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(inf, sup)-Hesitant Fuzzy Ideals of BCK/BCI-Algebras

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Abstract. In this paper, we introduce the concept of (inf, sup)-hesitant fuzzy ideals, which is a generalization of the concept of interval-valued fuzzy ideals, in BCK/BCI-algebras and its related properties are investigated. The concept is established in terms of sets, fuzzy sets, negative fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, bipolar fuzzy sets and hesitant fuzzy sets. Moreover, characterizations of ideals, fuzzy ideals, anti-fuzzy ideals, negative fuzzy ideals, Pythagorean fuzzy ideals and bipolar fuzzy ideals of BCK/BCI-algebras are discussed in terms of (inf, sup)-hesitant fuzzy ideals and interval-valued fuzzy ideals.

1. Introduction

The concept of fuzzy sets, introduced by Zadeh [3], has been widely and successfully applied in many branches: finite state machine, computer science, automata, artificial intelligence, expert, control engineering, robotics and theory of groups, semigroups, BCK/BCI-algebras and UP-algebras.

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Several general, extended and related concepts of fuzzy sets have been introduced and studied such as interval-valued fuzzy sets [4, 5], intuitionistic fuzzy sets [6, 7], Pythagorean fuzzy sets [10–12], negative fuzzy sets [13, 14], bipolar fuzzy sets [15, 16], hesitant fuzzy sets [17, 18, 20, 22] and so forth.

BCK and BCI-algebras are algebraic structures, introduced by Imai, Iséki and Tanaka, that describe fragments of the propositional calculus involving implication known as BCK and BCI logic (see [29–31]). In 1991, Xi [8] applied the concept of fuzzy sets to BCK-algebras. Later, a number of authors applied and discussed concept of fuzzy sets and its some general, extended and related concepts to BCK/BCI-algebras. Hong and Jun [9] introduced anti-fuzzy ideals of BCK-algebras and investigated their some useful properties. Subha and Dhanalakshmi [12] exposed and studied Pythagorean fuzzy ideals of BCK-algebras. Jun [5] introduced interval-valued fuzzy subalgebras and ideals of BCK-algebras, and investigated their related properties and characterizations. Lee [16] introduced bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, investigated their related properties, and considered equivalent relations on the set of all bipolar fuzzy ideals of BCK/BCI-algebras. Jun and Ahn [19] introduced hesitant fuzzy subalgebras and ideals of BCK/BCI-algebras, and investigated their related properties and important characterizations. Muhiuddin et al. [32] introduced hesitant fuzzy translations and hesitant fuzzy extensions of a hesitant fuzzy set on BCK/BCI-algebras, investigated related properties, and characterized hesitant fuzzy (subalgebras) ideals.

Studying hesitant fuzzy sets on algebraic structures in the meaning of the infimum or supremum of its images, Mosrijai et al. [33] introduced sup-hesitant fuzzy UP-subalgebras, UP-filters, UP-ideals, and strong UP-ideals of UP-algebras and investigated their related properties. Muhiuddin and Jun [34] Muhiuddin et al. [35] Muhiuddin et al. [38], Harizavi and Jun [37], Jun and Song [39] and Takallo et al. [36] used hesitant fuzzy sets related to the infimum or supremum of their images in study of BCK/BCI-algebras. Jittburus and Julatha [24, 25], Phummee et al. [28], and Jittburus et al. [27] used hesitant fuzzy sets related to the infimum or the supremum of their images in study of semigroups. Julatha and Iampan [21–23, 26] used hesitant fuzzy sets related to the infimum or the supremum of their images in study of ternary semigroups and Γ-semigroups.

As previously stated, it motivated us to study hesitant fuzzy set theory based on ideals of BCK/BCI-algebras in the meaning of infimum and supremum. On BCK/BCI-algebras, we introduce $(\inf, \sup)$-hesitant fuzzy ideals, show that it is a general concept of interval-valued fuzzy ideals, and investigate its related properties. Characterizations of $(\inf, \sup)$-hesitant fuzzy ideals are established in terms of sets, fuzzy sets, negative fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, bipolar fuzzy sets and hesitant fuzzy sets. Moreover, characterizations of ideals, fuzzy ideals, anti-fuzzy ideals, negative fuzzy ideals, Pythagorean fuzzy ideals and bipolar fuzzy ideals of BCK/BCI-algebras are discussed in terms of $(\inf, \sup)$-hesitant fuzzy ideals and interval-valued fuzzy ideals.

2. Preliminaries

An algebra $(\mathcal{X}; \boxtimes, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if the followings hold:

(I) $(\forall x, y, z \in \mathcal{X})(((x \boxtimes y) \boxtimes (x \boxtimes z)) \boxtimes (z \boxtimes y) = 0)$,
(II) $(\forall x, y \in \mathcal{X})(((x \boxtimes (x \boxtimes y)) \boxtimes y) = 0)$,
(III) $(\forall x \in \mathcal{X})(x \boxtimes x = 0)$,
(IV) $(\forall x, y \in \mathcal{X})(x \boxtimes y = 0 = y \boxtimes x \Rightarrow x = y)$.

By a *BCK-algebra* we mean a BCI-algebra $(\mathcal{X}; \boxtimes, 0)$ satisfies $0 \boxtimes x = 0$ for all $x \in \mathcal{X}$. For any $x, y \in \mathcal{X}$, we define $x \leq y$ by $x \boxtimes y = 0$. In a BCK/BCI-algebra $(\mathcal{X}; \boxtimes, 0)$, the following hold:

$$(\forall x \in \mathcal{X})(x \boxtimes 0 = x), \tag{2.1}$$

$$(\forall x, y, z \in \mathcal{X})((x \boxtimes y) \boxtimes z = (x \boxtimes z) \boxtimes y). \tag{2.2}$$

A nonempty subset $\mathcal{A}$ of a BCK/BCI-algebra $(\mathcal{X}; \boxtimes, 0)$ is called an *ideal* (Id) of $\mathcal{X}$ if it satisfies the following:

$$0 \in \mathcal{A}, \tag{2.3}$$

$$(\forall x \in \mathcal{X})(y \in \mathcal{A}, x \boxtimes y \in \mathcal{A} \Rightarrow x \in \mathcal{A}). \tag{2.4}$$

We refer the reader to the books [1, 2] for further information regarding BCK/BCI-algebras. In what follows, let $\mathcal{X}$ denote a BCK/BCI-algebra $(\mathcal{X}, \boxtimes, 0)$ and $\mathcal{Y}$ denote an arbitrary nonempty set unless otherwise specified.

A *fuzzy set* (FS) [3] in $\mathcal{Y}$ is an arbitrary function from $\mathcal{Y}$ into $[0, 1]$. For FSs ζ and ξ in $\mathcal{Y}$, we denote $\zeta \leq \xi$ in case that $\zeta(x) \leq \xi(x)$ for all $x \in \mathcal{Y}$. A FS ζ in $\mathcal{X}$ is call a *fuzzy ideal* (FId) [8] of $\mathcal{X}$ if it satisfies the following conditions:

$$(\forall x \in \mathcal{X})(\zeta(0) \geq \zeta(x)), \tag{2.5}$$

$$(\forall x, y \in \mathcal{X})(\zeta(x) \geq \min\{\zeta(x \boxtimes y), \zeta(y)\}) \tag{2.6}$$

and called an *anti-fuzzy ideal* (AFId) [9] of $\mathcal{X}$ if it satisfies the following conditions:

$$(\forall x \in \mathcal{X})(\zeta(0) \leq \zeta(x)), \tag{2.7}$$

$$(\forall x, y \in \mathcal{X})(\zeta(x) \leq \max\{\zeta(x \boxtimes y), \zeta(y)\}). \tag{2.8}$$

Then ζ is both a FId and an AFId of $\mathcal{X}$ if and only if it is a constant function.

A *Pythagorean fuzzy set* (PFS) [10, 11] on $\mathcal{Y}$ is an object having the form $P = \{(x, \zeta(x), \xi(x)) \mid x \in \mathcal{Y}\}$ when the functions $\zeta : \mathcal{Y} \to [0, 1]$ denote the degree of membership and $\xi : \mathcal{Y} \to [0, 1]$ denote the degree of nonmembership, and $0 \leq (\zeta(x))^2 + (\xi(x))^2 \leq 1$ for all $x \in \mathcal{Y}$. For the sake of simplicity, we will use the symbol (ζ, ξ) of the PFS $\{(x, \zeta(x), \xi(x)) \mid x \in \mathcal{Y}\}$. For a FS ζ in $\mathcal{Y}$, we define a FS $\frac{\zeta}{2}$ by $\frac{\zeta}{2}(x) = \frac{\zeta(x)}{2}$ for all $x \in \mathcal{Y}$. Then $(\frac{\zeta}{2}, \frac{\xi}{2})$ and $(\frac{\xi}{2}, \frac{\zeta}{2})$ are PFSs in $\mathcal{Y}$ for all FSs ζ and ξ in $\mathcal{Y}$. Thus the concept of PFSs is an extension of the concept of FSs. A PFS (ζ, ξ) on $\mathcal{X}$ is called a *Pythagorean fuzzy ideal* (PFId) [12] of $\mathcal{X}$ if ζ is a FId and ξ is an AFId of $\mathcal{X}$.

