International Journal of Analysis and Applications
ISSN 2291-8639
Volume 5, Number 1 (2014), 20-26
http://www.etamaths.com

SOME NEW INEQUALITIES OF QI TYPE FOR DEFINITE INTEGRALS

BO-YAN XI[1] AND FENG QI[2,*]

ABSTRACT. In the paper, the authors establish some new integral inequalities, from which some integral inequalities of Qi type may be derived.

1. INTRODUCTION

In [11] and its preprint [12], an interesting integral inequality below was obtained.

Theorem 1.1 ([11, 12]). *Let $n \in \mathbb{N}$ and the n-th order derivative of f be continuous on $[a,b] \subseteq \mathbb{R} = (-\infty, \infty)$, satisfying $f^{(i)}(a) \ge 0$ and $f^{(n)}(x) \ge n!$ for $0 \le i \le n-1$. Then*

$$\int_a^b f^{n+2}(x)\,\mathrm{d}\,x \ge \left[\int_a^b f(x)\,\mathrm{d}\,x\right]^{n+1}. \tag{1.1}$$

At the end of [11, 12], the following open problem was posed.

Open Problem 1.1 ([11, 12]). *Under what conditions does the inequality*

$$\int_a^b f^t(x)\,\mathrm{d}\,x \ge \left[\int_a^b f(x)\,\mathrm{d}\,x\right]^{t-1} \tag{1.2}$$

hold for some $t > 1$?

Thereafter, the following answer to Open Problem 1.1 was confirmed.

Theorem 1.2 ([14, 15]). *Let $t \ge 1$ and f be a continuous function on $[a,b] \subseteq \mathbb{R}$ such that*

$$\int_a^b f(x)\,\mathrm{d}\,x \ge (b-a)^{t-1}. \tag{1.3}$$

Then the inequality (1.2) is valid.

To the best of our knowledge, till now there have been many mathematicians and articles devoted to generalizing and applying the integral inequality (1.1) and to answering Open Problem 1.1. In these investigations, different and various tools, ideas, methods, and techniques, such as Jensen's inequality [6], convexity method [4], functional inequalities in abstract spaces [1, 4, 6], probability measures viewpoint [1, 7, 8], Hölder inequality and its reversed variants [10, 19], analytical

2010 *Mathematics Subject Classification.* 26D15.
Key words and phrases. generalization; integral inequality; Qi type integral inequality; convex function; Jensen's inequality; r-mean convex function; geometrically convex function; logarithmically convex function; open problem.

methods [15], Cauchy's mean value theorem [3, 13], and q-integral [2, 9, 20], have been created. Recently, this type of inequalities were generalized in [18] to double integrals. Importantly, the mathematical meanings in probability and statistics was found in [4]. For a much complete list of references appeared in recent years on this topic, please refer to [20].

The aim of this paper is to establish some new integral inequalities, from which some integral inequalities of Qi type may be derived. In other words, the integral inequality (1.1) will be generalized and some more answers to Open Problem 1.1 will be supplied in this paper.

2. Definitions and lemmas

Before establishing some new inequalities of Qi type, we state several definitions and lemmas.

Let $I \subseteq \mathbb{R}$ be an interval and $n \in \mathbb{N}$. For $f : I \to \mathbb{R}_+ = (0, \infty)$, $x_k \in I$ for $1 \le k \le n$, and $\lambda_k \ge 0$ satisfying $\sum_{i=1}^{n} \lambda_k = 1$, let

$$M_n(f(x), \lambda, r) = \begin{cases} \left[\displaystyle\sum_{k=1}^{n} \lambda_k f^r(x_k)\right]^{1/r}, & r \neq 0, \\ \displaystyle\prod_{k=1}^{n} f^{\lambda_k}(x_k), & r = 0. \end{cases} \tag{2.1}$$

Especially, for $x_k \in I \subseteq \mathbb{R}_+$, let

$$M_n(x, \lambda, r) = \begin{cases} \left(\displaystyle\sum_{k=1}^{n} \lambda_k x_k^r\right)^{1/r}, & r \neq 0, \\ \displaystyle\prod_{k=1}^{n} x_k^{\lambda_k}, & r = 0. \end{cases} \tag{2.2}$$

Definition 2.1 ([5, p. 348])**.** Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y) \tag{2.3}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. If the above inequality is reversed, then f is said to be concave on I.

