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A Plancherel Theorem On a Noncommutative Hypergroup

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Abstract. Let G be a locally compact hypergroup and let K be a compact sub-hypergroup of G. (G, K) is a Gelfand pair if $M_c(G//K)$, the algebra of measures with compact support on the double coset $G//K$, is commutative for the convolution. In this paper, assuming that (G, K) is a Gelfand pair, we define and study a Fourier transform on G and then establish a Plancherel theorem for the pair (G, K).

1. Introduction

Hypergroups generalize locally compact groups. They appear when the Banach space of all bounded Radon measures on a locally compact space carries a convolution having all properties of a group convolution apart from the fact that the convolution of two point measures is a probability measure with compact support and not necessarily a point measure. The intention was to unify harmonic analysis on duals of compact groups, double coset spaces $G//H$ (H a compact subgroup of a locally compact group G), and commutative convolution algebras associated with product linearization formulas of special functions. The notion of hypergroup has been sufficiently studied (see for example [2,4,6,7]). Harmonic analysis and probability theory on commutative hypergroups are well developed meanwhile where many results from group theory remain valid (see [1]). When G is a commutative hypergroup, the convolution algebra $M_c(G)$ consisting of measures with compact support on G is commutative. The typical example of commutative hypergroup is the double coset $G//K$ when G is a locally compact group, K is a compact subgroup of G such that (G, K) is a Gelfand pair. In [4], R. I. Jewett has shown the existence of a positive measure called Plancherel measure on the dual space $\widehat{G}$ of a commutative hypergroup G. When the hypergroup G is not commutative, it is possible to involve a

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compact sub-hypergroup K of G leading to a commutative subalgebra of $M_c(G)$. In fact, if K is a compact sub-hypergroup of a hypergroup G, the pair (G, K) is said to be a Gelfand pair if $M_c(G//K)$ the convolution algebra of measures with compact support on $G//K$ is commutative. The notion of Gelfand pairs for hypergroups is well-known (see [3, 8, 9]). The goal of this paper is to extend Jewett work's by obtaining a Plancherel theorem over Gelfand pair associated with non-commutative hypergroup. In the next section, we give notations and setup useful for the remainder of this paper. In section 3, we introduce first the notion of K-multiplicative functions and obtain some of their characterizations. Thanks to these results, we establish a one to one correspondence between the space of K-multiplicative functions and the dual space of G. Then, we define a Fourier tranform on $M_b(G)$, the algebra of bounded measures on G and on $\mathcal{K}(G)$, the algebra of continuous functions on G with compact support. Finally, using the fact that $G//K$ is a commutative hypergroup, we prove that there exists a nonnegative measure (the Plancherel measure) on the dual space of G.

2. Notations and preliminaries

We use the notations and setup of this section in the rest of the paper without mentioning. Let G be a locally compact space. We denote by:

- $C(G)$ (resp. $M(G)$) the space of continuous complex valued functions (resp. the space of Radon measures) on G,
- $C_b(G)$ (resp. $M_b(G)$) the space of bounded continuous functions (resp. the space of bounded Radon measures) on G,
- $\mathcal{K}(G)$ (resp. $M_c(G)$) the space of continuous functions (resp. the space of Radon measures) with compact support on G,
- $C_0(G)$ the space of elements in $C(G)$ which are zero at infinity,
- $\mathfrak{C}(G)$ the space of compact sub-space of G,
- δ_x the point measure at $x \in G$,
- $spt(f)$ the support of the function f.

Let us notice that the topology on $M(G)$ is the cône topology [4] and the topology on $\mathfrak{C}(G)$ is the topology of Michael [5].

Definition 2.1. G is said to be a *hypergroup* if the following assumptions are satisfied.

(H1) There is a binary operator $*$ named convolution on $M_b(G)$ under which $M_b(G)$ is an associative algebra such that:

i) the mapping $(\mu, \nu) \longmapsto \mu * \nu$ is continuous from $M_b(G) \times M_b(G)$ in $M_b(G)$.

ii) $\forall x, y \in G$, $\delta_x * \delta_y$ is a measure of probability with compact support.

iii) the mapping: $(x, y) \longmapsto \text{supp}(\delta_x * \delta_y)$ is continuous from $G \times G$ in $\mathfrak{C}(G)$.