A *bipolar fuzzy set* (BFS) [15] in $\mathcal{Y}$ is an object having the form $B = \{(x, \zeta(x), \xi(x)) \mid x \in \mathcal{Y}\}$, where $\zeta : \mathcal{Y} \to [-1, 0]$ is a negative fuzzy set (NFS) in $\mathcal{Y}$ and $\xi : \mathcal{Y} \to [0, 1]$ is a FS in $\mathcal{Y}$. We'll use the symbol $\langle\zeta,\xi\rangle$ for the BFS $\{(x,\zeta(x),\xi(x)) \mid x\in\mathcal{Y}\}$ for the purpose of simplicity. Let R be the set of all real numbers. For any element r of R and any function ζ from $\mathcal{Y}$ into R, define functions $r-\zeta$, $r+\zeta$, $r\zeta$ and $-\zeta$ by:

$$r-\zeta:\mathcal{Y}\to R, x\mapsto r-\zeta(x), \tag{2.9}$$

$$r+\zeta:\mathcal{Y}\to R, x\mapsto r+\zeta(x) \tag{2.10}$$

$$r\zeta:\mathcal{Y}\to R, x\mapsto r\zeta(x) \tag{2.11}$$

$$-\zeta:\mathcal{Y}\to R, x\mapsto -\zeta(x). \tag{2.12}$$

Then the followings hold:

(1) $\langle\zeta-1,\zeta\rangle$ is a BFS in $\mathcal{Y}$ for any FS ζ in $\mathcal{Y}$,
(2) $(\frac{1+\zeta}{2},\frac{\xi}{2})$ and $(\frac{\xi}{2},\frac{1+\zeta}{2})$ are PFSs in $\mathcal{Y}$ for any BFS $\langle\zeta,\xi\rangle$ in $\mathcal{Y}$,
(3) $\langle\zeta-1,\xi\rangle$ and $\langle\xi-1,\zeta\rangle$ are BFSs in $\mathcal{Y}$ for any PFS (ζ,ξ) in $\mathcal{Y}$.

Thus the concept of BFSs is an extension of the concept of FSs.

A BFS $B=\langle\zeta,\xi\rangle$ in $\mathcal{X}$ is called a *bipolar fuzzy ideal* (BFId) [16] of $\mathcal{X}$ if it satisfies the following conditions:

$$(\forall x\in\mathcal{X})(\zeta(0)\le\zeta(x)), \tag{2.13}$$

$$(\forall x\in\mathcal{X})(\xi(0)\ge\xi(x)), \tag{2.14}$$

$$(\forall x,y\in\mathcal{X})(\zeta(x)\le\max\{\zeta(x\boxtimes y),\zeta(y)\}), \tag{2.15}$$

$$(\forall x,y\in\mathcal{X})(\xi(x)\ge\min\{\xi(x\boxtimes y),\xi(y)\}). \tag{2.16}$$

By a *negative fuzzy ideal* (NFId) of $\mathcal{X}$ we mean a NFS ζ of $\mathcal{X}$ satisfies the conditions (2.13) and (2.15). Then a BFS $\langle\zeta,\xi\rangle$ of $\mathcal{X}$ is a BFId of $\mathcal{X}$ if and only if ζ is a NFId and ξ is a FId of $\mathcal{X}$.

By an interval number $\breve{a}$ we mean an interval $[a^-,a^+]$, where $0\le a^-\le a^+\le 1$. The set of all interval numbers is denoted by $\mathcal{D}([0,1])$. For two elements $\breve{a}=[a^-,a^+]$ and $\breve{b}=[b^-,b^+]$ in $\mathcal{D}([0,1])$, define the operations $\precsim$, $=$, $\prec$ and rmin in case of two elements in $\mathcal{D}([0,1])$ as follows:

(1) $\breve{a}\precsim\breve{b}\Leftrightarrow a^+\le b^+$ and $a^-\le b^-$,
(2) $\breve{a}=\breve{b}\Leftrightarrow a^+=b^+$ and $a^-=b^-$,
(3) $\breve{a}\prec\breve{b}\Leftrightarrow\breve{a}\precsim\breve{b}$ and $\breve{a}\neq\breve{b}$,
(4) $\mathrm{rmin}\{\breve{a},\breve{b}\}=[\min\{a^-,b^-\},\min\{a^+,b^+\}]$.

An *interval-valued fuzzy set* (IvFS) [4] on $\mathcal{Y}$ is defined to be a function $\breve{\lambda}:\mathcal{Y}\to\mathcal{D}([0,1])$, where $\breve{\lambda}(x)=[\breve{\lambda}^L(x),\breve{\lambda}^U(x)]$ for all $x\in\mathcal{Y}$, $\breve{\lambda}^L$ and $\breve{\lambda}^U$ are FSs in $\mathcal{Y}$ such that $\breve{\lambda}^L\le\breve{\lambda}^U$. Thus the concept of IvFSs is an extension of the concept of FSs. An IvFS $\breve{\lambda}$ on $\mathcal{X}$ is called an *interval-valued fuzzy ideal*

(IvFId) [5] of $\mathcal{X}$ if it satisfies:

$$(\forall x \in \mathcal{X})(\check{\lambda}(x) \precsim \check{\lambda}(0)), \tag{2.17}$$

$$(\forall x, y \in \mathcal{X})(\text{rmin}\{\check{\lambda}(x \boxtimes y), \check{\lambda}(y)\} \precsim \check{\lambda}(x)). \tag{2.18}$$

Remark 2.1. *an IvFS $\check{\lambda}$ on $\mathcal{X}$ is an IvFId of $\mathcal{X}$ if and only if $\check{\lambda}^L$ and $\check{\lambda}^U$ are FIds of $\mathcal{X}$.*

A *hesitant fuzzy set* (HFS) [17,18] on $\mathcal{Y}$ is defined to be a function $\widetilde{\omega} : \mathcal{Y} \to \wp([0,1])$ when $\wp([0,1])$ is the set of all subsets of $[0,1]$. Note that every IvFS on $\mathcal{Y}$ is a HFS on $\mathcal{Y}$. Then the concept of HFSs is a generalization of the concept of IvFSs, and the concept of HFSs is an extension of the concept of FSs. A HFS $\widetilde{\omega}$ is a *hesitant fuzzy ideal* (HFId) [19,20] of $\mathcal{X}$ if it satisfies the following:

$$(\forall x \in \mathcal{X})(\widetilde{\omega}(x) \subseteq \widetilde{\omega}(0)), \tag{2.19}$$

$$(\forall x, y \in \mathcal{X})(\widetilde{\omega}(x \boxtimes y) \cap \widetilde{\omega}(y) \subseteq \widetilde{\omega}(x)). \tag{2.20}$$

3. Main Results

For an element $\nabla \in \wp([0,1])$, define INF ∇ [24,27] and SUP ∇ [25,26] by

$$\text{INF}\,\nabla = \begin{cases} \inf \nabla & \text{if } \nabla \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\text{SUP}\,\nabla = \begin{cases} \sup \nabla & \text{if } \nabla \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

Definition 3.1. *A HFS $\widetilde{\omega}$ on $\mathcal{X}$ is called an $(\inf, \sup)$-hesitant fuzzy ideal ($(\inf, \sup)$-HFId) of $\mathcal{X}$ if the set $[\mathcal{X}, \widetilde{\omega}, \nabla]$ is an Id of $\mathcal{X}$ for all $\nabla \in \wp([0,1])$ when $[\mathcal{X}, \widetilde{\omega}, \nabla] := \{x \in \mathcal{X} \mid \text{INF}\,\widetilde{\omega}(x) \geq \text{INF}\,\nabla, \text{SUP}\,\widetilde{\omega}(x) \geq \text{SUP}\,\nabla\}$ is not empty.*

Example 3.1. *Let $\mathcal{X} = \{0, u, v, w, x\}$ be a BCI-algebra [1] with the following Cayley table:*

$\boxtimes$	0	u	v	w	x
0	0	0	v	w	x
u	u	0	v	w	x
v	v	v	0	x	w
w	w	w	x	0	v
x	x	x	w	v	0

Define a HFS $\widetilde{\omega}$ on $\mathcal{X}$ by $\widetilde{\omega}(0) = [0.6, 0.8], \widetilde{\omega}(u) = (0.5, 0.7), \widetilde{\omega}(v) = [0.5, 0.6] \cup \{0.7\}, \widetilde{\omega}(w) = \{0.3, 0.4\}, \widetilde{\omega}(z) = (0.3, 0.4)$. It is routine to verify that $\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.

