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Some Hermite-Hadamard Type Inequalities via Katugampola Fractional for pq-Convex on the Interval-Valued Coordinates

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Abstract. In this paper, we established the Hermite-Hadamard inequalities via Katugampola fractional. Meanwhile, interval analysis is a particular case of set-interval analysis. We established the fractional inequalities and these results are an extension of a previous research.

1. Introduction

The classical Hermite-Hadamard inequalities such that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(t)\,dt \leq \frac{f(a)+f(b)}{2}$$

where $f : I \to \mathbb{R}$ is a convex function on the closed bounded interval I of $\mathbb{R}$,and $a, b \in I$ with $a < b$. Since then, some improved and generalized results of Hermite-Hadamard inequality on convex function have been explored and study by many authors(e.g. [2], [7], [10–12], [20], [23,24], [27], [29]).

On the other hand, interval analysis is a particular case of set-valued analysis which is the study of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. An old example of interval enclosure is Archimede's method which is related to the computation of the circumference of circle. In 1966, the first book related to interval analysis was given by Moore who is known as the first user of intervals in computational mathematics. After this book, several scientists started to investigate theory and application of interval arithmetic.

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Nowadays, because of its application, interval analysis is a useful tool in various areas related to uncertain data. We can see applications in computer graphics, experimental and computational physics, error analysis, robotics and many others.

2. Interval Calculus

A real valued interval X is bounded, closed subset of $\mathbb{R}$ and is defined by

$$X = \left[\underline{X}, \overline{X}\right] = \left\{t \in \mathbb{R} : \underline{X} \leq t \leq \overline{X}\right\}$$

where $\underline{X}, \overline{X} \in \mathbb{R}$ and $\underline{X} \leq \overline{X}$. The number $\underline{X}$ and $\overline{X}$ are called the left and right endpoints of interval X, respectively. When $\underline{X} = \overline{X} = a$, the interval X is said to be degenerate and we use the form $X = a = [a, a]$. Also we call X positive if $\underline{X} > 0$ or negative if $\overline{X} < 0$. The set of all closed intervals of $\mathbb{R}$, the sets of all closed positive intervals of $\mathbb{R}$ and closed negative intervals of $\mathbb{R}$ is denoted by $\mathbb{R}_I, \mathbb{R}_I^+$ and $\mathbb{R}_I^-$, respectively. The Pompeiu-Hausdorff distance between the intervals X and Y is defined by

$$d(X, Y) = d\left(\left[\underline{X}, \overline{X}\right], \left[\underline{Y}, \overline{Y}\right]\right) = \max\left\{\left|\underline{X} - \underline{Y}\right|, \left|\overline{X} - \overline{Y}\right|\right\}.$$

It is known that $(\mathbb{R}_I, d)$ is a complete metric space.

Now, we give the definitions of basic interval arithmetic operations for the intervals X and Y as follows:

$$X + Y = \left[\underline{X} + \underline{Y}, \overline{X} + \overline{Y}\right],$$

$$X - Y = \left[\underline{X} - \overline{Y}, \overline{X} - \underline{Y}\right],$$

$$X \cdot Y = [\min S, \max S] \text{ where } S = \left\{\underline{XY}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{XY}\right\},$$

$$X/Y = [\min T, \max T] \text{ where } T = \left\{\underline{X}/\underline{Y}, \underline{X}/\overline{Y}, \overline{X}/\underline{Y}, \overline{X}/\overline{Y}\right\} \text{ and } 0 \notin Y.$$

Scalar multiplication of the interval X is defined by

$$\lambda X = \lambda\left[\underline{X}, \overline{X}\right] = \begin{cases} \left[\lambda\underline{X}, \lambda\overline{X}\right], & \lambda > 0, \\ 0, & \lambda = 0, \\ \left[\lambda\overline{X}, \lambda\underline{X}\right], & \lambda < 0, \end{cases}$$

where $\lambda \in \mathbb{R}$.

The opposits of the interval X is

$$-X := (-1)X = \left[-\overline{X}, -\underline{X}\right],$$

where $\lambda = -1$.

The subtraction is given by

$$X - Y = X + (-Y) = \left[\underline{X} - \overline{Y}, \overline{X} - \underline{Y}\right].$$

In general, $-X$ is not additive inverse for X, i.e. $X - X \neq 0$.

Use of monotonic functions

$$F(X) = \left[F(\underline{X}), F(\overline{X})\right].$$

The definitons of operations lead to a number of algebraic properties which allows $\mathbb{R}_I$ to be quasi-linear space. They can be listed as follows

(1)(Associativity of addition) $(X + Y) + Z = X + (Y + Z)$ for all $X, Y, Z \in \mathbb{R}_I$,

(2)(Additivity elemant) $X + 0 = 0 + X = X$ for all $X \in \mathbb{R}_I$,

(3)(Commutativity of addition) $X + Y = Y + X$ for all $X, Y \in \mathbb{R}_I$,

(4)(Cancellation law) $X + Z = Y + Z \Longrightarrow X = Y$ for all $X, Y, Z \in \mathbb{R}_I$,

(5)(Associativity of multiplication) $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$ for all $X, Y, Z \in \mathbb{R}_I$,

(6)(Commutativity of multiplication) $X \cdot Y = Y \cdot X$ for all $X, Y \in \mathbb{R}_I$,

(7)(Unity element) $X \cdot 1 = 1 \cdot X$ for all $X \in \mathbb{R}_I$,

(8)(Associativity law) $\lambda(\mu X) = (\lambda\mu) X$ for all $X \in \mathbb{R}_I$, and for all $\lambda, \mu \in \mathbb{R}$,

(9)(First distributiviyu law) $\lambda(X + Y) = \lambda X + \lambda Y$ for all $X, Y \in \mathbb{R}_I$, and for all $\lambda \in \mathbb{R}$,

(10)(Second distributiviyu law) $(\lambda + \mu) X = \lambda X + \mu X$ for all $X \in \mathbb{R}_I$, and for all $\lambda, \mu \in \mathbb{R}$.