(H2) There is a unique element e (called neutral element) in G such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x, \forall x \in G$.

(H3) There is an involutive homeomorphism: $x \longmapsto \overline{x}$ from G in G, named involution, such that:
i) $(\delta_x * \delta_y)^- = \delta_{\overline{y}} * \delta_{\overline{x}}, \forall x, y \in G$ with $\mu^-(f) = \mu(f^-)$ where $f^-(x) = f(\overline{x}), \forall f \in C(G)$ and $\mu \in M(G)$.
ii) $\forall x, y, z \in G$, $z \in \text{supp}(\delta_x * \delta_y)$ if and only if $x \in \text{supp}(\delta_z * \delta_{\overline{y}})$.

The hypergroup G is commutative if $\delta_x * \delta_y = \delta_y * \delta_x, \forall x, y \in G$. For $x, y \in G$, $x * y$ is the support of $\delta_x * \delta_y$ and for $f \in C(G)$,

$$f(x * y) \equiv (\delta_x * \delta_y)(f) = \int_G f(z) d(\delta_x * \delta_y)(z).$$

The convolution of two measures μ, ν in $M_b(G)$ is defined by: $\forall f \in C(G)$

$$(\mu * \nu)(f) = \int_G \int_G (\delta_x * \delta_y)(f) d\mu(x) d\nu(y) = \int_G \int_G f(x * y) d\mu(x) d\nu(y),$$

For μ in $M_b(G)$, $\mu^* = (\overline{\mu})^-$. So $M_b(G)$ is a *-Banach algebra.

Definition 2.2. $H \subset G$ is a sub-hypergroup of G if the following conditions are satisfied.

(1) H is non empty and closed in G,
(2) $\forall x \in H, \overline{x} \in H$,
(3) $\forall x, y \in H$, $\text{supp}(\delta_x * \delta_y) \subset H$.

Let us now consider a hypergroup G provided with a left Haar measure μ_G and K a compact sub-hypergroup of G with a normalized Haar measure ω_K. Let us put $M_{\mu_G}(G)$ the space of measures in $M_b(G)$ which are absolutely continuous with respect to μ_G. $M_{\mu_G}(G)$ is a closed self-adjoint ideal in $M_b(G)$. For $x \in G$, the double coset of x with respect to K is $K * \{x\} * K = \{k_1 * x * k_2; k_1, k_2 \in K\}$. We write simply KxK for a double coset and recall that $KxK = \bigcup_{k_1, k_2 \in K} \text{supp}(\delta_{k_1} * \delta_x * \delta_{k_2})$. All double coset form a partition of G and the quotient topology with respect to the corresponding equivalence relation equips the double cosets space $G//K$ with a locally topology ([1], page 53). The natural mapping $p_K : G \longrightarrow G//K$ defined by: $p_K(x) = KxK$, $x \in G$ is an open surjective continuous mapping. A function $f \in C(G)$ is said to be invariant by K or $K - invariant$ if $f(k_1 * x * k_2) = f(x)$ for all $x \in G$ and for all $k_1, k_2 \in K$. We denote by $C^\natural(G)$, (resp. $\mathcal{K}^\natural(G)$) the space of continuous functions (resp. continuous functions with compact support) which are $K-$invariant. For $f \in C^\natural(G)$, one defines the function $\widetilde{f}$ on $G//K$ by $\widetilde{f}(KxK) = f(x)\ \forall x \in G$. $\widetilde{f}$ is well defined and it is continuous on $G//K$. Conversely, for all continuous function φ on $G//K$, the function $f = \varphi \circ p_K \in C^\natural(G)$. One has the obvious consequence that the mapping $f \longmapsto \widetilde{f}$ sets up a topological isomorphism between the topological vector spaces $C^\natural(G)$ and $C(G//K)$ (see [8, 9]). So, for any f in $C^\natural(G)$, $f = \widetilde{f} \circ p_K$. Otherwise, we consider the K-projection $f \longmapsto f^\natural$ (by identifying $f^\natural$ and $\widetilde{f^\natural}$) from $C(G)$ into $C(G//K)$ where for $x \in G$, $f^\natural(x) = \int_K \int_K f(k_1 * x * k_2) d\omega_K(k_1) d\omega_K(k_2)$. If $f \in \mathcal{K}(G)$, then $f^\natural \in \mathcal{K}(G//K)$. For a measure $\mu \in M(G)$, one defines $\mu^\natural$ by $\mu^\natural(f) = \mu(f^\natural)$ for $f \in \mathcal{K}(G)$. μ is said to be $K-$invariant if $\mu^\natural = \mu$ and we denote by $M^\natural(G)$ the set of all those measures. Considering these properties, one defines a hypergroup operation on $G//K$ by: $\delta_{KxK} * \delta_{KyK}(\widetilde{f}) = \int_K f(x*k*y)d\omega_K(k)$ (see [2, p. 12]). This defines uniquely the convolution $(KxK)*(KyK)$ on $G//K$. The involution is defined by: $\overline{KxK} = K\overline{x}K$ and the neutral element is K. Let us put $m = \int_G \delta_{KxK} d\mu_G(x)$, m is a left Haar measure on $G//K$. We say that (G, K) is a Gelfand pair if the convolution algebra $M_c(G//K)$ is commutative. $M_c(G//K)$ is topologically isomorphic to $M_c^{\natural}(G)$. Considering the convolution product on $\mathcal{K}(G)$, $\mathcal{K}(G)$ is a convolution algebra and $\mathcal{K}^{\natural}(G)$ is a subalgebra. Thus (G, K) is a Gelfand pair if and only if $\mathcal{K}^{\natural}(G)$ is commutative ([3], theorem 3.2.2).