Example 3.2. *Let $\mathcal{X} = \{0, w, x, y, z\}$ be a BCK-algebra with the following Cayley table:*

$\boxtimes$	0	w	x	y	z
0	0	0	0	0	0
w	w	0	0	0	0
x	x	x	0	0	0
y	y	x	w	0	w
z	z	x	w	w	0

Define a HFS $\widetilde{\omega}$ on $\mathcal{X}$ by $\widetilde{\omega}(0) = \{0.8, 0.9, 1\}, \widetilde{\omega}(w) = (0.6, 0.8], \widetilde{\omega}(x) = \widetilde{\omega}(y) = \{0\}, \widetilde{\omega}(z) = \emptyset$. It is routine to verify that $\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$. Moreover, we know that $\widetilde{\omega}$ is not a HFId of $\mathcal{X}$ because $\widetilde{\omega}(w) \nsubseteq \widetilde{\omega}(0)$, and $\widetilde{\omega}$ is not an IvFId of $\mathcal{X}$ because it is not an IvFS.

For any HFS $\widetilde{\omega}$ on $\mathcal{Y}$, define the FSs $\mathcal{F}^{\widetilde{\omega}}$ and $\mathcal{F}_{\widetilde{\omega}}$ in $\mathcal{Y}$ by

$$(\forall x \in \mathcal{Y})(\mathcal{F}^{\widetilde{\omega}}(x) = \mathrm{SUP}\,\widetilde{\omega}(x)), \tag{3.1}$$

$$(\forall x \in \mathcal{Y})(\mathcal{F}_{\widetilde{\omega}}(x) = \mathrm{INF}\,\widetilde{\omega}(x)). \tag{3.2}$$

A HFS $\widetilde{\vartheta}$ on $\mathcal{Y}$ is called an *infimum complement* [21, 24] of $\widetilde{\omega}$ on $\mathcal{Y}$ if $\mathrm{INF}\,\widetilde{\vartheta}(x) = (1 - \mathcal{F}_{\widetilde{\omega}})(x)$ for all $x \in \mathcal{Y}$ and called a *supremum complement* of $\widetilde{\omega}$ on $\mathcal{Y}$ if $\mathrm{SUP}\,\widetilde{\vartheta}(x) = (1 - \mathcal{F}^{\widetilde{\omega}})(x)$ for all $x \in \mathcal{Y}$. Let $\mathrm{IC}(\widetilde{\omega})$ and $\mathrm{SC}(\widetilde{\omega})$ be the set of all infimum complements of $\widetilde{\omega}$ and the set of all supremum complements of $\widetilde{\omega}$, respectively. Define the HFSs $\widetilde{\omega}^{\pm}$ and $\widetilde{\omega}^{\mp}$ on $\mathcal{Y}$ by $\widetilde{\omega}^{\pm}(x) = \{(1 - \mathcal{F}_{\widetilde{\omega}})(x)\}$ and $\widetilde{\omega}^{\mp}(x) = \{(1 - \mathcal{F}^{\widetilde{\omega}})(x)\}$ for all $x \in \mathcal{Y}$. Then we have $\widetilde{\omega}^{\pm} \in \mathrm{IC}(\widetilde{\omega})$, $\mathcal{F}_{\widetilde{\omega}^{\pm}} = 1 - \mathcal{F}_{\widetilde{\omega}}$ and $\widetilde{\omega}^{\mp} \in \mathrm{SC}(\widetilde{\omega})$, $\mathcal{F}^{\widetilde{\omega}^{\mp}} = 1 - \mathcal{F}^{\widetilde{\omega}}$. Next, we investigate characterizations of $(\inf, \sup)$-HFIds of BCK/BCI-algebras in terms of Ids, FIds, AFIds and NFIds.

Lemma 3.1. *Let $\widetilde{\omega}$ be a HFS on $\mathcal{X}$. Then the followings are equivalent.*

1. *$\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*
2. *The set $[\mathcal{X}, \widetilde{\omega}, \breve{a}]$ is an Id of $\mathcal{X}$ for all $\breve{a} \in \mathcal{D}([0, 1])$ when $[\mathcal{X}, \widetilde{\omega}, \breve{a}]$ is not empty.*
3. *$\mathcal{F}_{\widetilde{\omega}}$ and $\mathcal{F}^{\widetilde{\omega}}$ are FIds of $\mathcal{X}$.*
4. *$\mathcal{F}_{\widetilde{\vartheta}}$ and $\mathcal{F}^{\widetilde{\theta}}$ are AFIds of $\mathcal{X}$ for all $\widetilde{\vartheta} \in \mathrm{IC}(\widetilde{\omega})$ and $\widetilde{\theta} \in \mathrm{SC}(\widetilde{\omega})$.*
5. *$\mathcal{F}_{\widetilde{\omega}^{\pm}}$ and $\mathcal{F}^{\widetilde{\omega}^{\mp}}$ are AFIds of $\mathcal{X}$.*
6. *$\mathcal{F}_{\widetilde{\vartheta}} - 1$ and $\mathcal{F}^{\widetilde{\theta}} - 1$ are NFIds of $\mathcal{X}$ for all $\widetilde{\vartheta} \in \mathrm{IC}(\widetilde{\omega})$ and $\widetilde{\theta} \in \mathrm{SC}(\widetilde{\omega})$.*
7. *$\mathcal{F}_{\widetilde{\omega}^{\pm}} - 1$ and $\mathcal{F}^{\widetilde{\omega}^{\mp}} - 1$ are NFIds of $\mathcal{X}$.*

Proof. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7). They are clear.

(2) $\Rightarrow$ (3). Let $x \in \mathcal{X}$ and $\breve{a} := \{t \in [0, 1] \mid \mathrm{INF}\,\widetilde{\omega}(x) \leq t \leq \mathrm{SUP}\,\widetilde{\omega}(x)\}$. Then $\breve{a} \in \mathcal{D}([0, 1])$ and $x \in [\mathcal{X}, \widetilde{\omega}, \breve{a}]$. By the assumption (2), we get $[\mathcal{X}, \widetilde{\omega}, \breve{a}]$ is an Id of $\mathcal{X}$ and so $0 \in [\mathcal{X}, \widetilde{\omega}, \breve{a}]$. Thus $\mathrm{SUP}\,\widetilde{\omega}(x) = a^{+} \leq \mathrm{SUP}\,\widetilde{\omega}(0)$ and $\mathrm{INF}\,\widetilde{\omega}(x) = a^{-} \leq \mathrm{INF}\,\widetilde{\omega}(0)$, which imply that $\mathcal{F}^{\widetilde{\omega}}(x) \leq \mathcal{F}^{\widetilde{\omega}}(0)$ and $\mathcal{F}_{\widetilde{\omega}}(x) \leq \mathcal{F}_{\widetilde{\omega}}(0)$. Hence, $\mathcal{F}^{\widetilde{\omega}}$ and $\mathcal{F}_{\widetilde{\omega}}$ satisfy the condition (2.5). To show that $\mathcal{F}^{\widetilde{\omega}}$ and $\mathcal{F}_{\widetilde{\omega}}$ satisfy the

condition (2.6), let $x, y \in \mathcal{X}$ and

$$\breve{b} := \{t \in [0,1] \mid \min\{\mathrm{INF}\,\widetilde{\omega}(y), \mathrm{INF}\,\widetilde{\omega}(x \boxtimes y)\} \leq t \leq \min\{\mathrm{SUP}\,\widetilde{\omega}(y), \mathrm{SUP}\,\widetilde{\omega}(x \boxtimes y)\}\}.$$

Then $\breve{b} \in \mathcal{D}([0,1])$ and $y, x \boxtimes y \in [\mathcal{X}, \widetilde{\omega}, \breve{b}]$. By the assumption (2), we have $x \in [\mathcal{X}, \widetilde{\omega}, \breve{b}]$. Thus

$$\begin{aligned}
\mathcal{F}^{\widetilde{\omega}}(x) &= \mathrm{SUP}\,\widetilde{\omega}(x) \\
&\geq b^{+} \\
&= \min\{\mathrm{SUP}\,\widetilde{\omega}(y), \mathrm{SUP}\,\widetilde{\omega}(x \boxtimes y)\} \\
&= \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\}, \\
\mathcal{F}_{\widetilde{\omega}}(x) &= \mathrm{INF}\,\widetilde{\omega}(x) \\
&\geq b^{-} \\
&= \min\{\mathrm{INF}\,\widetilde{\omega}(y), \mathrm{INF}\,\widetilde{\omega}(x \boxtimes y)\} \\
&= \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\}.
\end{aligned}$$

Hence, $\mathcal{F}^{\widetilde{\omega}}$ and $\mathcal{F}_{\widetilde{\omega}}$ satisfy the condition (2.6). Therefore, it follows from the conditions (2.5) and (2.6) that $\mathcal{F}^{\widetilde{\omega}}$ and $\mathcal{F}_{\widetilde{\omega}}$ are FIds of $\mathcal{X}$.

