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A Brief Note On QS/BP/BOI-Algebras

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Abstract. In this paper, the three types of the logical algebra of interest are QS-algebras, BOI-algebras, and BP-algebras, which are continuously studied by algebraists. Since the terms for each of the above logical algebras appear to be independent of each other, it may be understood that they are different. Therefore, the main purpose of this brief note is to show that QS-algebras, BOI-algebras, and BP-algebras satisfying (BM) as one.

1. Introduction and Preliminaries

Several algebras with one binary and one nullary operations were proposed to develop an algebraic replica of implication reduction in classical or non-classical propositional logics. In 1966, Imai and Iséki [4,5] introduced BCK-algebras and BCI-algebras, which were important algebras and also inspired the creation of new types of algebra. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1999, Ahn and Kim [2] introduced the concept of QS-algebras (see Definition 1.1) and thereafter in 2013 Ahn and Han [1] introduced the concept of BP-algebras (see Definition 1.2). In 2019, El-Gendy [3] introduced the concept of BOI-algebras (see Definition 1.3).

The three types of the logical algebra of interest in this paper are QS-algebras, BOI-algebras, and BP-algebras, which are continuously studied by algebraists. Since the terms for each of the above

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logical algebras appear to be independent of each other, it may be understood that they are different. Therefore, the main purpose of this brief note is to show that QS-algebras, BOI-algebras, and BP-algebras satisfying (BM) as one.

Before we begin the study, let's review the definitions of QS-algebras, BP-algebras, and BOI-algebras according to Definitions 1.1, 1.2, and 1.3, respectively.

Definition 1.1. *[2] A QS-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying axioms:*

$$(\forall x \in X)(x * x = 0), \tag{QS1}$$

$$(\forall x \in X)(x * 0 = x), \tag{QS2}$$

$$(\forall x, y, z \in X)((x * y) * z = (x * z) * y), \tag{QS3}$$

$$(\forall x, y, z \in X)((z * x) * (z * y) = y * x). \tag{BM}$$

Example 1.1. *[2] Let $X = \{0, 1, 2\}$ with the following Cayley table:*

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then X is a QS-algebra.

Ahn and Kim proved the following proposition (see [2]).

Proposition 1.1. *If $X = (X; *, 0)$ is a QS-algebra, then*

$$(\forall x, y \in X)((x * (x * y)) * y = 0), \tag{1.1}$$

$$(\forall x, y, z \in X)(((x * z) * (y * z)) * (x * y) = 0), \tag{1.2}$$

$$(\forall x \in X)(0 * (0 * (0 * x)) = 0 * x). \tag{1.3}$$

Definition 1.2. *[1] A BP-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying axioms:*

$$(\forall x \in X)(x * x = 0), \tag{QS1}$$

$$(\forall x, y \in X)(x * (x * y) = y), \tag{BP1}$$

$$(\forall x, y, z \in X)((x * z) * (y * z) = x * y). \tag{BP2}$$

Example 1.2. *[1] Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:*

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is a BP-algebra.

Ahn and Han proved the following proposition (see [1]).

Proposition 1.2. *If $X = (X; *, 0)$ is a BP-algebra, then*

$$(\forall x \in X)(0 * (0 * x) = x), \tag{1.4}$$

$$(\forall x, y \in X)(0 * (x * y) = y * x), \tag{1.5}$$

$$(\forall x \in X)(x * 0 = x). \tag{QS2}$$

Definition 1.3. *[3] A BOI-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying axioms:*

$$(\forall x \in X)(x * x = 0), \tag{QS1}$$

$$(\forall x, y \in X)(x * (x * y) = y), \tag{BP1}$$

$$(\forall x, y, z \in X)((x * y) * z = (x * z) * y). \tag{QS3}$$

Example 1.3. *[3] Let $X = \{0, 1, 2\}$ with the following Cayley table:*

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then X is a BOI-algebra.

El-Gendy proved the following proposition (see [3]).

Proposition 1.3. *If $X = (X; *, 0)$ is a BOI-algebra, then*

$$(\forall x \in X)(x * 0 = x), \tag{QS2}$$

$$(\forall x, y \in X)(0 * (x * y) = y * x), \tag{1.6}$$

$$(\forall x, y, z \in X)((z * x) * (z * y) = y * x), \tag{BM}$$

$$(\forall x, y, z \in X)((x * z) * (y * z) = x * y), \tag{BP2}$$

$$(\forall x, y \in X)((x * y) * (0 * y) = x), \tag{1.7}$$

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{1.8}$$

$$(\forall x, y \in X)((x * y) * x = 0 * y), \tag{1.9}$$

$$(\forall x, y, z \in X)((x * y) * z = (x * y) * (0 * z)). \tag{1.10}$$

2. Results

In this section, we will link that QS-algebras and BOI-algebras are the same, and BOI-algebras and BP-algebras satisfying (BM) are the same.

Theorem 2.1. *Every QS-algebra is a BOI-algebra.*

Proof. It only needs to show (BP1). Replacing x by 0 and z by x in (BM) and by (QS2), we obtain (BP1). □

Theorem 2.2. *Every BOI-algebra is a QS-algebra.*

Proof. It only needs to show (QS2) and (BM). It is immediately obtained by (QS2) and (BM). □

From Theorems 2.1 and 2.2, we get the following theorem.

Theorem 2.3. *QS-algebras and BOI-algebras are the same.*

Theorem 2.4. *Every BOI-algebra is a BP-algebra.*

Proof. It only needs to show (BP2). It is immediately obtained by (BP2). □

Theorem 2.5. *Every BP-algebra satisfying (BM) is a BOI-algebra.*

Proof. It only needs to show (QS3). Let $X = (X; *, 0)$ be a BP-algebra satisfying (BM). Let $x, y, z \in X$. Then

$$\begin{aligned}(x * y) * z &= (x * y) * (x * (x * z)) && \text{(by (BP1))}\\ &= (x * z) * y, && \text{(by (BM))}\end{aligned}$$

so (QS3) is satisfied. Hence, X is a BOI-algebra. □

From Theorems 2.4 and 2.5, we get the following theorem.

Theorem 2.6. *BOI-algebras and BP-algebras satisfying* (BM) *are the same.*

From Theorems 2.3 and 2.6, we get the following corollary.

Corollary 2.1. *QS-algebras, BOI-algebras, and BP-algebras satisfying* (BM) *are the same.*

We can summarize the diagram as follows.

$$\text{QS-algebras} \longleftrightarrow \text{BOI-algebras} \underset{+\,(\mathrm{BM})}{\overset{\longrightarrow}{\longleftarrow}} \text{BP-algebras}$$

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