3. Plancherel theorem

Let G be a locally compact hypergroup and let K be a compact sub-hypergroup of G. In this section, we assume that (G, K) is a Gelfand pair.

3.1. K-multiplicative functions.

Let us put $G_b^{\natural}$ the space of continuous, bounded function ϕ on G such that:

(i) ϕ is K- invariant,

(ii) $\phi(e) = 1$,

(ii) $\int_K \phi(x*k*y)dw_K(k) = \phi(x)\phi(y)\ \forall x, y \in G$.

Let $\widehat{G}$ be the sub-space of $G_b^{\natural}$ containing the elements ϕ in $G_b^{\natural}$ such that

$$\phi(\overline{x}) = \overline{\phi(x)}\ \forall x \in G.$$

$\widehat{G}$ is called the dual space of the hypergroup G.

Remark 3.1. (1) If $\phi \in \widehat{G}$, then $\phi^- \in \widehat{G}$.

(2) Equipped with the topology of uniform convergence on compacta, $\widehat{G}$ is a locally compact Hausdorff space.

(3) In general, $\widehat{G}$ is not a hypergroup.

Definition 3.2. A complex-valued function χ on G will be called a multiplicative (resp. K-multiplicative) function if χ is continuous and not identically zero, and has the property that:

$$\chi(x*y) = \chi(x)\chi(y)\ (resp. \int_K \chi(x*k*y)dw_K(k) = \chi(x)\chi(y))\ \forall x, y \in G.$$

A multiplicative (resp. K-multiplicative) function on $M_b(G)$ is a continuous complex-valued function F not identically zero on $M_b^{\natural}(G)$, and has the property that:

$$F(\mu*\nu) = F(\mu)F(\nu)\ (resp.\ F(\mu*w_K*\nu) = F(\mu)F(\nu))\ \forall \mu, \nu \in M_b(G).$$

For $\chi \in C_b(G)$, not identically zero, let put $F_\chi(\mu) = \int_G \overline{\chi} d\mu$ for $\mu \in M_b(G)$.