(3) $\Rightarrow$ (1). Let ∇ be an element of $\wp([0,1])$ such that $[\mathcal{X}, \widetilde{\omega}, \nabla] \neq \emptyset$. Let $x \in \mathcal{X}$ and $y, x \boxtimes y \in [\mathcal{X}, \widetilde{\omega}, \nabla]$. Then $\mathrm{SUP}\,\widetilde{\omega}(y) \geq \mathrm{SUP}\,\nabla$, $\mathrm{INF}\,\widetilde{\omega}(y) \geq \mathrm{INF}\,\nabla$, $\mathrm{SUP}\,\widetilde{\omega}(x \boxtimes y) \geq \mathrm{SUP}\,\nabla$ and $\mathrm{INF}\,\widetilde{\omega}(x \boxtimes y) \geq \mathrm{INF}\,\nabla$. By the assumption (3), we have

$$\mathrm{SUP}\,\widetilde{\omega}(0) = \mathcal{F}^{\widetilde{\omega}}(0) \geq \mathcal{F}^{\widetilde{\omega}}(y) = \mathrm{SUP}\,\widetilde{\omega}(y) \geq \mathrm{SUP}\,\nabla,$$

$$\mathrm{INF}\,\widetilde{\omega}(0) = \mathcal{F}_{\widetilde{\omega}}(0) \geq \mathcal{F}_{\widetilde{\omega}}(y) = \mathrm{INF}\,\widetilde{\omega}(y) \geq \mathrm{INF}\,\nabla,$$

$$\mathrm{SUP}\,\widetilde{\omega}(x) = \mathcal{F}^{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\} = \min\{\mathrm{SUP}\,\widetilde{\omega}(y), \mathrm{SUP}\,\widetilde{\omega}(x \boxtimes y)\} \geq \mathrm{SUP}\,\nabla,$$

and

$$\mathrm{INF}\,\widetilde{\omega}(x) = \mathcal{F}_{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\} = \min\{\mathrm{INF}\,\widetilde{\omega}(y), \mathrm{INF}\,\widetilde{\omega}(x \boxtimes y)\} \geq \mathrm{INF}\,\nabla.$$

Thus $0, x \in [\mathcal{X}, \widetilde{\omega}, \nabla]$. Hence, $[\mathcal{X}, \widetilde{\omega}, \nabla]$ is an Id of $\mathcal{X}$. Therefore, $\widetilde{\omega}$ is an (inf, sup)-HFId of $\mathcal{X}$.

(3) $\Rightarrow$ (4). Let $\widetilde{\vartheta} \in \mathrm{IC}(\widetilde{\omega})$ and $\widetilde{\theta} \in \mathrm{SC}(\widetilde{\omega})$. By the assumption (3), we obtain that $\mathcal{F}_{\widetilde{\vartheta}}$ and $\mathcal{F}^{\widetilde{\theta}}$ satisfy the conditions (2.5) and (2.6). Thus, for all $x, y \in \mathcal{X}$, we have

$$\begin{aligned}
\mathcal{F}^{\widetilde{\theta}}(0) &= 1 - \mathcal{F}^{\widetilde{\omega}}(0) \leq 1 - \mathcal{F}^{\widetilde{\omega}}(x) = \mathcal{F}^{\widetilde{\theta}}(x), \\
\mathcal{F}_{\widetilde{\vartheta}}(0) &= 1 - \mathcal{F}_{\widetilde{\omega}}(0) \leq 1 - \mathcal{F}_{\widetilde{\omega}}(x) = \mathcal{F}_{\widetilde{\vartheta}}(x), \\
\mathcal{F}^{\widetilde{\theta}}(x) &= 1 - \mathcal{F}^{\widetilde{\omega}}(x) \\
&\leq 1 - \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\} \\
&= \max\{1 - \mathcal{F}^{\widetilde{\omega}}(y), 1 - \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\}
\end{aligned}$$

$$
\begin{aligned}
&= \max\{\mathcal{F}^{\widetilde{\theta}}(y), \mathcal{F}^{\widetilde{\theta}}(x \boxtimes y)\}, \\
\mathcal{F}_{\widetilde{\vartheta}}(x) &= 1 - \mathcal{F}_{\widetilde{\omega}}(x) \\
&\leq 1 - \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\} \\
&= \max\{1 - \mathcal{F}_{\widetilde{\omega}}(y), 1 - \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\} \\
&= \max\{\mathcal{F}_{\widetilde{\vartheta}}(y), \mathcal{F}_{\widetilde{\vartheta}}(x \boxtimes y)\}.
\end{aligned}
$$

Hence, $\mathcal{F}_{\widetilde{\vartheta}}$ and $\mathcal{F}^{\widetilde{\theta}}$ satisfy that conditions (2.7) and (2.8) that they are AFIds of $\mathcal{X}$.

(4) $\Rightarrow$ (6). Let $\widetilde{\vartheta} \in \mathrm{IC}(\widetilde{\omega})$ and $\widetilde{\theta} \in \mathrm{SC}(\widetilde{\omega})$. It is clear that $\mathcal{F}_{\widetilde{\vartheta}} - 1$ and $\mathcal{F}^{\widetilde{\theta}} - 1$ are NFSs in $\mathcal{X}$. By the assumption (4), we get that $\mathcal{F}_{\widetilde{\vartheta}}$ and $\mathcal{F}^{\widetilde{\theta}}$ satisfy the conditions (2.7) and (2.8). Thus, for all $x, y \in \mathcal{X}$, we get

$$
\begin{aligned}
(\mathcal{F}^{\widetilde{\theta}} - 1)(0) &= \mathcal{F}^{\widetilde{\theta}}(0) - 1 \leq \mathcal{F}^{\widetilde{\theta}}(x) - 1 = (\mathcal{F}^{\widetilde{\theta}} - 1)(x), \\
(\mathcal{F}_{\widetilde{\vartheta}} - 1)(0) &= \mathcal{F}_{\widetilde{\vartheta}}(0) - 1 \leq \mathcal{F}_{\widetilde{\vartheta}}(x) - 1 = (\mathcal{F}_{\widetilde{\vartheta}} - 1)(x), \\
(\mathcal{F}^{\widetilde{\theta}} - 1)(x) &= \mathcal{F}^{\widetilde{\theta}}(x) - 1 \\
&\leq \max\{\mathcal{F}^{\widetilde{\theta}}(y), \mathcal{F}^{\widetilde{\theta}}(x \boxtimes y)\} - 1 \\
&= \max\{\mathcal{F}^{\widetilde{\theta}}(y) - 1, \mathcal{F}^{\widetilde{\theta}}(x \boxtimes y) - 1\} \\
&= \max\{(\mathcal{F}^{\widetilde{\theta}} - 1)(y), (\mathcal{F}^{\widetilde{\theta}} - 1)(x \boxtimes y)\}, \\
(\mathcal{F}_{\widetilde{\vartheta}} - 1)(x) &= \mathcal{F}_{\widetilde{\vartheta}}(x) - 1 \\
&\leq \max\{\mathcal{F}_{\widetilde{\vartheta}}(y), \mathcal{F}_{\widetilde{\vartheta}}(x \boxtimes y)\} - 1 \\
&= \max\{\mathcal{F}_{\widetilde{\vartheta}}(y) - 1, \mathcal{F}_{\widetilde{\vartheta}}(x \boxtimes y) - 1\} \\
&= \max\{(\mathcal{F}_{\widetilde{\vartheta}} - 1)(y), (\mathcal{F}_{\widetilde{\vartheta}} - 1)(x \boxtimes y)\}.
\end{aligned}
$$

Hence, $\mathcal{F}_{\widetilde{\vartheta}} - 1$ and $\mathcal{F}^{\widetilde{\theta}} - 1$ satisfy that conditions (2.13) and (2.15) that they are NFIds of $\mathcal{X}$.

(5) $\Rightarrow$ (7). It is similar to prove (4) $\Rightarrow$ (6).

