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FIXED POINT THEOREM ON MULTI-VALUED MAPPINGS

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ABSTRACT. In this paper, we prove a common fixed point theorem for two multivalued self-mappings in complete metric spaces.

1. INTRODUCTION AND PRELIMINARIES:

The study of fixed points for set valued contractions and nonexpansive maps using the Hausdorff metric was initiated by Markin. Later, an interesting and rich fixed point theory for such maps has been developed. The theory of set valued maps has applications in control theory, convex optimization, differential inclusions and economics.

Following the Banach contraction principle Nadler introduced the concept of set valued contractions and established that a set valued contraction possesses a fixed point in a complete metric space. Subsequently many authors generalized Nadlers fixed point theorem in different ways[[1],[2]].

Definition 1.1. *Let X and Y be nonempty sets. T is said to be a multi-valued mapping from X to Y if T is a function from X to the power set of Y. We denote a multi-valued map by $T : X \to 2^Y$.*

Definition 1.2. *A point $x_0 \in X$ is said to be a fixed point of the multi-valued mapping T if $x_0 \in Tx_0$.*

Example 1.3. *Every single valued mapping can be viewed as a multi-valued mapping. Let $f : X \to Y$ be a single valued mapping.*
Define $T : X \to 2^Y$ by $Tx = \{f(x)\}$.
Note that T is multi-valued mapping iff for each $x \in X, Tx \subseteq Y$. Unless otherwise stated we always assume Tx is non-empty for each $x \in X$.

Definition 1.4. *Let (X, d) be a metric space. A map $T : X \to X$ is called contraction if there exists $0 \le \lambda < 1$ such that $d(Tx, Ty) \le \lambda d(x, y)$, for all $x, y \in X$.*

Definition 1.5. *Let (X, d) be a metric space. We define the Hausdorff metric on $CB(X)$ induced by d. That is*

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \quad \sup_{y \in B} d(y, A) \right\}$$

for all $A, B \in CB(X)$, where $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X and $d(x, B) = inf\{d(x, b) : b \in B\}$, for all $x \in X$.

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Definition 1.6. *Let (X, d) be a metric space. A map $T : X \to CB(X)$ is said to be multi valued contraction if there exists $0 \leq \lambda < 1$ such that $H(Tx, Ty) \leq \lambda d(x, y)$, for all $x, y \in X$.*

Lemma 1.7. [3] *If $A, B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

2. Main Results

Theorem 2.1. *Let(X, d) be complete metric space and let $S, T : X \to CB(X)$ be multivalued maps satisfying $H(Tx, Sy) \leq ad(x, Ty) + b(d(x, Sy) + d(Ty, Tx))$, where $0 < a + 2b < 1$, $a, b \geq 0$, for all $x, y \in X$. Then $F(T) = F(S) \neq \emptyset$ and $Tx = Sx = F(T)$, for all $x \in F(T)$.*

Proof. Fix any $x \in X$. Define $x_0 = x$ and let $x_1 \in Tx_0$. By lemma(1.7), we may choose $x_2 \in Sx_0$ such that $d(x_1, x_2) \leq H(Tx_0, Sx_0) + (a + b)$. Now

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Sx_0) + (a + b) \\ &\leq ad(x_0, x_1) + b(d(x_0, x_1) + d(x_1, x_2)) + (a + b) \\ &\leq \frac{a + b}{1 - b} d(x_0, x_1) + \frac{a + b}{1 - b}. \end{aligned}$$

By lemma(1.7), there exists $x_3 \in Tx_2$ such that

$$d(x_3, x_2) \leq H(Tx_2, Sx_0) + \frac{(a + b)^2}{1 - b}.$$

Now

$$\begin{aligned} d(x_3, x_2) &\leq H(Tx_2, Sx_0) + \frac{(a + b)^2}{1 - b} \\ &\leq ad(x_2, x_1) + b(d(x_2, x_2) + d(x_1, x_3)) + \frac{(a + b)^2}{1 - b} \\ &\leq \frac{a + b}{1 - b} d(x_2, x_1) + (\frac{a + b}{1 - b})^2 \\ &\leq (\frac{a + b}{1 - b})^2 d(x_0, x_1) + 2(\frac{a + b}{1 - b})^2 \end{aligned}$$

Continuing this process, we obtain by induction a sequence $\{x_n\}$ such that $x_{2n} \in Sx_{2n-2}$, $x_{2n+1} \in Tx_{2n}$, such that
$d(x_{2n+1}, x_{2n+2}) \leq H(Tx_{2n}, Sx_{2n}) + \dfrac{(a + b)^{2n+1}}{(1 - b)^{2n}}$,
$d(x_{2n}, x_{2n+1}) \leq H(Sx_{2n-2}, Tx_{2n}) + \dfrac{(a + b)^{2n}}{(1 - b)^{2n-1}}$,

Now,

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq H(Sx_{2n-2}, Tx_{2n}) + \frac{(a+b)^{2n}}{(1-b)^{2n-1}} \\ &\leq ad(x_{2n}, Tx_{2n-2}) + b(d(x_{2n}, Sx_{2n-2}) + d(Tx_{2n-2}, Tx_{2n}) \\ &\quad + \frac{(a+b)^{2n}}{(1-b)^{2n-1}} \\ &\leq ad(x_{2n}, x_{2n-1}) + b(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}) \\ &\quad + \frac{(a+b)^{2n}}{(1-b)^{2n-1}} \\ &\leq \frac{(a+b)}{(1-b)} d(x_{2n-1}, x_{2n}) + \frac{(a+b)^{2n}}{(1-b)^{2n}} \end{aligned}$$