Proposition 3.3. *Let F be a K-multiplicative function on $M_b(G)$, then:*
i) F is multiplicative on $M_b^\natural(G)$.
ii) $F(w_K) = F(\delta_e) = 1$.
iii) $\forall \mu \in M_b(G)$, $F(\mu^\natural) = F(\mu)$
iv) $\forall k \in K$, $F(\delta_k) = 1$.

Proof. i) Just remember that $\mu * w_K = \mu, \forall \mu \in M_b^\natural(G)$.

ii) Let $\nu \in M_b^\natural(G)$ such that $F(\nu) \neq 0$.

$$F(\nu) = F(\nu * w_K) = F(\nu)F(w_K) \implies F(w_K) = 1.$$
$$F(\nu) = F(\nu * w_K * \delta_e) = F(\nu)F(\delta_e) \implies F(\delta_e) = 1.$$

iii) Let $\mu \in M_b(G)$. Since $\mu^\natural = w_K * \mu * w_K$, we have

$$\begin{aligned} F(\mu^\natural) &= F(w_K * \mu * w_K) \\ &= F(w_K * \mu * w_K * w_K) \\ &= F(w_K * \mu) \\ &= F(\delta_e * w_K * \mu) \\ &= F(\mu). \end{aligned}$$

iv) If $k \in K$, $\delta_K^\natural = w_K$. Using (ii) and (iii), we have $F(\delta_k) = 1$.

□

Proposition 3.4. *Let $\phi \in G_b^\natural$.*

i) *F_ϕ is a bounded linear K-multiplicative function on $M_b(G)$.*

ii) *F_ϕ is not identically zero on $M_{\mu_G}^\natural(G)$.*

Proof. i) That is clear that F_ϕ is linear and bounded. Let $\mu, \nu \in M_b(G)$. We have

$$\begin{aligned} F_\phi(\mu * w_K * \nu) &= \int_G \int_K \int_G \overline{\phi}(x * k * y) d\mu(x) dw_K(k) d\nu(y) \\ &= \int_G \overline{\phi}(x) d\mu(x) \int_G \overline{\phi}(x) d\nu(y) \\ &= F_\phi(\mu) F_\phi(\nu). \end{aligned}$$

Morever, $F_\phi(w_K) = \int_K \overline{\phi}(k) dw_K(k) = 1 \neq 0$.

ii) If $\mu \in M_{\mu_G}(G)$, then $\mu^\natural = w_K * \mu * w_K \in M_{\mu_G}^\natural(G)$. Let $f \in \mathcal{K}(G)$ with $spt(f) \subset K$ such that $\int_G f(x) du_G(x) = 1$. $f^\natural \mu_G \in M_{\mu_G}^\natural(G)$ and

$$\begin{aligned} F_\phi(f^\natural \mu_G) &= F_\phi(f \mu_G) \\ &= \int_G \overline{\phi}(x) f(x) du_G(x) \\ &= \int_K f(x) du_G(x) \\ &= 1 \neq 0. \end{aligned}$$

□

Theorem 3.5. 1) *Let E be a multiplicative linear function on $M^{\natural}_{\mu_G}(G)$ not identically zero. There exists a unique K-multiplicative linear function F on $M_b(G)$ such that $F = E$ on $M^{\natural}_{\mu_G}(G)$.*

2) *Let F be a bounded linear K-multiplicative function on $M_b(G)$ not identically zero on $M^{\natural}_{\mu_G}(G)$. There exists a unique function ϕ in $G^{\natural}_b$ such that $F = F_\phi$.*

Proof. 1) Let $\nu \in M^{\natural}_{\mu_G}(G)$ such that $E(\nu) \neq 0$ and put

$$F(\mu) = \frac{E(\mu^{\natural} * \nu)}{E(\nu)},\ for\ \mu \in M_b(G)$$

F is well defined since $M_{\mu_G}(G)$ is an ideal in $M_b(G)$. Let us first see that F is multiplicative on $M^{\natural}_b(G)$. For μ and μ' in $M^{\natural}_b(G)$, we have

$$\begin{aligned} F(\mu * \mu') &= \frac{E(\mu * \mu' * \nu)}{E(\nu)} \\ &= \frac{E(\nu * \mu * \mu' * \nu)}{E(\nu)^2} \\ &= \frac{E(\nu * \mu)}{E(\nu)} \frac{E(\mu' * \nu)}{E(\nu)} \\ &= \frac{E(\nu * \mu * \nu)}{E(\nu)^2} F(\mu') \\ &= F(\mu)F(\mu'). \end{aligned}$$