(7) $\Rightarrow$ (3). Let $x, y \in \mathcal{X}$. Since $\mathcal{F}_{\widetilde{\omega}^{\pm}} - 1 = -\mathcal{F}_{\widetilde{\omega}}$, $\mathcal{F}^{\widetilde{\omega}^{\mp}} - 1 = -\mathcal{F}^{\widetilde{\omega}}$ and by the assumption (7), we have $-\mathcal{F}^{\widetilde{\omega}}(0) \leq -\mathcal{F}^{\widetilde{\omega}}(x)$, $-\mathcal{F}_{\widetilde{\omega}}(0) \leq -\mathcal{F}_{\widetilde{\omega}}(x)$, and

$$
\begin{aligned}
-\mathcal{F}^{\widetilde{\omega}}(x) &\leq \max\{-\mathcal{F}^{\widetilde{\omega}}(y), -\mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\} &&= -(\min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\}), \\
-\mathcal{F}_{\widetilde{\omega}}(x) &\leq \max\{-\mathcal{F}_{\widetilde{\omega}}(y), -\mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\} &&= -(\min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\}).
\end{aligned}
$$

Thus $\mathcal{F}^{\widetilde{\omega}}(0) \geq \mathcal{F}^{\widetilde{\omega}}(x)$, $\mathcal{F}_{\widetilde{\omega}}(0) \geq \mathcal{F}_{\widetilde{\omega}}(x)$, $\mathcal{F}^{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\}$ and $\mathcal{F}_{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\}$. Hence, $\mathcal{F}_{\widetilde{\omega}}$ and $\mathcal{F}^{\widetilde{\omega}}$ satisfy the conditions (2.5) and (2.6). Therefore, $\mathcal{F}_{\widetilde{\omega}}$ and $\mathcal{F}^{\widetilde{\omega}}$ are FIds of $\mathcal{X}$. □

Proposition 3.1. *Every IvFId of $\mathcal{X}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*

Proof. It follows from Remark 2.1 and Lemma 3.1 □

The converse of Proposition 3.1 is not generally true, which can see in Example 3.2. By Proposition 3.1 and Example 3.2, we obtain that an (inf, sup)-HFId of a BCK/BCI-algebra $\mathcal{X}$ is a generalization of the concept of an IvFId of $\mathcal{X}$.

Theorem 3.1. *Let $\check{\lambda}$ be an IvFS on $\mathcal{X}$. Then the followings are equivalent.*

(1) *$\check{\lambda}$ is an IvFId of $\mathcal{X}$.*
(2) *The set $[\mathcal{X}, \check{\lambda}, \breve{a}]$ is an Id of $\mathcal{X}$ for all $\breve{a} \in \mathcal{D}([0,1])$ when $[\mathcal{X}, \check{\lambda}, \breve{a}]$ is not empty.*
(3) *$\check{\lambda}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*

Proof. It follows from Remark 2.1, Lemma 3.1 and Proposition 3.1. □

Theorem 3.2. *Let $\widetilde{\omega}$ be a HFS on $\mathcal{X}$. The followings are equivalent.*

(1) *$\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*
(2) *$\check{\lambda}$ is an IvFId of $\mathcal{X}$ when $\check{\lambda}$ is an IvFS on $\mathcal{X}$ such that $\check{\lambda}^L = F_{\widetilde{\omega}}$ and $\check{\lambda}^U = \mathcal{F}^{\widetilde{\omega}}$.*
(3) *$\widetilde{\vartheta}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$ for all HFS $\widetilde{\vartheta}$ on $\mathcal{X}$ such that $F_{\widetilde{\vartheta}} = F_{\widetilde{\omega}}$ and $\mathcal{F}^{\widetilde{\vartheta}} = \mathcal{F}^{\widetilde{\omega}}$.*

Proof. It follows from Lemma 3.1 and Theorem 3.1. □

Proposition 3.2. *Let $\widetilde{\omega}$ be an $(\inf, \sup)$-HFId of $\mathcal{X}$ and $x, y, z \in \mathcal{X}$ such that $x \boxtimes y \leq z$. Then $\mathcal{F}^{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(z)\}$ and $\mathcal{F}_{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(z)\}$.*

Proof. Since $x \boxtimes y \leq z$, we have $(x \boxtimes y) \boxtimes z = 0$. Thus

$$\begin{aligned}
\mathcal{F}^{\widetilde{\omega}}(x) &\geq \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\} \\
&\geq \min\{\mathcal{F}^{\widetilde{\omega}}(y), \min\{\mathcal{F}^{\widetilde{\omega}}(z), \mathcal{F}^{\widetilde{\omega}}((x \boxtimes y) \boxtimes z)\}\} \\
&= \min\{\mathcal{F}^{\widetilde{\omega}}(y), \min\{\mathcal{F}^{\widetilde{\omega}}(z), \mathcal{F}^{\widetilde{\omega}}(0)\}\} \\
&= \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(z)\}
\end{aligned}$$

and similarly, we hve $\mathcal{F}_{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(z)\}$. □

Corollary 3.1. *Let $\check{\lambda}$ be an IvFId of $\mathcal{X}$ and $x, y, z \in \mathcal{X}$ such that $x \boxtimes y \leq z$. Then $\mathrm{rmin}\{\check{\lambda}(y), \check{\lambda}(z)\} \precsim \check{\lambda}(x)$.*

Proof. It follows from Proposition 3.2 and Theorem 3.1. □

Proposition 3.3. *Let $\widetilde{\omega}$ be an $(\inf, \sup)$-HFId of $\mathcal{X}$ and $x, y \in \mathcal{X}$ such that $x \leq y$. Then $\mathcal{F}^{\widetilde{\omega}}(x) \geq \mathcal{F}^{\widetilde{\omega}}(y)$ and $\mathcal{F}_{\widetilde{\omega}}(x) \geq \mathcal{F}_{\widetilde{\omega}}(y)$.*

Proof. Since $x \leq y$, we have $x \boxtimes y = 0$. Then

$$\mathcal{F}^{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\} = \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(0)\} = \mathcal{F}^{\widetilde{\omega}}(y),$$

$$\mathcal{F}_{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\} = \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(0)\} = \mathcal{F}_{\widetilde{\omega}}(y).$$

Hence, $\mathcal{F}^{\widetilde{\omega}}(x) \geq \mathcal{F}^{\widetilde{\omega}}(y)$ and $\mathcal{F}_{\widetilde{\omega}}(x) \geq \mathcal{F}_{\widetilde{\omega}}(y)$. □

Corollary 3.2. *Let $\check{\lambda}$ be an IvFId of $\mathcal{X}$ and $x, y \in \mathcal{X}$ such that $x \leq y$. Then $\check{\lambda}(y) \precsim \check{\lambda}(x)$.*

Proof. It follows from Proposition 3.3 and Theorem 3.1. □

For any subset A of $\mathcal{Y}$ and $\nabla, \Delta \in \wp([0,1])$, define a map $\mathcal{C}(A, \nabla, \Delta)$ [21, 23] as follows:

$$\mathcal{C}(A, \nabla, \Delta) : \mathcal{Y} \to \wp([0,1]), x \mapsto \begin{cases} \Delta & \text{if } x \in A, \\ \nabla & \text{otherwise.} \end{cases}$$

We denote $\mathcal{C}(A)$ for $\mathcal{C}(A, [0,0], [1,1])$ and it is called the *characteristic interval-valued fuzzy set* of A on $\mathcal{X}$.

Theorem 3.3. *Let A be a nonempty subset of $\mathcal{X}$ and $\nabla, \Delta \in \wp([0,1])$ such that $\mathrm{SUP}\,\nabla < \mathrm{SUP}\,\Delta, \mathrm{INF}\,\nabla \leq \mathrm{INF}\,\Delta$ or $\mathrm{SUP}\,\nabla \leq \mathrm{SUP}\,\Delta, \mathrm{INF}\,\nabla < \mathrm{INF}\,\Delta$. Then A is an Id of $\mathcal{X}$ if and only if $\mathcal{C}(A, \nabla, \Delta)$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*

Proof. Since A is an Id of $\mathcal{X}$, we have $0 \in A$. Then

$$\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(0) = \mathrm{SUP}\,\Delta = \max\{\mathrm{SUP}\,\Delta, \mathrm{SUP}\,\nabla\} \geq \mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(x),$$

$$\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(0) = \mathrm{INF}\,\Delta = \max\{\mathrm{INF}\,\Delta, \mathrm{INF}\,\nabla\} \geq \mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(x)$$

for all $x \in \mathcal{X}$. Thus $\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}$ and $\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}$ satisfy the condition (2.5).

To show that $\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}$ and $\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}$ satisfy the condition (2.6), let $x, y \in \mathcal{X}$. If $y \notin A$ or $x \boxtimes y \notin A$, then

$$\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(x) \geq \mathrm{SUP}\,\nabla = \min\{\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(y), \mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(x \boxtimes y)\},$$

$$\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(x) \geq \mathrm{INF}\,\nabla = \min\{\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(y), \mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(x \boxtimes y)\}.$$

On the other hand, suppose that $y, x \boxtimes y \in A$. Since A is an Id of $\mathcal{X}$, we have $x \in A$. Thus

$$\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(x) = \mathrm{SUP}\,\Delta = \min\{\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(y), \mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(x \boxtimes y)\},$$

$$\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(x) = \mathrm{INF}\,\Delta = \min\{\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(y), \mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(x \boxtimes y)\}.$$

Hence, $\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}$ and $\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}$ satisfy the condition (2.6). Therefore, $\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}$ and $\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}$ are Ids of $\mathcal{X}$ and by Lemma 3.1, we obtain that $\mathcal{C}(A, \nabla, \Delta)$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.