But, this law holds in certain cases. If $Y \cdot Z > 0$, then

$$X \cdot Y + Z = X \cdot Y + X \cdot Z.$$

What's more, one of the set property is the inclusion $\subseteq$ that is given by

$$X \subseteq Y \Longleftrightarrow \underline{Y} \leq \underline{X} \text{ and } \overline{X} \leq \overline{Y}.$$

Considering together with arthmetic operations and inclusion, one has the following property which is called inclusion isotone of interval operations:

Let $\odot$ be the addition, multiplication, subtraction or division. If X, Y, Z and T areintervals such that

$$X \subseteq Y \text{ and } Z \subseteq T,$$

then the following relation is valid

$$X \odot Z \subseteq Y \odot T.$$

3. Intgral of Interval-Valued Functions

In this section, the notion of integral is mentioned for interval-valued functions. Before the definition of integral, the necessary concepts will be given as the following:

A function F is said to be an interval-valued function of t on $[a, b]$, if it assigns a nonempty interval to each $t \in [a, b]$,

$$F(t) = \left[\underline{F}(t), \overline{F}(t)\right].$$

A partition of $[a, b]$ is any finite ordered subset P having the form:

$$P : a = t_0 < t_1 < ... < t_n = b.$$

The mesh of a partition P defined by

$$mesh(P) = \max\{t_i - t_{i-1} : i = 1, 2, ..., n\}.$$

We denoted by $P([a, b])$ the set of all partition of $[a, b]$. Let $P(\delta, [a, b])$ be the set of all $P \in P([a, b])$ such that $mesh(P) < \delta$. Choose an arbitrary point ξ_i in interval $[t_{i-1}, t_i], (i = 1, 2, ..., n)$ and let us define the sum

$$S(F, P, \delta) = \sum_{i=1}^{n} F(\xi_i)[t_i - t_{i-1}],$$

where $F : [a, b] \to \mathbb{R}_I$. We call $S(F, P, \delta)$ a Riemann sum of F corresponding to $P \in P(\delta, [a, b])$.

Definition 3.1

A function $F : [a, b] \to \mathbb{R}_I$ is called interval Riemann intrgrable ((IR)-integrable) on $[a, b]$, if there exists $A \in \mathbb{R}_I$ such that, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(S(F, P, \delta), A) < \epsilon$$

for every Riemann sum S of F corresponding to each $P \in P(\delta, [a, b])$ and independent from choice of $\xi_i \in [t_{i-1}, t_i]$ for all $1 \leq i \leq n$. In this case, A is called the (IR)-integral of F on $[a, b]$ and is denoted by

$$A = (IR)\int_a^b F(t)\,dt.$$

The collection of all functions that are (IR)-integrable on $[a, b]$ will be denoted by $IR_{([a,b])}$.

The following theorem gives relation between $(IR)-$integrable and Riemann integrable (R)-integrable.

Theorem 3.2

Let $F : [a, b] \to \mathbb{R}_I$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. $F \in IR_{([a,b])}$ if and only if $\underline{F}(t), \overline{F}(t) \in R_{([a,b])}$ and

$$(IR)\int_a^b F(t)\,dt = \left[(R)\int_a^b \underline{F}(t)\,dt, (R)\int_a^b \overline{F}(t)\,dt\right],$$

where $R_{([a,b])}$ denoted the all R-integrable functions.

It is seen easily that, if $F(t) \subseteq G(t)$ for all $t \in [a,b]$, then

$$(IR)\int_a^b F(t)\,dt \subseteq (IR)\int_a^b G(t)\,dt.$$

Furthermore, if $\{t_{i-1}, t_i\}_{i=1}^{m}$ is a δ-fine P_1 of $[a,b]$ and if $\{s_{j-1}, s_j\}_{j=1}^{n}$ is a δ-fine P_2 of $[c,d]$, then retangles

$$\triangle_{i,j} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$$

are the partition of retangle $\triangle = [a,b] \times [c,d]$ and the point (ξ_i, η_j) are inside the retangles $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$. And we denote the set of all δ-fine partition P of $\triangle$ with $P_1 \times P_2$, where $P_1 \in P(\delta, [a,b])$ and $P_2 \in P(\delta, [c,d])$. Let $\triangle A_{i,j}$ be the area retangle $\triangle_{i,j}$, where $1 \leq i \leq m, 1 \leq j \leq n$, choose arbitrary (ξ_i, η_j) and get

$$S(F, P, \delta, \triangle) = \sum_{i=1}^{m}\sum_{j=1}^{n} F(\xi_i, \eta_j) \triangle A_{i,j}.$$

Definition 3.3

A function $F : \triangle \to \mathbb{R}_I$ is called interval double Riemann integrable ((ID)-integrable) on $\triangle = [a,b] \times [c,d]$ with the ID-integral $I = (ID)\iint_{\triangle} F(t,s)\,dA$, if there exists $I \in \mathbb{R}_I$ such that, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(S(F, P, \delta, \triangle), I) < \epsilon$$

for each $P \in P(\delta, \triangle)$. We denote by $IR_{(\triangle)}$ the set of all ID-integrable function on $\triangle$, and by $R_{([a,b])}, IR_{([a,b])}$, the set of all R-integrable and IR-integrable functions on $[a,b]$,respectively.

Theorem 3.4

Let $\triangle = [a,b] \times [c,d]$. If $F : \triangle \to \mathbb{R}_I$ is ID-integrable on $\triangle$, then we have

$$(ID)\iint_{\triangle} F(t,s)\,dA = (IR)\int_a^b (IR)\int_c^d F(s,t)\,ds\,dt.$$

In [21], Sadowska obtained the following Hermite-Hadamard inequality for interval-valued functions:

Theorem 3.5

Let $F : [a,b] \to \mathbb{R}_I^+$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $F \in IR_{([a,b])}$. Then

$$\frac{F(a)+F(b)}{2} \subseteq \frac{1}{b-a}(IR)\int_a^b F(t)\,dt \subseteq F\left(\frac{a+b}{2}\right).$$

4. Fractional Integrals

In [2], Katugampola introduced a new fractional which generalizes the Riemann-Liouville and the Hadamard fractional integrals into a single form as follow.

Definition 4.1

Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integralsod order $\alpha > 0$ of $f \in X_c^p(a, b)$ are defined by

$$^{\rho}I_{a^+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)\, dt$$

and

$$^{\rho}I_{b^-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)\, dt$$

where $a < x < b$ and $\rho > 0$, if the integral exists.