Moreover $F(w_K) = \dfrac{E(w_K * \nu)}{E(\nu)} = \dfrac{E(\nu)}{E(\nu)} = 1$. So for μ and μ' in $M_b(G)$, we have

$$\begin{aligned} F(\mu * w_K * \mu') &= F(w_K * (w_K * \mu * w_K) * (w_K * \mu' * w_K) * w_K) \\ &= F((w_K * \mu * w_K) * (w_K * \mu' * w_K)) \\ &= F(\mu^{\natural} * \mu'^{\natural}) \\ &= F(\mu)F(\mu'). \end{aligned}$$

The uniqueness stems from proposition 3.3.

2) Let F be a bounded linear K-multiplicative function on $M_b(G)$. Let $\nu \in M^{\natural}_{\mu_G}(G)$ such that $F(\nu) \neq 0$. If $\mu_1, \mu_2 \in M_b(G)$ then

$$\begin{aligned} |F(\mu_1) - F(\mu_2)| &= \left|F(\mu_1^{\natural}) - F(\mu_2^{\natural})\right| \\ &= \frac{\left|F(\mu_1^{\natural} * \nu) - F(\mu_2^{\natural} * \nu)\right|}{|F(\nu)|} \\ &= \frac{\left|F((\mu_1 * \nu - \mu_2 * \nu)^{\natural})\right|}{F(\nu)} \\ &\leq \frac{\|F\|}{F(\nu)} \|\mu_1 * \nu - \mu_2 * \nu\|. \end{aligned}$$

Thus F is positive-continuous by ([4], Theorem 5.6B). By ([4], Theorem 2.2D) there exists a bounded continuous function h on G such that $F(\mu) = \int_G h(x)d\mu(x)$. So $\phi = \overline{h}$.

□

3.2. **Fourier transform on** $M_b(G)$**.**

Definition 3.6. Let $\mu \in M_b(G)$, the Fourier transform of μ is the map $\widehat{\mu} : \widehat{G} \longrightarrow \mathbb{C}$ defined by: $\widehat{\mu}(\phi) = \int_G \phi(\overline{x})d\mu(x)$.

Proposition 3.7. i) *For* $\mu \in M_b(G)$, $\widehat{\mu} \in C_b(\widehat{G})$.

ii) *For* $\mu \in M_b(G), \widehat{\mu} = \widehat{\mu^{\natural}}$.

iii) *For* $\mu \in M_{\mu_G}(G), \widehat{\mu} \in C_0(\widehat{G})$.

iv) *If* $\mu \in M_b^{\natural}(G)$ *and* $\nu \in M_b(G)$, *then* $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$.

Proof. i) We can see that, $\widehat{\mu}(\phi) = \mu(\overline{\phi})\ \forall \phi \in \widehat{G}$.

ii) For $\phi \in \widehat{G}$, we have $\widehat{\mu}(\phi) = F_{\phi^-}(\mu)$. So $\widehat{\mu^{\natural}}(\phi) = F_{\phi^-}(\mu^{\natural}) = F_{\phi^-}(\mu) = \widehat{\mu}(\phi)$.