Definition 2.2 ([16, 17])**.** Let $I \subseteq \mathbb{R}_+$ be an interval and $r \in \mathbb{R}$. A function $f : I \to \mathbb{R}_+$ is said to be r-mean convex on I if

$$f(M_2(x, \lambda, r)) \le M_2(f(x), \lambda, r) \tag{2.4}$$

for all $x_1, x_2 \in I$ and $\lambda \in [0, 1]$. If the above inequality is reversed, then we say that the function f is r-mean concave on I.

When $r = 0$, the r-mean convex (r-mean concave, respectively) functions are called geometrically convex (geometrically concave, respectively) functions.

Definition 2.3 ([5, p. 349])**.** Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}_+$ is said to be logarithmically convex on I if

$$f(\lambda x + (1 - \lambda)y) \le [f(x)]^{\lambda}[f(y)]^{1-\lambda} \tag{2.5}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. If the above inequality is reversed, then the function f is said to be logarithmically concave on I.

Lemma 2.1 (Jensen's Inequality)**.** *Let $I \subseteq \mathbb{R}_+$ be an interval, $r \in \mathbb{R}$, and $f : I \to \mathbb{R}_+$. Then f is r-mean convex* (*r-mean concave, respectively*) *on I if and only if*

$$f(M_n(x,\lambda,r)) \leq M_n(f(x),\lambda,r) \tag{2.6}$$

holds for all $x = (x_1, x_2, \dots, x_n) \in I^n$ and $\lambda_k \geq 0$ satisfying $\sum_{k=1}^n \lambda_k = 1$.

Proof. This may be found in [16, 17]. □

The following lemmas are useful for us.

Lemma 2.2. *For $x, y \in \mathbb{R}_+$, if either $xy \leq 4$, or $x \leq 1$, or $y \leq 1$, then $xy \leq x + y$.*

Proof. The proof is elementary. □

3. Some new integral inequalities of Qi type

Now we are in a position to establish some new integral inequalities of Qi type.

Theorem 3.1. *Suppose $I \subseteq \mathbb{R}_0 = [0,\infty)$ is an interval, $f : [a,b] \to I$ is continuous and not identically zero, $g : I \to \mathbb{R}_0$ is convex* (*or concave, respectively*)*, and*

$$g((b-a)u) \lesseqgtr g(b-a)g(u) \tag{3.1}$$

for $u \in I$, and

$$\int_a^b f(x)\,\mathrm{d}\,x \gtreqless \frac{g(b-a)}{b-a}. \tag{3.2}$$

Then we have

$$\int_a^b g(f(x))\,\mathrm{d}\,x \gtreqless \frac{g\big(\int_a^b f(x)\,\mathrm{d}\,x\big)}{\int_a^b f(x)\,\mathrm{d}\,x}. \tag{3.3}$$

Proof. Let

$$x_k = a + \frac{k}{n}(b-a), \quad 1 \leq k \leq n. \tag{3.4}$$

If $g(u)$ is a convex function on I, then it is easy to see that $M_n\big(f(x), \frac{1}{n}, 1\big) \in I$, and, by Jensen's inequality (2.6) and corresponding conditions,

$$\begin{aligned}
g\left[\int_a^b f(x)\,\mathrm{d}\,x\right] &= g\left((b-a)\lim_{n\to\infty} M_n\left(f(x), \frac{1}{n}, 1\right)\right) \\
&\leq g(b-a)\lim_{n\to\infty} g\left(M_n\left(f(x), \frac{1}{n}, 1\right)\right) \\
&\leq g(b-a)\lim_{n\to\infty} M_n\left(g\left(f(x), \frac{1}{n}, 1\right)\right) \\
&= \frac{g(b-a)}{b-a}\int_a^b g(f(x))\,\mathrm{d}\,x.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&\int_a^b f(x)\,\mathrm{d}\,x \int_a^b g(f(x))\,\mathrm{d}\,x - g\left[\int_a^b f(x)\,\mathrm{d}\,x\right] \\
&= \int_a^b g(f(x))\,\mathrm{d}\,x\left[\int_a^b f(x)\,\mathrm{d}\,x - \frac{g(b-a)}{b-a}\right] \\
&\geq 0.
\end{aligned}$$

Thus, the inequality (3.3) in the direction $\geq$ is true.

If $g(u)$ is a concave function on I, the proof is similar. This completes the proof of Theorem 3.1. $\square$

Applying Theorem 3.1 to special cases of $g(u)$ result in the following corollaries, which show that Theorem 3.1 and Theorem 3.2 and 3.3 below are generalizations of the inequality (1.1) and answers of Open Problem 1.1.