Theorem 4.2

Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$,

1. $\lim\limits_{\rho \to 1} {}^{\rho}I_{a^+}^{\alpha} f(x) = J_{a^+}^{\alpha} f(x)$,
2. $\lim\limits_{\rho \to 0^+} {}^{\rho}I_{a^+}^{\alpha} f(x) = H_{a^+}^{\alpha} f(x)$.

Similar results also hold for right-sided operators.

Theorem 4.3

Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a positive function with $0 \le a \le b$ and $f \in X_c^p(a, b)$. If f is also a convex function on $[a, b]$, then the following inequalities hold:

$$f\left(\frac{a^\rho + b^\rho}{2}, \frac{c^\rho + d^\rho}{2}\right) \le \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[{}^{\rho}I_{a^+}^{\alpha} f(b^\rho) + {}^{\rho} I_{b^-}^{\alpha} f(a^\rho)\right] \le \frac{f(a^\rho) + f(b^\rho)}{2}$$

where the fractional integral are considered for the function $f(x^\rho)$ and evaluated at a and b, respectively.

In [28], Yaldiz established the new definitions and theorem related Katugampola fractional integrals for two variables functions:

Definition 4.4

Let $f \in L_1([a^\rho, b^\rho] \times [c^\rho, d^\rho])$. The Katugampola fractional integrals ${}^{\rho}I_{a^+,c^+}^{\alpha,\beta} f(x, y)$, ${}^{\rho}I_{a^+,d^-}^{\alpha,\beta} f(x, y)$, ${}^{\rho} I_{b^-,c^+}^{\alpha,\beta} f(x, y)$ and ${}^{\rho}I_{b^-,d^-}^{\alpha,\beta} f(x, y)$ of $\alpha, \beta > 0$ are defined by

$$\begin{aligned} {}^{\rho}I_{a^+,c^+}^{\alpha,\beta} f(x, y) &: = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_a^x \int_c^y \frac{t^{\rho-1} s^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha} (y^\rho - s^\rho)^{1-\beta}} f(t, s)\, ds dt, \\ x &> a, y > c, \end{aligned}$$

$$\begin{aligned} {}^{\rho}I_{a^+,d^-}^{\alpha,\beta} f(x, y) &: = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_a^x \int_y^d \frac{t^{\rho-1} s^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha} (s^\rho - y^\rho)^{1-\beta}} f(t, s)\, ds dt, \\ x &> a, y < d, \end{aligned}$$

$$\begin{aligned} {}^{\rho}I^{\alpha,\beta}_{b^-,c^+}f\left(x,y\right) &:= \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\frac{\rho^{1-\beta}}{\Gamma\left(\beta\right)}\int_x^b\int_c^y \frac{t^{\rho-1}s^{\rho-1}}{\left(t^\rho-x^\rho\right)^{1-\alpha}\left(y^\rho-s^\rho\right)^{1-\beta}}f\left(t,s\right)dsdt, \\ x &< b, y>c, \end{aligned}$$

$$\begin{aligned} {}^{\rho}I^{\alpha,\beta}_{b^-,d^-}f\left(x,y\right) &:= \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\frac{\rho^{1-\beta}}{\Gamma\left(\beta\right)}\int_x^b\int_y^d \frac{t^{\rho-1}s^{\rho-1}}{\left(t^\rho-x^\rho\right)^{1-\alpha}\left(s^\rho-y^\rho\right)^{1-\beta}}f\left(t,s\right)dsdt, \\ x &< b, y<d, \end{aligned}$$

with $a<x<b$ and $c<y<d$ with $\rho>0$.

Similarly, we introduce the following integrals:

$${}^{\rho}I^{\alpha}_{a^+}f\left(x,\frac{c+d}{2}\right) := \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_a^x \frac{t^{\rho-1}}{\left(x^\rho-t^\rho\right)^{1-\alpha}}f\left(t,\frac{c+d}{2}\right)dt, \quad x>a,$$

$${}^{\rho}I^{\alpha}_{b^-}f\left(x,\frac{c+d}{2}\right) := \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_x^b \frac{t^{\rho-1}}{\left(t^\rho-x^\rho\right)^{1-\alpha}}f\left(t,\frac{c+d}{2}\right)dt, \quad x<b,$$

$${}^{\rho}I^{\beta}_{c^+}f\left(\frac{a+b}{2},y\right) := \frac{\rho^{1-\beta}}{\Gamma\left(\beta\right)}\int_c^y \frac{s^{\rho-1}}{\left(y^\rho-s^\rho\right)^{1-\beta}}f\left(\frac{a+b}{2},s\right)ds, \quad y>c,$$

and

$${}^{\rho}I^{\alpha}_{d^-}f\left(\frac{a+b}{2},y\right) := \frac{\rho^{1-\beta}}{\Gamma\left(\beta\right)}\int_a^x \frac{s^{\rho-1}}{\left(s^\rho-y^\rho\right)^{1-\beta}}f\left(\frac{a+b}{2},s\right)ds, \quad y<d.$$

Theorem 4.5

Let $\alpha,\beta>0$ and $\rho>0$. Let $f:\triangle^\rho\subset\mathbb{R}^2\to\mathbb{R}$ be a coordinated convex on $\triangle^\rho:=[a^\rho,b^\rho]\times[c^\rho,d^\rho]$ in $\mathbb{R}^2$ with $0\leq a\leq b, 0\leq c\leq d$ and $f\in L_1\left(\triangle^\rho\right)$. Then the following inequalities hold:

$$\begin{aligned} & f\left(\frac{a^\rho+b^\rho}{2},\frac{c^\rho+d^\rho}{2}\right) \\ \leq\ & \frac{\rho^{\alpha+\beta}\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}{4\left(b^\rho-a^\rho\right)^\alpha\left(d^\rho-c^\rho\right)^\beta} \\ & \times\left[{}^{\rho}I^{\alpha,\beta}_{a^+,c^+}f\left(b,d\right)+{}^{\rho}I^{\alpha,\beta}_{a^+,d^-}f\left(b,c\right)+{}^{\rho}I^{\alpha,\beta}_{b^-,c^+}f\left(a,d\right)+{}^{\rho}I^{\alpha,\beta}_{b^-,d^-}f\left(a,c\right)\right] \\ \leq\ & \frac{f\left(a^\rho,c^\rho\right)+f\left(a^\rho,d^\rho\right)+f\left(b^\rho,c^\rho\right)+f\left(b^\rho,d^\rho\right)}{4} \end{aligned}$$

with $a<x<b$ and $c<y<d$.