iii) This comes from theorem 3.5 and ([4], theorem 6.3G)

iv) Let ϕ belongs to $\widehat{G}$, we have

$$\begin{aligned}
\widehat{\mu * \nu}(\phi) &= \int_G \phi^-(x)d\mu * \nu(x) \\
&= \int_G \int_G \phi^-(x * y)d\mu(x)d\nu(y) \\
&= \int_G \left[\int_G (\int_K \int_K \phi^-(k_1 * x * k_2 * y)d\omega_K(k_1)d\omega_K(k_2))d\mu(x) \right] d\nu(y) \\
&= \int_G \left[\int_G (\int_K (\int_K \phi^-((k_1 * x) * k_2 * y)d\omega_K(k_2))d\omega_K(k_1))d\mu(x) \right] d\nu(y) \\
&= \int_G \phi^-(y) \left[\int_G (\int_K \phi^-(k_1 * x)d\omega_K(k_1))d\mu(x) \right] d\nu(y) \\
&= \int_G \phi^-(y) \left[\int_G (\phi^-(x)d\mu(x) \right] d\nu(y) \\
&= \int_G \phi^-(x)d\mu(x) \int_G \phi^-(y)d\nu(y) \\
&= \widehat{\mu}(\phi)\widehat{\nu}(\phi).
\end{aligned}$$

□

Remark 3.8. By the definition, the mapping $\mu \longmapsto \widehat{\mu}$ from $M_b(G)$ to $C_b(\widehat{G})$ is continuous.

3.3. **Fourier transform on** G**.**

Definition 3.9. Let $f \in \mathcal{K}^{\natural}(G)$, the Fourier transform of f is the map $\widehat{f} : \widehat{G} \longrightarrow \mathbb{C}$ defined by: $\widehat{f}(\phi) = \int_G \phi(\overline{x})f(x)du_G(x)$

Proposition 3.10. i) *For $f \in \mathcal{K}(G)$, $\widehat{f^\natural} = \widehat{f\mu_G} \in C_0(\widehat{G})$.*

ii) *If $f \in \mathcal{K}^\natural(G)$ and $g \in \mathcal{K}(G)$, then $\widehat{f * g} = \widehat{f}\widehat{g^\natural}$.*

Proof. i) For any f in $\mathcal{K}(G)$, we have

$$\begin{aligned}
\widehat{f^\natural}(\phi) &= \int_G \phi^-(x)(\int_K \int_K f(k_1 * x * k_2) d\omega_K(k_1) d\omega_K(k_2)) du_G(x) \\
&= \int_G f(x)(\int_K \int_K \phi^-(k_1 * x * k_2) d\omega_K(k_1) d\omega_K(k_2)) du_G(x) \\
&= \int_G \phi(\overline{x}) f(x) du_G(x) = \widehat{f\mu_G}(\phi) \ \forall \phi \in \widehat{G}
\end{aligned}$$

Since $f\mu_G \in M_{\mu_G}(G)$, then $\widehat{f\mu_G} \in C_0(\widehat{G})$.

ii) Let $f \in \mathcal{K}^\natural(G)$ and $g \in \mathcal{K}(G)$. For $\phi \in \widehat{G}$, we have

$$\begin{aligned}
\widehat{f * g}(\phi) &= \int_G \phi^-(x) f * g(x) d\mu_G(x) \\
&= \int_G \phi^-(x)(\int_G f(x * y) g(\overline{y}) d\mu_G(y)) d\mu_G(x) \\
&= \int_G g(\overline{y})(\int_G \phi^-(x * \overline{y}) f(x) d\mu_G(x)) d\mu_G(y) \\
&= \int_G g(\overline{y}) \int_K \int_K \int_G \phi^-(k_1 * x * k_2 * \overline{y}) f(x) d\mu_G(x) d\omega_K(k_1) d\omega_K(k_2) d\mu_G(y) \\
&= \int_G g(\overline{y}) \phi^-(\overline{y}) d\mu_G(y) \int_G f(x) \int_K \phi^-(k_1 * x) d\omega_K(k_1) d\mu_G(x) \\
&= \int_G \phi^-(y) g(y) d\mu_G(y) \int_G \phi^-(x) f(x) d\omega_K(k_1) d\mu_G(x) \\
&= \widehat{f}(\phi)\widehat{g}(\phi).
\end{aligned}$$

□

We therefore extend the spherical Fourier transform to all $\mathcal{K}(G)$ with $\widehat{f} = \widehat{f^\natural}$ for any $f \in \mathcal{K}(G)$ and to $L^1(G, \mu_G)$ and $L^2(G, \mu_G)$. We have the following result.

