fuzzy Bi-Ideals in Semigroups, Fuzzy Sets Syst. 5 (1981), 203–215. https://doi.org/10.1016/0165-0114(81)90018-X.

[7] S. Kar, P. Sarkar, Fuzzy Ideals of Ternary Semigroups, Fuzzy Inf. Eng. 4 (2012), 181–193. https://doi.org/10.1007/s12543-012-0110-4

[8] D.H. Lehmer, A Ternary Analogue of Abelian Groups, Amer. J. Math. 54 (1932), 329–338. https://doi.org/10.2307/2370997.

[9] J. Los, On the Extending of Models (I), Fundam. Math. 42 (1955), 38–54. https://bibliotekanauki.pl/articles/1383224.

[10] Z. Pawlak, Rough Sets, Int. J. Computer Inf. Sci. 11 (1982), 341–356. https://doi.org/10.1007/bf01001956.

[11] P. Petchkhaew, R. Chinram, Fuzzy, Rough and Rough Fuzzy Ideals in Ternary Semigroups, Int. J. Pure Appl. Math. 56 (2009), 21–36.

[12] F.M. Siosan, Ideal Theory in Ternary Semigroups, Math. Japon. 10 (1965), 63–84.

[13] Q. Wang, J. Zhan, Rough Semigroups and Rough Fuzzy Semigroups Based on Fuzzy Ideals, Open Math. 14 (2016), 1114–1121. https://doi.org/10.1515/math-2016-0102.

[14] L. A. Zadeh, Fuzzy Sets, Inf. Control, 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[15] S. Zeng, A. Hussain, T. Mahmood, M. I. Ali, S. Ashraf, M. Munir, Covering-Based Spherical Fuzzy Rough Set Model Hybrid With TOPSIS for Multi-Attribute Decision-Making, Symmetry 11 (2019), 547. https://doi.org/10.3390/sym11040547.