Definition 4.6

Let $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\backslash\{0\}$. A function $f:I\to\mathbb{R}$ is said to be a p-convex function, if

$$f\left(\left[tx^p+(1-t)\,y^p\right]^{\frac{1}{p}}\right)\leq tf\left(x\right)+\left(1-t\right)f\left(y\right)$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality is reserved, then f is said to be p-concave.

In [5], Fang and Shi established the following inequlaity

Theorem 4.7

Let $f : I \to \mathbb{R}$ be a p-convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then we have

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a) + f(b)}{2}.$$

In [27], Toplu et al. established the following inequality

Theorem 4.8

Let $f : I \to \mathbb{R}$ be a p-convex function, $p > 0$, $\alpha > 0$ and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequality for fractional integrals holds:

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p^\alpha \Gamma(\alpha + 1)}{b^p - a^p}\left[{}^pI^\alpha_{a^+} f(b) + {}^p I^\alpha_{b^-} f(a)\right] \leq \frac{f(a) + f(b)}{2}.$$

In this paper, we can give a different version of the definition of the pq-convex function as below.

Definition 4.9

Let $I \subset (0, \infty) \times (0, \infty)$be a real interval and $p, q \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a pq-convex function, if

$$\begin{aligned} & f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}, \left[\lambda z^q + (1-\lambda)w^q\right]^{\frac{1}{q}}\right) \\ \leq \;& t\lambda f(x, z) + t(1-\lambda) f(x, w) + (1-t)\lambda f(y, z) + (1-t)(1-\lambda) f(y, w) \end{aligned}$$

for all $(x, z), (x, w), (y, z), (y, w) \in I$ and $t, \lambda \in [0, 1]$. If the inequality is reserved, then f is said to be pq-concave.

We recall the following special functions and inequalities.

(1)The Gamma Function:

The Gamma Γ function is defined by

$$\Gamma(z) = \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$$

for all complex numbers z with $\mathrm{Re}(z) > 0$, respectively. The gamma function is a natural extension of the factorial from integers n to real (and complex) numbers z.

(2)The Beta Function:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x, y > 0.$$

(3)The Hypergeometric Function:

$${}_2F_1(a, b; c, z) = \frac{1}{\beta(b, c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \;\; c > b > 0, |z| < 1.$$

5. Main Result

Theorem 5.1

Let $f : I \times I \to \mathbb{R}$ be an interval-valued pq-convex function such that $f(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ and the order $p, q > 0$, $\alpha, \beta > 0$ and $a, b, c, d \in I$ with $a < b$ and $c < d$. If $f \in ID_{([a,b]\times[c,d])}$, then the following inequality for fractional integrals holds:

$$
\begin{aligned}
& f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}, \left[\frac{c^q+d^q}{2}\right]^{\frac{1}{q}}\right) \\
\supseteq\ & \frac{p^\alpha q^\beta \Gamma(\alpha+1)\Gamma(\beta+1)}{(b^p-a^p)^\alpha (d^q-c^q)^\beta} \\
& \times \left[{}^{p,q}I^{\alpha,\beta}_{a^+,c^+} f(b,d) + {}^{p,q}I^{\alpha,\beta}_{a^+,d^-} f(b,c) + {}^{p,q}I^{\alpha,\beta}_{b^-,c^+} f(a,d) + {}^{p,q}I^{\alpha,\beta}_{b^-,d^-} f(a,c)\right] \\
\supseteq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

Proof :

Since f is pq-convex function on $[a,b]\times[c,d]$, we have for all $(x,z), (x,w), (y,z), (y,w) \in [a,b]\times[c,d]$ (with $t, \lambda = \frac{1}{2}$)

$$
\begin{aligned}
& f\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}, \left[\frac{z^q+w^q}{2}\right]^{\frac{1}{q}}\right) \\
\supseteq\ & \frac{f(x,z)+f(x,w)+f(y,z)+f(y,w)}{4}.
\end{aligned}
$$

By choosing $x = [ta^p + (1-t)b^p]^{\frac{1}{p}}$, $y = [(1-t)a^p + tb^p]^{\frac{1}{p}}$, $z = [\lambda c^q + (1-\lambda)d^q]^{\frac{1}{q}}$ and $w = [(1-\lambda)c^q + \lambda d^q]^{\frac{1}{q}}$, then we get

$$
\begin{aligned}
& f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}, \left[\frac{c^q+d^q}{2}\right]^{\frac{1}{q}}\right) \\
\supseteq\ & \frac{1}{4}\left[f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}, [\lambda c^q+(1-\lambda)d^q]^{\frac{1}{q}}\right) + f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}, [(1-\lambda)c^q+\lambda d^q]^{\frac{1}{q}}\right)\right. \\
& \left. + f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}, [\lambda c^q+(1-\lambda)d^q]^{\frac{1}{q}}\right) + f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}, [(1-\lambda)c^q+\lambda d^q]^{\frac{1}{q}}\right)\right].
\end{aligned}
$$

Multiplying both sides of the inequality by $t^{\alpha-1}\lambda^{\beta-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$ and with respect to λ over $[0,1]$, then we obtain,

$$
\begin{aligned}
& \frac{4}{\alpha\beta} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}, \left[\frac{c^q+d^q}{2}\right]^{\frac{1}{q}}\right) \\
\supseteq\ & (ID)\int_0^1\int_0^1 \left[f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}, [\lambda c^q+(1-\lambda)d^q]^{\frac{1}{q}}\right) + f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}, [(1-\lambda)c^q+\lambda d^q]^{\frac{1}{q}}\right)\right. \\
& \left. + f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}, [\lambda c^q+(1-\lambda)d^q]^{\frac{1}{q}}\right) + f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}, [(1-\lambda)c^q+\lambda d^q]^{\frac{1}{q}}\right)\right] dt d\lambda
\end{aligned}
$$