Theorem 3.11. *There exists a unique nonnegative measure π on $\widehat{G}$ such that*

$$\int_G |f(x)|^2 \, d\mu_G(x) = \int_{\widehat{G}} \left|\widehat{f}(\phi)\right|^2 d\pi(\phi) \text{ for all } f \text{ in } L^1(G, \mu_G) \cap L^2(G, \mu_G).$$

The space $\left\{\widehat{f} : f \in \mathcal{K}(G)\right\}$ is dense in $L^2(\widehat{G}, \pi)$.

Proof. Considering the space $\widehat{G//K}$ defined by [4], $\widetilde{\phi} \in \widehat{G//K}$ if and only if $\phi = \widetilde{\phi} \circ p_K \in \widehat{G}$. Let $\widetilde{\varphi}$ belongs to $C_b(\widehat{G//K})$. Let us consider $\varphi : \widehat{G} \longrightarrow \mathbb{C}$ defined by:

$$\varphi(\phi) = \widetilde{\varphi}(\widetilde{\phi}).$$

$\varphi \in C_b(\widehat{G})$ and the mapping

$$\begin{aligned} C_b(\widehat{G//K}) &\longrightarrow C_b(\widehat{G}) \\ \widetilde{\varphi} &\longmapsto \varphi \end{aligned}$$

is a linear bijection, specificaly $\varphi \in \mathcal{K}(\widehat{G}) \iff \widetilde{\varphi} \in \mathcal{K}(\widehat{G//K})$. By ([4], theorem. 7.3I), there exist a unique nonnegative measure $\widetilde{\pi}$ on $\widehat{G//K}$ such that $\int_{G//K} \left|\widetilde{f}(KxK)\right|^2 dm(KxK) = \int_{\widehat{G//K}} \left|\widehat{\widetilde{f}}(\widetilde{\phi})\right|^2 d\widetilde{\pi}(\widetilde{\phi})$ for $\widetilde{f} \in L^1(G//K, m) \cap L^2(G//K, m)$. Let us consider the mapping π defined by $\pi(\varphi) = \widetilde{\pi}(\widetilde{\varphi})$ for $\varphi \in \mathcal{K}(\widehat{G}).\pi$ is a measure on $\widehat{G}$. Since $\widetilde{\pi}$ is nonnegative, then π is nonnegative. Otherwise, note that $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$ for $f \in \mathcal{K}^\natural(G)$. Indeed since $f \in \mathcal{K}^\natural(G)$ then $\widetilde{f} \in \mathcal{K}(G//K)$ and $\widehat{f} \in C_b(\widehat{G})$. So $\widehat{\widetilde{f}}$ and $\widetilde{\widehat{f}}$ belong to $C_b(\widehat{G//K})$. For $\widetilde{\phi} \in \widehat{G//K}$, we have

$$\begin{aligned} \widehat{\widetilde{f}}(\widetilde{\phi}) &= \int_{G//K} \widetilde{\phi}(K\overline{x}K)\widetilde{f}(KxK)dm(KxK) \\ &= \int_{G//K} \widetilde{\phi^-}(KxK)\widetilde{f}(KxK)dm(KxK) \\ &= \int_G \phi^-(x)f(x)du_G(x) \\ &= \widehat{f}(\phi) = \widetilde{\widehat{f}}(\widetilde{\phi}) \end{aligned}$$

Let $f \in \mathcal{K}^\natural(G)$. We have

$$\begin{aligned} \int_{\widehat{G}} \left|\widehat{f}(\phi)\right|^2 d\pi(\phi) &= \int_{\widehat{G//K}} \left|\widetilde{\widehat{f}}(\widetilde{\phi})\right|^2 d\widetilde{\pi}(\widetilde{\phi}) \\ &= \int_{\widehat{G//K}} \left|\widehat{\widetilde{f}}(\widetilde{\phi})\right|^2 d\widetilde{\pi}(\widetilde{\phi}) \\ &= \int_{G//K} \left|\widetilde{f}(KxK)\right|^2 dm(KxK) \\ &= \int_G |f(x)|^2 d\mu_G(x). \end{aligned}$$