Conversely, let $x \in \mathcal{X}$ and $y, x \boxtimes y \in A$. Then $\mathcal{C}(A, \nabla, \Delta)(y) = \Delta = \mathcal{C}(A, \nabla, \Delta)(x \boxtimes y)$. If $\mathrm{SUP}\,\nabla < \mathrm{SUP}\,\Delta$ and $\mathrm{INF}\,\nabla \leq \mathrm{INF}\,\Delta$, then by Lemma 3.1, we have

$$\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(0) \geq \mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(x) \geq \min\{\mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(y), \mathcal{F}^{\mathcal{C}(A,\nabla,\Delta)}(x \boxtimes y)\} = \mathrm{SUP}\,\Delta > \mathrm{SUP}\,\nabla.$$

Thus $0, x \in A$. In the case that $\mathrm{SUP}\,\nabla \leq \mathrm{SUP}\,\Delta$ and $\mathrm{INF}\,\nabla < \mathrm{INF}\,\Delta$, then by Lemma 3.1, we get

$$\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(0) \geq \mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(x) \geq \min\{\mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(y), \mathcal{F}_{\mathcal{C}(A,\nabla,\Delta)}(x \boxtimes y)\} = \mathrm{INF}\,\Delta > \mathrm{INF}\,\nabla.$$

Thus $0, x \in A$. Therefore, A is an Id of $\mathcal{X}$. □

Theorem 3.4. *Let A be a nonempty subset of $\mathcal{X}$. The followings are equivalent.*

(1) *A is an Id of $\mathcal{X}$.*
(2) *$\mathcal{C}(A, \breve{a}, \breve{b})$ is an IvFId of $\mathcal{X}$ when $\breve{a}, \breve{b} \in \mathcal{D}([0,1])$ and $\breve{a} \prec \breve{b}$.*
(3) *$\mathcal{C}(A)$ is an IvFId of $\mathcal{X}$.*

Proof. It follows from Theorem 3.3 and Theorem 3.1. □

For a FS ζ in $\mathcal{Y}$ and a positive integer n, we define the HFS $\mathcal{H}(\zeta, n)$ and the IvFS $\mathcal{I}(\zeta, n)$ on $\mathcal{Y}$ as follows:

$$\mathcal{H}(\zeta, n) : \mathcal{Y} \to \wp([0,1]), x \mapsto \{\frac{\zeta}{1+n}(x), \frac{n+\zeta}{1+n}(x)\}$$

and

$$\mathcal{I}(\zeta, n) : \mathcal{Y} \to \mathcal{D}([0,1]), x \mapsto \{t \in [0,1] \mid \frac{\zeta}{1+n}(x) \le t \le \frac{n+\zeta}{1+n}(x)\}.$$

Then the followings are true.

(1) $\mathrm{SUP}\,\mathcal{H}(\zeta, n)(x) = \mathrm{SUP}\,\mathcal{I}(\zeta, n)(x)$, $\mathrm{INF}\,\mathcal{H}(\zeta, n)(x) = \mathrm{INF}\,\mathcal{I}(\zeta, n)(x)$ and $\mathcal{H}(\zeta, n)(x) \subseteq \mathcal{I}(\zeta, n)(x)$ for all $x \in \mathcal{Y}$.
(2) $\mathcal{H}(\zeta, 1)(x) = \{\frac{\zeta}{2}(x), \frac{1+\zeta}{2}(x)\}$ and $\mathcal{I}(\zeta, 1)(x) = \{t \in [0,1] \mid \frac{\zeta}{2}(x) \le t \le \frac{1+\zeta}{2}(x)\}$ for all $x \in \mathcal{Y}$.
(3) $\mathcal{H}(-\zeta, n)$ is a HFS and $\mathcal{I}(-\zeta, n)$ is an IvFS on $\mathcal{Y}$ for all NFS ζ in $\mathcal{Y}$.

Next, we use $(\inf, \sup)$-HFIds and IvFIds of BCK/BCI-algebras to characterize FIds in Theorem 3.5, AFIds in Theorem 3.6 and NFIds in Theorem 3.7.

Theorem 3.5. *Let ζ be a FS in $\mathcal{X}$. The followings are equivalent.*

(1) *ζ is a FId of $\mathcal{X}$.*
(2) *$\mathcal{I}(\zeta, n)$ is an IvFId of $\mathcal{X}$ for all positive integer n.*
(3) *$\mathcal{H}(\zeta, n)$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$ for all positive integer n.*
(4) *$\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$ for all HFS $\widetilde{\omega}$ on $\mathcal{X}$ and positive integer n such that $\mathcal{F}_{\widetilde{\omega}} = \frac{\zeta}{1+n}$ and $\mathcal{F}^{\widetilde{\omega}} = \frac{n+\zeta}{1+n}$.*

Proof. By using Theorem 3.2, the conditions (2), (3) and (4) are equivalent. Next, we show that (1) and (4) are equivalent. Let $\widetilde{\omega}$ be a HFS on $\mathcal{X}$ and n be a positive integer such that $\mathcal{F}_{\widetilde{\omega}} = \frac{\zeta}{1+n}$ and $\mathcal{F}^{\widetilde{\omega}} = \frac{n+\zeta}{1+n}$. By the assumption (1), we have

$$\begin{aligned}
\mathcal{F}_{\widetilde{\omega}}(0) &= \frac{\zeta(0)}{1+n} \ge \frac{\zeta(x)}{1+n} = \mathcal{F}_{\widetilde{\omega}}(x),\\
\mathcal{F}^{\widetilde{\omega}}(0) &= \frac{n+\zeta(0)}{1+n} \ge \frac{n+\zeta(x)}{1+n} = \mathcal{F}^{\widetilde{\omega}}(x),\\
\mathcal{F}_{\widetilde{\omega}}(x) &= \frac{\zeta(x)}{1+n} \ge \frac{\min\{\zeta(y), \zeta(x \boxtimes y)\}}{1+n} = \min\{\frac{\zeta(y)}{1+n}, \frac{\zeta(x \boxtimes y)}{1+n}\}\\
&= \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\},\\
\mathcal{F}^{\widetilde{\omega}}(x) &= \frac{n+\zeta(x)}{1+n} \ge \frac{n+\min\{\zeta(y), \zeta(x \boxtimes y)\}}{1+n} = \min\{\frac{n+\zeta(y)}{1+n}, \frac{n+\zeta(x \boxtimes y)}{1+n}\}\\
&= \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\}
\end{aligned}$$

for all $x, y \in \mathcal{X}$. Hence, $\mathcal{F}^{\widetilde{\omega}}$ and $\mathcal{F}_{\widetilde{\omega}}$ are FIds of $\mathcal{X}$ and by using Lemma 3.1, we obtain that $\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$. Therefore, (4) is true.

Conversely, assume that (4) is true. Let $\widetilde{\omega}$ be a HFS on $\mathcal{X}$ such that $\mathcal{F}_{\widetilde{\omega}} = \frac{\zeta}{2}$ and $\mathcal{F}^{\widetilde{\omega}} = \frac{1+\zeta}{2}$. By the assumption (4) and Lemma 3.1, we obtain that $\mathcal{F}_{\widetilde{\omega}} = \frac{\zeta}{2}$ is a FId of $\mathcal{X}$. Then for all $x, y \in \mathcal{X}$, we get $\zeta(0) = 2(\frac{\zeta(0)}{2}) \geq 2(\frac{\zeta(x)}{2}) = \zeta(x)$ and

$$\zeta(x) = 2(\frac{\zeta(x)}{2}) \geq 2(\frac{\min\{\zeta(y), \zeta(x \boxtimes y)\}}{2}) = \min\{\zeta(y), \zeta(x \boxtimes y)\}.$$

Hence, ζ is an Id of $\mathcal{X}$, that is (1) is true. □

Lemma 3.2. *A FS ζ in $\mathcal{X}$ is an AFId of $\mathcal{X}$ if and only if $1-\zeta$ is a FId of $\mathcal{X}$.*

Proof. Assume that ζ is an AFId of $\mathcal{X}$. Then for all $x, y \in \mathcal{X}$, we get $1-\zeta(0) \geq 1-\zeta(x)$ and

$$1-\zeta(x) \geq 1-\max\{\zeta(y), \zeta(x \boxtimes y)\} = \min\{1-\zeta(y), 1-\zeta(x \boxtimes y)\}.$$

Then $1-\zeta$ is a FId of $\mathcal{X}$.