$$
\begin{aligned}
&\supseteq (ID)\int_c^d\int_a^b \left(\frac{b^p-x^p}{b^p-a^p}\right)^{\alpha-1}\left(\frac{d^q-y^q}{d^q-c^q}\right)^{\beta-1} f(x,y)\,\frac{px^{p-1}}{b^p-a^p}\frac{qy^{q-1}}{d^q-c^q}dxdy\\
&+(ID)\int_c^d\int_a^b \left(\frac{b^p-x^p}{b^p-a^p}\right)^{\alpha-1}\left(\frac{y^q-c^q}{d^q-c^q}\right)^{\beta-1} f(x,y)\,\frac{px^{p-1}}{b^p-a^p}\frac{qy^{q-1}}{d^q-c^q}dxdy\\
&+(ID)\int_c^d\int_a^b \left(\frac{x^p-a^p}{b^p-a^p}\right)^{\alpha-1}\left(\frac{d^q-y^q}{d^q-c^q}\right)^{\beta-1} f(x,y)\,\frac{px^{p-1}}{b^p-a^p}\frac{qy^{q-1}}{d^q-c^q}dxdy\\
&+(ID)\int_c^d\int_a^b \left(\frac{x^p-a^p}{b^p-a^p}\right)^{\alpha-1}\left(\frac{y^q-c^q}{d^q-c^q}\right)^{\beta-1} f(x,y)\,\frac{px^{p-1}}{b^p-a^p}\frac{qy^{q-1}}{d^q-c^q}dxdy\\
= \;&\frac{pq}{(b^p-a^p)^{\alpha}(d^q-c^q)^{\beta}}\left[{}^{p,q}I^{\alpha,\beta}_{a^+,c^+}f(b,d)+{}^{p,q}I^{\alpha,\beta}_{a^+,d^-}f(b,c)+{}^{p,q}I^{\alpha,\beta}_{b^-,c^+}f(a,d)+{}^{p,q}I^{\alpha,\beta}_{b^-,d^-}f(a,c)\right].
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\left[\frac{c^q+d^q}{2}\right]^{\frac{1}{q}}\right)\\
\supseteq\;&\frac{p^{\alpha}q^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b^p-a^p)^{\alpha}(d^q-c^q)^{\beta}}\\
&\times\left[{}^{p,q}I^{\alpha,\beta}_{a^+,c^+}f(b,d)+{}^{p,q}I^{\alpha,\beta}_{a^+,d^-}f(b,c)+{}^{p,q}I^{\alpha,\beta}_{b^-,c^+}f(a,d)+{}^{p,q}I^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]
\end{aligned}
$$

which completes the proof of the first inequality.

For the second inequalitiy, br using pq-convexity of f, we have

$$
\begin{aligned}
&f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}},\left[\lambda c^q+(1-\lambda)d^q\right]^{\frac{1}{q}}\right)\\
\supseteq\;& t\lambda f(a,c)+t(1-\lambda)f(a,d)+(1-t)\lambda f(b,c)+(1-t)(1-\lambda)f(b,d),
\end{aligned}
$$

$$
\begin{aligned}
&f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}},\left[\lambda c^q+(1-\lambda)d^q\right]^{\frac{1}{q}}\right)\\
\supseteq\;&(1-t)\lambda f(a,c)+(1-t)(1-\lambda)f(a,d)+t\lambda f(b,c)+t(1-\lambda)f(b,d),
\end{aligned}
$$

$$
\begin{aligned}
&f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}},\left[(1-\lambda)c^q+\lambda d^q\right]^{\frac{1}{q}}\right)\\
\supseteq\;& t(1-\lambda)f(a,c)+t\lambda f(a,d)+(1-t)(1-\lambda)f(b,c)+(1-t)\lambda f(b,d),
\end{aligned}
$$

and

$$
\begin{aligned}
&f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}},\left[(1-\lambda)c^q+\lambda d^q\right]^{\frac{1}{q}}\right)\\
\supseteq\;&(1-t)(1-\lambda)f(a,c)+(1-t)\lambda f(a,d)+t(1-\lambda)f(b,c)+t\lambda f(b,d).
\end{aligned}
$$

By adding these inequalities, then, we have

$$\begin{aligned} & f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}},\left[\lambda c^q+(1-\lambda)\,d^q\right]^{\frac{1}{q}}\right)+f\left(\left[(1-t)\,a^p+tb^p\right]^{\frac{1}{p}},\left[\lambda c^q+(1-\lambda)\,d^q\right]^{\frac{1}{q}}\right) \\ & +f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}},\left[(1-\lambda)\,c^q+\lambda d^q\right]^{\frac{1}{q}}\right)+f\left(\left[(1-t)\,a^p+tb^p\right]^{\frac{1}{p}},\left[(1-\lambda)\,c^q+\lambda d^q\right]^{\frac{1}{q}}\right) \\ \supseteq \quad & f\,(a,c)+f\,(a,d)+f\,(b,c)+f\,(b,d)\,. \end{aligned}$$

Multiplying both sides of the inequality by $t^{\alpha-1}\lambda^{\beta-1}$, $\alpha>0,\beta>0$ and then integrating the resulting inequality with respect to t over $[0,1]$ and with respect to λ over $[0,1]$, then we similarly obtain,

$$\begin{aligned} & \frac{p^{\alpha}q^{\beta}\Gamma(\alpha+1)\,\Gamma(\beta+1)}{(b^p-a^p)^{\alpha}\,(d^q-c^q)^{\beta}} \\ & \times\left[{}^{p,q}I^{\alpha,\beta}_{a^+,c^+}f\,(b,d)+{}^{p,q}\,I^{\alpha,\beta}_{a^+,d^-}f\,(b,c)+{}^{p,q}\,I^{\alpha,\beta}_{b^-,c^+}f\,(a,d)+{}^{p,q}\,I^{\alpha,\beta}_{b^-,d^-}f\,(a,c)\right] \\ \supseteq \quad & \frac{f\,(a,c)+f\,(a,d)+f\,(b,c)+f\,(b,d)}{4}. \end{aligned}$$

Lemma 5.2

Let $f: I\times I\to\mathbb{R}$ be a partial differentiable function with $0\le a<b$ and $0\le c<d$. Then the equality holds.