Corollary 3.1. *Let $f(x)$ is a positive continuous function on an interval $[a,b]\subseteq\mathbb{R}$.*

(1) *If $t\notin[0,1)$ and $\int_a^b f(x)\,\mathrm{d}\,x\geq(b-a)^{t-1}$, then the inequality* (1.2) *is valid;*
(2) *If $0<t\leq 1$ and $\int_a^b f(x)\,\mathrm{d}\,x\leq(b-a)^{t-1}$, then the inequality* (1.2) *is reversed.*

Corollary 3.2. *Let $f(x)$ be a positive continuous function on $[a,b]\subseteq\mathbb{R}$.*

(1) *If $t\notin[0,1)$ and $f(x)\geq(b-a)^{t-2}$, then the inequality* (1.2) *is valid;*
(2) *If $0<t\leq 1$ and $f(x)\leq(b-a)^{t-2}$, then the inequality* (1.2) *is reversed.*

Corollary 3.3. *Suppose $f(x)$ is a positive continuous function on $[a,b]\subseteq\mathbb{R}$.*

(1) *If $t\geq 2$ and $f(x)\geq(t-1)(x-a)^{t-2}$, then the inequality* (1.2) *is valid;*
(2) *If $2<t\leq 3$ and $f'(x)\geq(b-a)(t-1)(x-a)^{t-2}$ on $[a,b]$, then the inequality* (1.2) *is also valid;*
(3) *If $t>3$ and $f'(x)\geq(t-1)(t-2)(x-a)^{t-3}$ on $[a,b]$, then the inequality* (1.2) *is still valid.*

Corollary 3.4. *Suppose $f(x)$ is a positive continuous function on $[a,b]\subseteq\mathbb{R}$, and suppose that either $0<f(x)\leq\frac{4}{b-a}$, or $0<f(x)\leq 1$, or $0<b-a\leq 1$. If $c>1$ and*

$$\int_a^b f(x)\,\mathrm{d}\,x\geq\frac{c^{b-a}}{b-a},$$

then

$$\int_a^b c^{f(x)}\,\mathrm{d}\,x\geq\frac{c^{\int_a^b f(x)\,\mathrm{d}\,x}}{\int_a^b f(x)\,\mathrm{d}\,x}. \tag{3.5}$$

In particular,

$$\int_a^b e^{f(x)}\,\mathrm{d}\,x\geq\frac{\exp\bigl[\int_a^b f(x)\,\mathrm{d}\,x\bigr]}{\int_a^b f(x)\,\mathrm{d}\,x}. \tag{3.6}$$

Proof. From Lemma 2.2, when $x,y>0$ and either $xy\leq 4$ or $x\leq 1$, it follows that $c^{x+y}\geq c^{xy}$. By choosing $g(u)=c^u$ in Theorem 3.1, Corollary 3.4 follows. $\square$

Theorem 3.2. *Suppose $I\subseteq\mathbb{R}_+$ is an interval, $f:[a,b]\to I$ is a continuous function and not identically zero, and $g:I\to\mathbb{R}_+$.*

(1) *For $r\neq 0$, if $g(u)$ is r-mean convex (or r-mean concave, respectively) on I, and*

$$g\bigl((b-a)^{1/r}u\bigr)\lesseqgtr g\bigl((b-a)^{1/r}\bigr)g(u) \tag{3.7}$$

for $u\in I$, and

$$\int_a^b f(x)\,\mathrm{d}\,x\gtreqless\frac{g\bigl((b-a)^{1/r}\bigr)}{(b-a)^{1/r}}, \tag{3.8}$$

then

$$\left[\int_a^b g^r(f(x))\,\mathrm{d}\,x\right]^{1/r} \gtreqless \frac{g\Big(\big(\int_a^b f^r(x)\,\mathrm{d}\,x\big)^{1/r}\Big)}{\int_a^b f(x)\,\mathrm{d}\,x}. \tag{3.9}$$

(2) *If $g(u)$ is a geometrically convex (or geometrically concave, respectively) on I, satisfying*

$$g\big(e^{(b-a)u}\big) \lesseqgtr g\big(e^{b-a}\big)g(e^u), \quad u \in I \tag{3.10}$$

and

$$\int_a^b f(x)\,\mathrm{d}\,x \gtreqless g\big(e^{b-a}\big), \tag{3.11}$$

then

$$\exp\left(\frac{1}{b-a}\int_a^b \ln g(f(x))\,\mathrm{d}\,x\right) \gtreqless \frac{g\big(\exp\big(\int_a^b \ln f(x)\,\mathrm{d}\,x\big)\big)}{\int_a^b f(x)\,\mathrm{d}\,x}. \tag{3.12}$$