Conversely, assume that $1-\zeta$ is a FId of $\mathcal{X}$. Then $1-(1-\zeta)(0) \leq 1-(1-\zeta)(x)$ and

$$1-(1-\zeta)(x) \leq 1-\min\{(1-\zeta)(y), (1-\zeta)(x \boxtimes y)\} = \max\{1-(1-\zeta)(y), 1-(1-\zeta)(x \boxtimes y)\}$$

for all $x, y \in \mathcal{X}$. Since $\zeta = 1-(1-\zeta)$, we obtain that ζ is an AFId of $\mathcal{X}$. □

Theorem 3.6. *Let ζ be a FS in $\mathcal{X}$. The followings are equivalent.*

(1) *ζ is an AFId of $\mathcal{X}$.*
(2) *$\mathcal{I}(1-\zeta, n)$ is an IvFId of $\mathcal{X}$ for all positive integer n.*
(3) *$\mathcal{H}(1-\zeta, n)$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$ for all positive integer n.*
(4) *$\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$ for all HFS $\widetilde{\omega}$ on $\mathcal{X}$ and positive integer n such that $\mathcal{F}_{\widetilde{\omega}} = \frac{1-\zeta}{1+n}$ and $\mathcal{F}^{\widetilde{\omega}} = 1 + \frac{-\zeta}{1+n}$.*

Proof. It follows from Lemma 3.2 and Theorem 3.5. □

Lemma 3.3. *A NFS ζ in $\mathcal{X}$ is a NFId of $\mathcal{X}$ if and only if $-\zeta$ is a FId of $\mathcal{X}$.*

Proof. Assume that ζ is a NFId of $\mathcal{X}$. Let $x, y \in \mathcal{X}$. Then $\zeta(0) \leq \zeta(x)$ and $\zeta(x) \leq \max\{\zeta(y), \zeta(x \boxtimes y)\}$. Thus $-\zeta(0) \geq -\zeta(x)$ and

$$-\zeta(x) \geq -(\max\{\zeta(y), \zeta(x \boxtimes y)\}) = \min\{-\zeta(y), -\zeta(x \boxtimes y)\}.$$

Hence, $-\zeta$ is a FId of $\mathcal{X}$.

Conversely, assume that $-\zeta$ is a FId of $\mathcal{X}$. Then $\zeta(0) = -(-\zeta(0)) \leq -(-\zeta(x)) = \zeta(x)$ and

$$\begin{aligned}\zeta(x) &= -(-\zeta(x)) \\ &\leq -(\min\{-\zeta(y), -\zeta(x \boxtimes y)\}) \\ &= \max\{-(-\zeta(y)), -(-\zeta(x \boxtimes y))\} \\ &= \max\{\zeta(y), \zeta(x \boxtimes y)\}\end{aligned}$$

for all $x, y \in \mathcal{X}$. Hence, ζ is a NFId of $\mathcal{X}$. □

Theorem 3.7. *Let ζ be a NFS in $\mathcal{X}$. The followings are equivalent.*

(1) *ζ is a NFId of $\mathcal{X}$.*
(2) *$\mathcal{I}(-\zeta, n)$ is an IvFId of $\mathcal{X}$ for all positive integer n.*
(3) *$\mathcal{H}(-\zeta, n)$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$ for all positive integer n.*
(4) *$\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$ for all HFS $\widetilde{\omega}$ on $\mathcal{X}$ and positive integer n such that $\mathcal{F}_{\widetilde{\omega}} = \frac{-\zeta}{1+n}$ and $\mathcal{F}^{\widetilde{\omega}} = \frac{n-\zeta}{1+n}$.*

Proof. It follows from Lemma 3.3 and Theorem 3.5. □

For any HFS $\widetilde{\omega}$ on $\mathcal{Y}$ and any element ∇ of $\wp([0, 1])$, define the HFS $\mathcal{H}^{\widetilde{\omega}}_{\nabla}$ on $\mathcal{Y}$ by

$$\mathcal{H}^{\widetilde{\omega}}_{\nabla}(x) = \{t \in \nabla \mid \tfrac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x) \leq t \leq \tfrac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x)\} \text{ for all } x \in \mathcal{Y}.$$

We denote $\mathcal{H}^{\widetilde{\omega}}$ for $\mathcal{H}^{\widetilde{\omega}}_{[0,1]}$. Then $\mathcal{H}^{\widetilde{\omega}}_{\nabla}(x) \subseteq \mathcal{H}^{\widetilde{\omega}}_{\Delta}(x) \subseteq \mathcal{H}^{\widetilde{\omega}}(x)$ when $x \in \mathcal{Y}$ and $\nabla \subseteq \Delta \subseteq [0, 1]$.

Theorem 3.8. *Let $\widetilde{\omega}$ be a HFS on $\mathcal{X}$. The followings are equivalent.*

(1) *$\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*
(2) *$\mathcal{H}^{\widetilde{\omega}}_{\nabla}$ is a HFId of $\mathcal{X}$ for all $\nabla \in \wp([0, 1])$.*
(3) *$\mathcal{H}^{\widetilde{\omega}}$ is a HFId of $\mathcal{X}$.*

Proof. (1) ⇒ (2). Let $x \in \mathcal{X}$, $\nabla \in \wp([0, 1])$ and $t \in \mathcal{H}^{\widetilde{\omega}}_{\nabla}(x)$. Then $t \in \nabla$ and $\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x) \leq t \leq \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x)$. By the assumption (1) and Lemma 3.1, we get $\mathcal{F}_{\widetilde{\omega}^{\pm}}(x) \geq \mathcal{F}_{\widetilde{\omega}^{\pm}}(0)$ and $\mathcal{F}^{\widetilde{\omega}}(x) \leq \mathcal{F}^{\widetilde{\omega}}(0)$. Thus

$$\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(0) \leq \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x) \leq t \leq \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x) \leq \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(0)$$

and so $t \in \mathcal{H}^{\widetilde{\omega}}(0)$. Hence, $\mathcal{H}^{\widetilde{\omega}}(x) \subseteq \mathcal{H}^{\widetilde{\omega}}(0)$. Therefore, $\mathcal{H}^{\widetilde{\omega}}$ satisfies the condition (2.19).

To show that $\mathcal{H}^{\widetilde{\omega}}$ satisfies the condition (2.20), let $x, y \in \mathcal{X}$, $\nabla \in \wp([0, 1])$ and $t \in \mathcal{H}^{\widetilde{\omega}}_{\nabla}(y) \cap \mathcal{H}^{\widetilde{\omega}}_{\nabla}(x \boxtimes y)$. Then

$$t \in \nabla, \tfrac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(y) \leq t \leq \tfrac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(y) \text{ and } \tfrac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x \boxtimes y) \leq t \leq \tfrac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x \boxtimes y).$$

By the assumption (1) and Lemma 3.1, we have $\mathcal{F}_{\widetilde{\omega}^{\pm}}(x) \leq \max\{\mathcal{F}_{\widetilde{\omega}^{\pm}}(y), \mathcal{F}_{\widetilde{\omega}^{\pm}}(x \boxtimes y)\}$ and $\mathcal{F}^{\widetilde{\omega}}(x) \geq \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\}$. Thus

$$\begin{aligned}
\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x) &\leq \max\{\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(y), \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x \boxtimes y)\} \\
&\leq t \\
&\leq \min\{\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(y), \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x \boxtimes y)\} \\
&\leq \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x),
\end{aligned}$$

and so $t \in \mathcal{H}^{\widetilde{\omega}}_{\nabla}(x)$. Hence, $\mathcal{H}^{\widetilde{\omega}}_{\nabla}(y) \cap \mathcal{H}^{\widetilde{\omega}}_{\nabla}(x \boxtimes y) \subseteq \mathcal{H}^{\widetilde{\omega}}_{\nabla}(x)$. It is showed that $\mathcal{H}^{\widetilde{\omega}}_{\nabla}$ satisfies the condition (2.20). Therefore, it follows from the conditions (2.19) and (2.20) that $\mathcal{H}^{\widetilde{\omega}}_{\nabla}$ is a HFId of $\mathcal{X}$ for all $\nabla \in \wp([0,1])$.

(2) ⇒ (3). It is clear.