$$\begin{aligned} & K_f\,(\alpha,\beta,a,b,c,d) \\ = \quad & \frac{(b^p-a^p)\,(d^q-c^q)}{4pq}\left\{\int_0^1\int_0^1\frac{\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]\frac{\partial^2}{\partial t\partial\lambda}f\,(M_p\,(a,b,t)\,,M_q\,(c,d,\lambda))}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}dtd\lambda\right\} \\ = \quad & \frac{1}{4}\left\{f\,(a,c)+f\,(a,d)+f\,(b,c)+f\,(b,d)\right. \\ & +\frac{p^{\alpha}q^{\beta}\Gamma(\alpha+1)\,\Gamma(\beta+1)}{(b^p-a^p)^{\alpha}\,(d^q-c^q)^{\beta}}\left[{}^{p,q}I^{\alpha,\beta}_{a^+,c^+}f\,(b,d)+{}^{p,q}\,I^{\alpha,\beta}_{a^+,d^-}f\,(b,c)+{}^{p,q}\,I^{\alpha,\beta}_{b^-,c^+}f\,(a,d)+{}^{p,q}\,I^{\alpha,\beta}_{b^-,d^-}f\,(a,c)\right] \\ & -\frac{p^{\alpha}\Gamma(\alpha+1)}{(b^p-a^p)^{\alpha}}\left[{}^{p}I^{\alpha}_{a^+}f\,(x,c)+{}^{p}\,I^{\alpha}_{b^-}f\,(x,c)+{}^{p}\,I^{\alpha}_{a^+}f\,(x,d)+{}^{p}\,I^{\alpha}_{b^-}f\,(x,d)\right] \\ & \left.-\frac{q^{\beta}\Gamma(\beta+1)}{(d^q-c^q)^{\beta}}\left[{}^{q}I^{\beta}_{c^+}f\,(a,y)+{}^{q}\,I^{\beta}_{d^-}f\,(a,y)+{}^{q}\,I^{\beta}_{c^+}f\,(b,y)+{}^{q}\,I^{\beta}_{d^-}f\,(b,y)\right]\right\} \end{aligned}$$

where $M_p\,(a,b,t)=\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}$ and $M_q\,(c,d,\lambda)=\left[\lambda c^q+(1-\lambda)\,d^q\right]^{\frac{1}{q}}$.

proof :

Let

$$I_1=\frac{(b^p-a^p)\,(d^q-c^q)}{4pq}\left\{\int_0^1\int_0^1\frac{(1-t)^{\alpha}\,(1-\lambda)^{\beta}\,\frac{\partial^2}{\partial t\partial\lambda}f\,(M_p\,(a,b,t)\,,M_q\,(c,d,\lambda))}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}dtd\lambda\right\}$$

$$I_2=\frac{(b^p-a^p)\,(d^q-c^q)}{4pq}\left\{\int_0^1\int_0^1\frac{(1-t)^{\alpha}\,\lambda^{\beta}\,\frac{\partial^2}{\partial t\partial\lambda}f\,(M_p\,(a,b,t)\,,M_q\,(c,d,\lambda))}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}dtd\lambda\right\}$$

$$I_3 = \frac{(b^p - a^p)(d^q - c^q)}{4pq}\left\{\int_0^1\int_0^1 \frac{t^\alpha (1-\lambda)^\beta \frac{\partial^2}{\partial t \partial \lambda} f(M_p(a,b,t), M_q(c,d,\lambda))}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q + (1-\lambda)d^q]^{1-\frac{1}{q}}} dtd\lambda\right\}$$

and

$$I_4 = \frac{(b^p - a^p)(d^q - c^q)}{4pq}\left\{\int_0^1\int_0^1 \frac{t^\alpha \lambda^\beta \frac{\partial^2}{\partial t \partial \lambda} f(M_p(a,b,t), M_q(c,d,\lambda))}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q + (1-\lambda)d^q]^{1-\frac{1}{q}}} dtd\lambda\right\}.$$

By using integrating by part, we have,

$$\begin{aligned} I_1 &= \frac{1}{4}\left\{f(b,d) - \frac{\beta q}{(d^q - c^q)}\int_c^d \frac{y^{q-1}}{(y^q - c^q)^{1-\beta}} f(b,y)\,dy - \frac{\alpha p}{(b^p - a^p)}\int_a^b \frac{x^{p-1}}{(x^p - a^p)^{1-\alpha}} f(x,d)\,dx \right. \\ &\left. + \frac{\alpha p}{(b^p - a^p)}\frac{\beta q}{(d^q - c^q)}\int_c^d\int_a^b \frac{x^{p-1}}{(x^p - a^p)^{1-\alpha}}\frac{y^{q-1}}{(y^q - c^q)^{1-\beta}} f(x,y)\,dxdy\right\} \end{aligned}$$

and similarly we get

$$\begin{aligned} I_2 &= \frac{-1}{4}\left\{f(b,c) - \frac{\beta q}{(d^q - c^q)}\int_c^d \frac{y^{q-1}}{(d^q - y^q)^{1-\beta}} f(b,y)\,dy - \frac{\alpha p}{(b^p - a^p)}\int_a^b \frac{x^{p-1}}{(x^p - a^p)^{1-\alpha}} f(x,c)\,dx \right. \\ &\left. + \frac{\alpha p}{(b^p - a^p)}\frac{\beta q}{(d^q - c^q)}\int_c^d\int_a^b \frac{x^{p-1}}{(x^p - a^p)^{1-\alpha}}\frac{y^{q-1}}{(d^q - y^q)^{1-\beta}} f(x,y)\,dxdy\right\} \end{aligned}$$

$$\begin{aligned} I_3 &= \frac{-1}{4}\left\{f(a,d) - \frac{\beta q}{(d^q - c^q)}\int_c^d \frac{y^{q-1}}{(y^q - c^q)^{1-\beta}} f(b,y)\,dy - \frac{\alpha p}{(b^p - a^p)}\int_a^b \frac{x^{p-1}}{(b^p - x^p)^{1-\alpha}} f(x,d)\,dx \right. \\ &\left. + \frac{\alpha p}{(b^p - a^p)}\frac{\beta q}{(d^q - c^q)}\int_c^d\int_a^b \frac{x^{p-1}}{(b^p - x^p)^{1-\alpha}}\frac{y^{q-1}}{(y^q - c^q)^{1-\beta}} f(x,y)\,dxdy\right\} \end{aligned}$$