Proof. Let $g(u)$ be a r-mean convex function on I and adopt the notations in (3.4). Utilizing $M_n\big(f(x), \frac{1}{n}, 1\big) \in I$ and Jensen's inequality (2.6) leads to

$$\begin{aligned}
g\left(\left[\int_a^b f^r(x)\,\mathrm{d}\,x\right]^{1/r}\right) &= g\left((b-a)^{1/r}\lim_{n\to\infty} M_n\left(f(x), \frac{1}{n}, r\right)\right)\\
&\le g\big((b-a)^{1/r}\big)\lim_{n\to\infty} g\left(M_n\left(f(x), \frac{1}{n}, r\right)\right)\\
&\le g\big((b-a)^{1/r}\big)\lim_{n\to\infty} M_n\left(g(f(x)), \frac{1}{n}, r\right)\\
&= \frac{g\big((b-a)^{1/r}\big)}{(b-a)^{1/r}}\left[\int_a^b g^r(f(x))\,\mathrm{d}\,x\right]^{1/r},
\end{aligned}$$

hence, the inequality (3.9) is true.

Let $g(u)$ be a geometrically convex function on I. Making use of Jensen's inequality (2.6) results in

$$\begin{aligned}
g\left(\exp\left(\int_a^b \ln f(x)\,\mathrm{d}\,x\right)\right) &= g\left(\exp\left((b-a)\lim_{n\to\infty} M_n\left(\ln f(x), \frac{1}{n}, 1\right)\right)\right)\\
&\le g\big(e^{b-a}\big)\lim_{n\to\infty} g\left(M_n\left(f(x), \frac{1}{n}, 0\right)\right)\\
&\le g\big(e^{b-a}\big)\lim_{n\to\infty} M_n\left(g(f(x)), \frac{1}{n}, 0\right)\\
&= g\big(e^{b-a}\big)\exp\left(\frac{1}{b-a}\int_a^b \ln g(f(x))\,\mathrm{d}\,x\right),
\end{aligned}$$

therefore, the inequality (3.12) is true.

The rest can be proved similarly. The proof of Theorem 3.2 is complete. □

Theorem 3.3. *Suppose $I \subseteq \mathbb{R}_+$ is an interval, $f : [a,b] \to I$ is a continuous function and not identically zero, and $g : I \to \mathbb{R}_+$ is a logarithmically convex (or logarithmically concave, respectively) function, satisfying*

$$g((b-a)u) \lesseqgtr g(b-a)g(u), \quad u \in I \tag{3.13}$$

and

$$\int_a^b f(x)\,\mathrm{d}\,x \gtreqless g(b-a). \tag{3.14}$$

Then we have

$$\exp\left(\frac{1}{b-a}\int_a^b \ln g(f(x))\,\mathrm{d}\,x\right) \gtreqless \frac{g\left(\int_a^b f(x)\,\mathrm{d}\,x\right)}{\int_a^b f(x)\,\mathrm{d}\,x}. \tag{3.15}$$

Proof. When $g(u)$ is a logarithmically convex function on I, Jensen's inequality (2.6) gives

$$\begin{aligned} g\left(\int_a^b f(x)\,\mathrm{d}\,x\right) &= g\left((b-a)\lim_{n\to\infty} M_n\left(f(x),\frac{1}{n},1\right)\right) \\ &\le g(b-a)\lim_{n\to\infty} g\left(M_n\left(f(x),\frac{1}{n},1\right)\right) \\ &\le g(b-a)\lim_{n\to\infty} M_n\left(g(f(x)),\frac{1}{n},0)\right) \\ &= g\left(e^{b-a}\right)\exp\left(\lim_{n\to\infty} M_n\left(\ln g(f(x)),\frac{1}{n},1\right)\right) \\ &= g\left(e^{b-a}\right)\exp\left(\frac{1}{b-a}\int_a^b \ln g(f(x))\,\mathrm{d}\,x\right), \end{aligned}$$

as a result, the inequality (3.15) is true.