(3) ⇒ (1). Let $x, y \in \mathcal{X}$. Then $\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x), \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x) \in \mathcal{H}^{\widetilde{\omega}}(x)$ and $\max\{\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(y), \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x \boxtimes y)\}, \min\{\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(y), \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x \boxtimes y)\} \in \mathcal{H}^{\widetilde{\omega}}(y) \cap \mathcal{H}^{\widetilde{\omega}}(x \boxtimes y)$. By the assumption (3), we get $\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x), \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x) \in \mathcal{H}^{\widetilde{\omega}}(0)$ and $\max\{\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(y), \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x \boxtimes y)\}, \min\{\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(y), \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x \boxtimes y)\} \in \mathcal{H}^{\widetilde{\omega}}(x)$. Thus $\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x) \geq \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(0)$, $\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x) \leq \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(0)$, $\max\{\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(y), \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x \boxtimes y)\} \geq \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x)$ and $\min\{\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(y), \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x \boxtimes y)\} \leq \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x)$. Since $\mathcal{F}_{\widetilde{\omega}} = 1 - 2(\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2})$ and $\mathcal{F}^{\widetilde{\omega}} = 2(\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}) - 1$, we have

$$\begin{aligned}
\mathcal{F}^{\widetilde{\omega}}(0) &= 2(\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(0)) - 1 \geq 2(\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x)) - 1 = \mathcal{F}^{\widetilde{\omega}}(x), \\
\mathcal{F}_{\widetilde{\omega}}(0) &= 1 - 2(\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(0)) \geq 1 - 2(\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x)) = \mathcal{F}_{\widetilde{\omega}}(x), \\
\mathcal{F}^{\widetilde{\omega}}(x) &= 2(\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x)) - 1 \\
&\geq 2(\min\{\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(y), \frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x \boxtimes y)\}) - 1 \\
&= \min\{2(\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(y)) - 1, 2(\frac{1+\mathcal{F}^{\widetilde{\omega}}}{2}(x \boxtimes y)) - 1\} \\
&= \min\{\mathcal{F}^{\widetilde{\omega}}(y), \mathcal{F}^{\widetilde{\omega}}(x \boxtimes y)\}, \\
\mathcal{F}_{\widetilde{\omega}}(x) &= 1 - 2(\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x)) \\
&\geq 1 - 2(\max\{\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(y), \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x \boxtimes y)\}) \\
&= \min\{1 - 2(\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(y)), 1 - 2(\frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2}(x \boxtimes y))\} \\
&= \min\{\mathcal{F}_{\widetilde{\omega}}(y), \mathcal{F}_{\widetilde{\omega}}(x \boxtimes y)\}.
\end{aligned}$$

Hence, $\mathcal{F}_{\widetilde{\omega}}$ and $\mathcal{F}^{\widetilde{\omega}}$ are FIds of $\mathcal{X}$ and by using Lemma 3.1, we obtain that $\widetilde{\omega}$ is an (inf, sup)-HFId of $\mathcal{X}$. □

Theorem 3.9. *Let $\widetilde{\omega}$ be a HFS on $\mathcal{X}$. The followings are equivalent.*

(1) *$\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*
(2) *$(\mathcal{F}_{\widetilde{\omega}}, \mathcal{F}^{\widetilde{\theta}})$ is a PFId of $\mathcal{X}$ for all $\widetilde{\theta} \in \mathrm{SC}(\widetilde{\omega})$.*
(3) *$(\mathcal{F}_{\widetilde{\omega}}, \mathcal{F}^{\widetilde{\omega}^{\mp}})$ is a PFId of $\mathcal{X}$.*
(4) *$(\frac{\mathcal{F}^{\widetilde{\omega}}}{2}, \frac{\mathcal{F}_{\widetilde{\vartheta}}}{2})$ is a PFId of $\mathcal{X}$ for all $\widetilde{\vartheta} \in \mathrm{IC}(\widetilde{\omega})$.*
(5) *$(\frac{\mathcal{F}^{\widetilde{\omega}}}{2}, \frac{\mathcal{F}_{\widetilde{\omega}^{\pm}}}{2})$ is a PFId of $\mathcal{X}$.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4). They follow from Lemma 3.1.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). They are clear.

(3) $\Rightarrow$ (1). By the assumption (3), we obtain that $\mathcal{F}_{\widetilde{\omega}}$ is a FId and $\mathcal{F}^{\widetilde{\omega}^{\mp}}$ is an AFId of $\mathcal{X}$. Since $\mathcal{F}^{\widetilde{\omega}} = 1 - \mathcal{F}^{\widetilde{\omega}^{\mp}}$ and Lemma 3.2, we get $\mathcal{F}^{\widetilde{\omega}}$ is a FId of $\mathcal{X}$. Hence, $\mathcal{F}_{\widetilde{\omega}}$ and $\mathcal{F}^{\widetilde{\omega}}$ are FIds of $\mathcal{X}$ and by using Lemma 3.1, we have that $\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.

(5) $\Rightarrow$ (1). It is similar to prove in the case (3) $\Rightarrow$ (1). □

For any PFS $P = (\zeta, \xi)$ in $\mathcal{Y}$, define the HFS $\mathcal{H}(P)$ on $\mathcal{Y}$ by

$$\mathcal{H}(P)(x) = \{t \in [0, 1] \mid \tfrac{1-\xi}{2}(x) \leq t \leq \tfrac{1+\zeta}{2}(x)\} \text{ for all } x \in \mathcal{Y}.$$

Theorem 3.10. *Let $P = (\zeta, \xi)$ be a PFS in $\mathcal{X}$. The followings are equivalent.*

(1) *P is a PFId of $\mathcal{X}$.*
(2) *$\mathcal{H}(P)$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*
(3) *$\mathcal{H}(P)$ is an IvFId of $\mathcal{X}$.*

Proof. It follows from Theorem 3.2 and Lemmas 3.1 and 3.2. □

Theorem 3.11. *Let $\widetilde{\omega}$ be a HFS on $\mathcal{X}$. The followings are equivalent.*

(1) *$\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$.*
(2) *$\langle \mathcal{F}_{\widetilde{\vartheta}} - 1, \mathcal{F}^{\widetilde{\omega}} \rangle$ is a BFId of $\mathcal{X}$ for all $\widetilde{\vartheta} \in \mathrm{IC}(\widetilde{\omega})$.*
(3) *$\langle \mathcal{F}_{\widetilde{\omega}^{\pm}} - 1, \mathcal{F}^{\widetilde{\omega}} \rangle$ is a BFId of $\mathcal{X}$.*

Proof. (1) $\Rightarrow$ (2). It follows from Lemma 3.1.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). By the assumption (3), we have that $\mathcal{F}^{\widetilde{\omega}}$ is a FId and $\mathcal{F}_{\widetilde{\omega}^{\pm}} - 1$ is a NFId of $\mathcal{X}$. Since $\mathcal{F}_{\widetilde{\omega}} = -(\mathcal{F}_{\widetilde{\omega}^{\pm}} - 1)$ and Lemma 3.3, we get $\mathcal{F}_{\widetilde{\omega}}$ is a FId of $\mathcal{X}$. Thus $\mathcal{F}_{\widetilde{\omega}}$ and $\mathcal{F}^{\widetilde{\omega}}$ are FIds of $\mathcal{X}$ and by using Lemma 3.1, we obtain that $\widetilde{\omega}$ is an $(\inf, \sup)$-HFId of $\mathcal{X}$. □

For any BFS $B = \langle \zeta, \xi \rangle$ on $\mathcal{Y}$, define the HFS $\mathcal{H}\langle B \rangle$ on $\mathcal{Y}$ by

$$\mathcal{H}\langle B \rangle(x) = \{t \in [0, 1] \mid \tfrac{-\zeta}{2}(x) \leq t \leq \tfrac{1+\xi}{2}(x)\} \text{ for all } x \in \mathcal{Y}.$$

Theorem 3.12. *Let $B = \langle \zeta, \xi \rangle$ be a BFS in $\mathcal{X}$. The followings are equivalent.*

(1) *B is a BFId of $\mathcal{X}$.*

(2) $\mathcal{H}\langle B\rangle$ *is an* $(\inf,\sup)$*-HFId of* $\mathcal{X}$*.*

(3) $\mathcal{H}\langle B\rangle$ *is an IvFId of* $\mathcal{X}$*.*

Proof. It follows from Theorem 3.2 and Lemmas 3.1 and 3.3. □

4. Conclusions

In present paper, we have introduced an $(\inf,\sup)$-HFId, which is one of genaral concepts of an IvFId, in BCK/BCI-algebras, and investigated its some important properties. As important study results, characterizations of $(\inf,\sup)$-HFIds have been discussed by sets, FSs, NFSs, PFSs, HFSs, IvFSs and BFSs. Also, we use concepts of $(\inf,\sup)$-HFIds and IvFIds to study characterizations of Ids, FIds, AFIds, NFIds, PFIds and BFIds.

In our future study of BCK/BCI-algebras and other algebras, the following objectives considered:

- to get more results of HFSs in the meaning of the infimum and supremum of its images,
- to define neutrosophic sets in BCK/BCI-algebras and related structures by means of HFSs in the meaning of the infimum and supremum of its images,
- to define $(\inf,\sup)$-type of HFSs baded on subalgebras, H-ideals and p-ideals of BCK/BCI-algebras,
- to introduce $(\inf,\sup)$-HFIds in UP-algebras, BE-algebras, semigroups and LA-semigroups.

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