and

$$\begin{aligned} I_4 &= \frac{1}{4}\left\{f(a,c) - \frac{\beta q}{(d^q - c^q)}\int_c^d \frac{y^{q-1}}{(d^q - y^q)^{1-\beta}} f(b,y)\,dy - \frac{\alpha p}{(b^p - a^p)}\int_a^b \frac{x^{p-1}}{(b^p - x^p)^{1-\alpha}} f(x,d)\,dx \right. \\ &\left. + \frac{\alpha p}{(b^p - a^p)}\frac{\beta q}{(d^q - c^q)}\int_c^d\int_a^b \frac{x^{p-1}}{(b^p - x^p)^{1-\alpha}}\frac{y^{q-1}}{(d^q - y^q)^{1-\beta}} f(x,y)\,dxdy\right\}. \end{aligned}$$

So that we combine $I_1 - I_2 - I_3 + I_4$, we will get the equality.

Theorem 5.3

Let $f : I \times I \to \mathbb{R}$ be an interval-valued pq-convex function such that $f(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ and the order $p, q > 0$, $\alpha, \beta > 0$ and $a, b, c, d \in I$ with $a < b$ and $c < d$. If $f \in ID_{([a,b]\times[c,d])}$ and $\left|\frac{\partial^2}{\partial t \partial \lambda} f\right|^m$ is $pq - convex$ on $[a, b] \times [c, d]$ for $m \geq 1$, then the following inequality for fractional integrals holds:

$$\left|\frac{1}{4}\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\right.$$
$$+\frac{p^{\alpha}q^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b^p-a^p)^{\alpha}(d^q-c^q)^{\beta}}\left[{}^{p,q}I^{\alpha,\beta}_{a^+,c^+}f(b,d)+{}^{p,q}I^{\alpha,\beta}_{a^+,d^-}f(b,c)+{}^{p,q}I^{\alpha,\beta}_{b^-,c^+}f(a,d)+{}^{p,q}I^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]$$
$$-\frac{p^{\alpha}\Gamma(\alpha+1)}{(b^p-a^p)^{\alpha}}\left[{}^{p}I^{\alpha}_{a^+}f(x,c)+{}^{p}I^{\alpha}_{b^-}f(x,c)+{}^{p}I^{\alpha}_{a^+}f(x,d)+{}^{p}I^{\alpha}_{b^-}f(x,d)\right]$$
$$\left.-\frac{q^{\beta}\Gamma(\beta+1)}{(d^q-c^q)^{\beta}}\left[{}^{q}I^{\beta}_{c^+}f(a,y)+{}^{q}I^{\beta}_{d^-}f(a,y)+{}^{q}I^{\beta}_{c^+}f(b,y)+{}^{q}I^{\beta}_{d^-}f(b,y)\right]\right|$$
$$\supseteq \frac{(b^p-a^p)(d^q-c^q)}{4pq}M_1^{1-\frac{1}{m}}(\alpha,\beta)\times\left\{M_2(\alpha,\beta)\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,c)\right|^m+M_3(\alpha,\beta)\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,d)\right|^m\right.$$
$$\left.+M_4(\alpha,\beta)\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,c)\right|^m+M_5(\alpha,\beta)\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,d)\right|^m\right\}^{\frac{1}{m}}.$$

Proof :

From Lemma by using the property of the modulus, the power mean inequality and the pq-convexity of $\left|\frac{\partial^2}{\partial t\partial\lambda}f\right|^m$, then we have

$$\left|\frac{(b^p-a^p)(d^q-c^q)}{4pq}\left\{(ID)\int_0^1\int_0^1\frac{[(1-t)^{\alpha}-t^{\alpha}]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]\frac{\partial^2}{\partial t\partial\lambda}f(M_p(a,b,t),M_q(c,d,\lambda))}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}dtd\lambda\right\}\right|$$
$$\supseteq \frac{(b^p-a^p)(d^q-c^q)}{4pq}$$
$$\times\left\{(ID)\int_0^1\int_0^1\frac{\left|[(1-t)^{\alpha}-t^{\alpha}]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]\right|\cdot\left|\frac{\partial^2}{\partial t\partial\lambda}f(M_p(a,b,t),M_q(c,d,\lambda))\right|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}dtd\lambda\right\}$$
$$\supseteq \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left\{(ID)\int_0^1\int_0^1\frac{\left|[(1-t)^{\alpha}-t^{\alpha}]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]\right|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}dtd\lambda\right\}^{1-\frac{1}{m}}$$
$$\times\left\{(ID)\int_0^1\int_0^1\frac{\left|[(1-t)^{\alpha}-t^{\alpha}]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]\right|\cdot\left|\frac{\partial^2}{\partial t\partial\lambda}f(M_p(a,b,t),M_q(c,d,\lambda))\right|^m}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}dtd\lambda\right\}^{\frac{1}{m}}$$
$$\supseteq \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left\{(ID)\int_0^1\int_0^1\frac{\left|[(1-t)^{\alpha}-t^{\alpha}]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]\right|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}dtd\lambda\right\}^{1-\frac{1}{m}}$$
$$\times\left\{(ID)\int_0^1\int_0^1\frac{\left|[(1-t)^{\alpha}-t^{\alpha}]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]\right|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}\right.$$
$$\times\left[t\lambda\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,c)\right|^m+t(1-\lambda)\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,d)\right|\right.$$
$$\left.+(1-t)\lambda\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,c)\right|^m+(1-t)(1-\lambda)\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,d)\right|^m dtd\lambda\right\}^{\frac{1}{m}}$$
$$= \frac{(b^p-a^p)(d^q-c^q)}{4pq}M_1^{1-\frac{1}{q}}(\alpha,\beta)\times\left\{M_2(\alpha,\beta)\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,c)\right|^m+M_3(\alpha,\beta)\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,d)\right|^m\right.$$
$$\left.+M_4(\alpha,\beta)\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,c)\right|^m+M_5(\alpha,\beta)\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,d)\right|^m\right\}^{\frac{1}{m}},$$