As $\widehat{f} = \widehat{f^\natural}$ $\forall f \in \mathcal{K}(G)$ and G unimodular, we deduce that $\int_{\widehat{G}} \left|\widehat{f}(\phi)\right|^2 d\pi(\phi) = \int_G |f(x)|^2 d\mu_G(x)$ $\forall f \in \mathcal{K}(G)$. By the continuity of the Fourier transform and by application of the dominated convergence theorem, we conclude that $\int_G |f(x)|^2 d\mu_G(x) = \int_{\widehat{G}} \left|\widehat{f}(\phi)\right|^2 d\pi(\phi)$ for any f belongs to $L^1(G, \mu_G) \cap L^2(G, \mu_G)$. Let π' a nonnegative measure on $\widehat{G}$ such that $\int_G |f(x)|^2 d\mu_G(x) = \int_{\widehat{G}} \left|\widehat{f}(\phi)\right|^2 d\pi'(\phi)$ for all f in $L^1(G, \mu_G) \cap L^2(G, \mu_G)$. As above but in reverse order π' defines a nonnegative measure $\widetilde{\pi}'$ on $\widehat{G//K}$ such that $\int_{G//K} \left|\widetilde{f}(KxK)\right|^2 dm(KxK) = \int_{\widehat{G//K}} \left|\widehat{\widetilde{f}}(\widetilde{\phi})\right|^2 d\widetilde{\pi}(\widetilde{\phi})$ for $\widetilde{f} \in L^1(G//K, m) \cap L^2(G//K, m)$. That is $\widetilde{\pi}' = \widetilde{\pi}$ seen the uniqueness of $\widetilde{\pi}$, so $\pi = \pi'$. Let us put $\mathcal{F}(\mathcal{K}(G)) =$

$\left\{\widehat{f};\ f \in \mathcal{K}(G\right\}$. Let $\varphi \in \mathcal{K}(\widehat{G})$ such that $\left\langle \widehat{f}, \varphi \right\rangle = \int_{\widehat{G}} \overline{\widehat{f}(\phi)}\varphi(\phi)d\pi(\phi) = 0\ \forall f \in \mathcal{K}^{\natural}(G)$. We have

$$\begin{aligned}
\left\langle \widehat{f}, \varphi \right\rangle = 0\ \forall f \in \mathcal{K}^{\natural}(G) &\implies \int_{\widehat{G}} \overline{\widehat{f}(\phi)}\varphi(\phi)d\pi(\phi) = 0\ \forall f \in \mathcal{K}^{\natural}(G) \\
&\implies \int_{\widehat{G}} \overline{\widetilde{\widehat{f}}(\widetilde{\phi})}\widetilde{\varphi}(\widetilde{\phi})d\widetilde{\pi}(\widetilde{\phi}) = 0\ \forall f \in \mathcal{K}^{\natural}(G) \\
&\implies \left\langle \widetilde{\widehat{f}}, \widetilde{\varphi} \right\rangle = 0\ \forall f \in \mathcal{K}(G) \\
&\implies \left\langle \widehat{\widetilde{f}}, \widetilde{\varphi} \right\rangle = 0\ \forall \widetilde{f} \in \mathcal{K}(G//K) \\
&\implies \widetilde{\varphi} = 0 \text{ since } \mathcal{F}(\mathcal{K}(G//K)) \text{ is dense in } L^2(\widehat{G//K}, \widetilde{\pi}) \\
&\implies \varphi = 0.
\end{aligned}$$

So $(\mathcal{F}(\mathcal{K}(G)))^{\perp} \cap \mathcal{K}(\widehat{G}) = \{0\}$. Since $\mathcal{K}(\widehat{G})$ is dense in $L^2(\widehat{G}, \pi)$, then $(\mathcal{F}(\mathcal{K}(G)))^{\perp} = \{0\}$ and $\mathcal{F}(\mathcal{K}(G))$ is dense in $L^2(\widehat{G}, \pi)$. □

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