Also,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq H(Tx_{2n}, Sx_{2n}) + \frac{(a+b)^{2n+1}}{(1-b)^{2n}} \\ &\leq ad(x_{2n}, Tx_{2n}) + b(d(x_{2n}, Sx_{2n}) + d(Tx_{2n}, Tx_{2n}) \\ &\quad + \frac{(a+b)^{2n+1}}{(1-b)^{2n}} \\ &\leq ad(x_{2n}, x_{2n+1}) + b(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \\ &\quad + \frac{(a+b)^{2n+1}}{(1-b)^{2n}} \\ &\leq \frac{(a+b)}{(1-b)} d(x_{2n}, x_{2n+1}) + \frac{(a+b)^{2n+1}}{(1-b)^{2n+1}} \end{aligned}$$

Therefore,

$$d(x_n, x_{n+1}) \leq \frac{(a+b)}{(1-b)} d(x_{n-1}, x_n) + \frac{(a+b)^n}{(1-b)^n}$$

for all $n \in \mathbf{N}$ and let $k = \frac{(a+b)}{(1-b)}$

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_{n-1}, x_n) + k^n \\ &\leq k(kd(x_{n-2}, x_{n-1}) + k^{n-1}) + k^n \\ &= k^2(d(x_{n-2}, x_{n-1})) + kk^{n-1} + k^n \\ &\leq ... \\ &\leq k^n d(x_0, x_1) + nk^n. \end{aligned}$$

Since $k < 1$, $\sum k^n$ and $\sum nk^n$ have same radius of convergence, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$.

$$\begin{aligned} d(Tz, z) &\leq d(z, x_{2n+2}) + d(x_{2n+2}, Tz) \\ &\leq d(z, x_{2n+2}) + H(Tz, Sx_{2n}) \\ &\leq d(z, x_{2n+2}) + [ad(z, Tx_{2n}) + b(d(z, Sx_{2n}) + d(Tx_{2n}, Tz))] \\ &\leq d(z, x_{2n+2}) + [ad(z, x_{2n+1}) + b(d(z, x_{2n+2}) + d(x_{2n+1}, Tz))] \\ &\to ad(z, z) + b[d(z, z) + d(z, Tz)] \text{ as } n \to \infty. \end{aligned}$$

Therefore $d(Tz, z)(1-b) \leq 0$. Hence

$$d(Tz, z) = 0$$

$$\begin{aligned} H(Tz,Sz) &\leq ad(z,Tz)+b(d(z,Sz)+d(Tz,Tz))] \\ &= ad(z,Tz)+bd(z,Sz) \\ &\leq ad(z,Tz)+bd(z,Tz)+bd(Tz,Sz) \\ &\leq (a+b)d(z,Tz)+bH(Tz,Sz) \\ H(Tz,Sz) &\leq (\frac{a+b}{1-b})d(z,Tz) \end{aligned}$$

Hence, $H(Tz,Sz)=0$, $z \in Tz = Sz$ and therefore $z \in F(T) \neq \emptyset$, $z \in F(S) \neq \emptyset$, To complete the proof, it is enough to show following four cases:
(i) $F(T) \subseteq Tz$ and $Sx = Tx$ for all $x \in F(T)$.
(ii) $Tz \subseteq F(T)$
(iii) $Tx = Tz$ for all $x \in F(T)$
(iv) $F(S) \subseteq Tz$
For any $x \in F(T)$,

$$\begin{aligned} d(x,Tz) &\leq H(Tx,Sz) \\ &\leq (\frac{a+b}{1-b})d(x,Tz) \end{aligned}$$

This shows that $d(x,Tz)=0$ and $x \in Tz$.Further

$$H(Sx,Tx) \leq (\frac{a+b}{1-b})d(x,Tx)=0$$

and $x \in Sx = Tx$.For any $x \in Tz$

$$d(x,Tx) \leq H(Sz,Tx) \leq (\frac{a+b}{1-b})d(Tz,x)=0$$

This shows that $x \in Tx$. Now, we see that$Tz = F(T) \subseteq F(S)$and $Sx = Tx$ for all $x \in F(T)$. For any $x \in F(T)$,

$$\begin{aligned} H(Tx,Sz) &\leq (\frac{a+b}{1-b})d(x,Tz) \\ &= (\frac{a+b}{1-b})d(x,F(T))=0 \end{aligned}$$

Hence, $Tx = Sz = Tz$. It remains to show that $F(S) \subseteq Tz = F(T)$. For any $x \in F(S)$,

$$\begin{aligned} d(x,Tz) &\leq H(Tx,Sz) \\ &\leq (\frac{a+b}{1-b})d(Tx,z) \\ &\leq (\frac{a+b}{1-b})H(Tx,Sz) \\ &\leq (\frac{a+b}{1-b})^2 d(x,Tz) \end{aligned}$$

Hence, $d(x,Tz)=0$. Then $x \in Tz$ and $F(S) \subseteq Tz$. □

In what follows, let $\multimap$ denote multimap.

Corollary 2.1. *Let $T : X \multimap X$ be a multivalued map with nonempty compact values and $r \in [0,1)$ such that*

$$H(Tx,T^2y) \leq rd(x,Ty),$$

for all $x, y \in X$. Then, $F(T) \neq \emptyset$ and $Tx = F(T)$ for all $x \in F(T)$.

Remark 2.2. *Let S be a self mapping (multi valued or single valued) defined on X, we denote $F(S)$ the collection of all fixed points of S.*
If one of S and T in Theorem 2.1 is single valued, then the set $F(T) = F(S)$ is singleton and the maps S and T have a unique common fixed point in X.

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