The rest can be proved similarly. The proof of Theorem 3.3 is complete. □

Acknowledgements

The author was partially supported by the NNSF under Grant No. 11361038 of China and by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZY13159, China.

References

[1] M. Akkouchi, *On an integral inequality of Feng Qi*, Divulg. Mat. **13** (2005), no. 1, 11–19.
[2] K. Brahim, N. Bettaibi, and M. Sellemi, *On some Feng Qi type q-integral inequalities*, J. Inequal. Pure Appl. Math. **9** (2008), no. 2, Art. 43; Available online at http://www.emis.de/journals/JIPAM/article975.html.
[3] Y. Chen and J. Kimball, *Note on an open problem of Feng Qi*, J. Inequal. Pure Appl. Math. **7** (2006), no. 1, Art. 4; Available online at http://www.emis.de/journals/JIPAM/article621.html.
[4] V. Csiszár and T. F. Móri, *The convexity method of proving moment-type inequalities*, Statist. Probab. Lett. **66** (2004), no. 3, 303–313; Available online at http://dx.doi.org/10.1016/j.spl.2003.11.007.
[5] J.-C. Kuang, *Applied Inequalities,* 3rd edition, Shandong Science and Technology Press, Ji'nan City, China, 2004. (Chinese)
[6] S. Mazouzi and F. Qi, *On an open problem regarding an integral inequality*, J. Inequal. Pure Appl. Math. **4** (2003), no. 2, Art. 31; Available online at http://www.emis.de/journals/JIPAM/article269.html.
[7] Y. Miao, *Further development of Qi-type integral inequality*, J. Inequal. Pure Appl. Math. **7** (2006), no. 4, Art. 144; Available online at http://www.emis.de/journals/JIPAM/article763.html.

[8] Y. Miao and J.-F. Liu, *Discrete results of Qi-type inequality*, Bull. Korean Math. Soc. **46** (2009), no. 1, 125–134; Available online at http://dx.doi.org/10.4134/BKMS.2009.46.1.125.

[9] Y. Miao and F. Qi, *Several q-integral inequalities*, J. Math. Inequal. **3** (2009), no. 1, 115–121; Available online at http://dx.doi.org/10.7153/jmi-03-11.

[10] T. K. Pogány, *On an open problem of F. Qi*, J. Inequal. Pure Appl. Math. **3** (2002), no. 4, Art. 54; Available online at http://www.emis.de/journals/JIPAM/article206.html.

[11] F. Qi, *Several integral inequalities*, J. Inequal. Pure Appl. Math. **1** (2000), no. 2, Art. 19; Available online at http://www.emis.de/journals/JIPAM/article113.html.

[12] F. Qi, *Several integral inequalities*, RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 9, 1039–1042; Available online at http://rgmia.org/v2n7.php.

[13] F. Qi, A.-J. Li, W.-Z. Zhao, D.-W. Niu, and J.-Cao, *Extensions of several integral inequalities*, J. Inequal. Pure Appl. Math. **7** (2006), no. 3, Art. 107; Available online at http://www.emis.de/journals/JIPAM/article706.html.

[14] F. Qi and K.-W. Yu, *Note on an integral inequality*, J. Math. Anal. Approx. Theory **2** (2007), no. 1, 96–98.

[15] N. Towghi, *Notes on integral inequalities*, RGMIA Res. Rep. Coll. **4** (2001), no. 2, Art. 12, 277–278; Available online at http://rgmia.org/v4n2.php.

[16] S.-H. Wu, *r_P-convex function and Jensen's type inequality*, Math. Pract. Theory **35** (2005), no. 3, 220–228. (Chinese)

[17] B.-Y. Xi and T.-Y. Bao, *On some properties of r-mean convex function*, Math. Pract. Theory **38** (2008), no. 12, 113–119. (Chinese)

[18] B.-Y. Xi and F. Qi, *Some inequalities of Qi type for double integrals*, J. Egyptian Math. Soc. (2014), in press; Available online at http://dx.doi.org/10.1016/j.joems.2013.11.002.

[19] L. Yin, Q.-M. Luo, and F. Qi, *Several integral inequalities on time scales*, J. Math. Inequal. **6** (2012), no. 3, 419–429; Available online at http://dx.doi.org/10.7153/jmi-06-39.

[20] L. Yin and F. Qi, *Some integral inequalities on time scales*, Results Math. **64** (2013), no. 3, 371–381; Available online at http://dx.doi.org/10.1007/s00025-013-0320-z.

[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

[2]Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

[*]Corresponding author