where by simple computation, we obtain,

$$\begin{aligned}
& M_1(\alpha,\beta) \\
= & \int_0^1\int_0^1 \frac{\left[(1-t)^\alpha-t^\alpha\right]\left[(1-\lambda)^\beta-\lambda^\beta\right]}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}\,dtd\lambda \\
= & \int_0^1 \frac{\left[(1-t)^\alpha-t^\alpha\right]}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\,dt\int_0^1 \frac{\left[(1-\lambda)^\beta-\lambda^\beta\right]}{\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}\,d\lambda \\
= & \left\{\frac{b^{1-p}}{\alpha+1}\left[{}_2F_1\left(1-\frac{1}{p},1;\alpha+2,1-\frac{a^p}{b^p}\right)+{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+2,1-\frac{a^p}{b^p}\right)\right]\right\} \\
& \times\left\{\frac{d^{1-q}}{\beta+1}\left[{}_2F_1\left(1-\frac{1}{q},1;\beta+2,1-\frac{c^q}{d^q}\right)+{}_2F_1\left(1-\frac{1}{q},\beta+1;\beta+2,1-\frac{c^q}{d^q}\right)\right]\right\} \\
& M_2(\alpha,\beta) \\
= & \int_0^1\int_0^1 \frac{\left[(1-t)^\alpha-t^\alpha\right]\left[(1-\lambda)^\beta-\lambda^\beta\right]}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}\,t\lambda dtd\lambda \\
= & \int_0^1 \frac{\left[(1-t)^\alpha-t^\alpha\right]}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\,tdt\int_0^1 \frac{\left[(1-\lambda)^\beta-\lambda^\beta\right]}{\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}\,\lambda d\lambda \\
= & \left\{\frac{b^{1-p}}{\alpha+2}\left[\frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3,1-\frac{a^p}{b^p}\right)+{}_2F_1\left(1-\frac{1}{p},\alpha+2;\alpha+3,1-\frac{a^p}{b^p}\right)\right]\right\} \\
& \times\left\{\frac{d^{1-q}}{\beta+2}\left[\frac{1}{\beta+1}{}_2F_1\left(1-\frac{1}{q},2;\beta+3,1-\frac{c^q}{d^q}\right)+{}_2F_1\left(1-\frac{1}{q},\beta+2;\beta+3,1-\frac{c^q}{d^q}\right)\right]\right\} \\
& M_3(\alpha,\beta) \\
= & \int_0^1\int_0^1 \frac{\left[(1-t)^\alpha-t^\alpha\right]\left[(1-\lambda)^\beta-\lambda^\beta\right]}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}\,t\,(1-\lambda)\,dtd\lambda \\
= & \int_0^1 \frac{\left[(1-t)^\alpha-t^\alpha\right]}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\,tdt\int_0^1 \frac{\left[(1-\lambda)^\beta-\lambda^\beta\right]}{\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}\,(1-\lambda)\,d\lambda \\
= & \left\{\frac{b^{1-p}}{\alpha+2}\left[\frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3,1-\frac{a^p}{b^p}\right)+{}_2F_1\left(1-\frac{1}{p},\alpha+2;\alpha+3,1-\frac{a^p}{b^p}\right)\right]\right\} \\
& \times\left\{\frac{d^{1-q}}{\beta+1}\left[{}_2F_1\left(1-\frac{1}{q},1;\beta+3,1-\frac{c^q}{d^q}\right)+\frac{1}{\beta+1}{}_2F_1\left(1-\frac{1}{q},\beta+1;\beta+3,1-\frac{c^q}{d^q}\right)\right]\right\} \\
& M_4(\alpha,\beta) \\
= & \int_0^1\int_0^1 \frac{\left[(1-t)^\alpha-t^\alpha\right]\left[(1-\lambda)^\beta-\lambda^\beta\right]}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}\,(1-t)\,\lambda dtd\lambda \\
= & \int_0^1 \frac{\left[(1-t)^\alpha-t^\alpha\right]}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\,(1-t)\,dt\int_0^1 \frac{\left[(1-\lambda)^\beta-\lambda^\beta\right]}{\left[\lambda c^q+(1-\lambda)\,d^q\right]^{1-\frac{1}{q}}}\,\lambda d\lambda \\
= & \left\{\frac{b^{1-p}}{\alpha+1}\left[{}_2F_1\left(1-\frac{1}{p},1;\alpha+3,1-\frac{a^p}{b^p}\right)+\frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+3,1-\frac{a^p}{b^p}\right)\right]\right\} \\
& \times\left\{\frac{d^{1-q}}{\beta+2}\left[\frac{1}{\beta+1}{}_2F_1\left(1-\frac{1}{q},2;\beta+3,1-\frac{c^q}{d^q}\right)+{}_2F_1\left(1-\frac{1}{q},\beta+2;\beta+3,1-\frac{c^q}{d^q}\right)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
& M_5(\alpha,\beta) \\
= & \int_0^1\int_0^1 \frac{\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}(1-t)(1-\lambda)\,dtd\lambda \\
= & \int_0^1 \frac{\left[(1-t)^{\alpha}-t^{\alpha}\right]}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}}(1-t)\,dt\int_0^1 \frac{\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]}{\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}(1-\lambda)\,d\lambda \\
= & \left\{\frac{b^{1-p}}{\alpha+1}\left[{}_2F_1\left(1-\frac{1}{p},1;\alpha+3,1-\frac{a^p}{b^p}\right)+\frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+3,1-\frac{a^p}{b^p}\right)\right]\right\} \\
& \times\left\{\frac{d^{1-q}}{\beta+1}\left[{}_2F_1\left(1-\frac{1}{q},1;\beta+3,1-\frac{c^q}{d^q}\right)+\frac{1}{\beta+1}{}_2F_1\left(1-\frac{1}{q},\beta+1;\beta+3,1-\frac{c^q}{d^q}\right)\right]\right\}.
\end{aligned}
$$

Theorem 5.4

Let $f: I\times I\to\mathbb{R}$ be an interval-valued pq-convex function such that $f(t)=\left[\underline{f}(t),\overline{f}(t)\right]$ and the order $p,q>0$, $\alpha,\beta>0$ and $a,b,c,d\in I$ with $a<b$ and $c<d$. If $f\in ID_{([a,b]\times[c,d])}$ and $\left|\frac{\partial^2}{\partial t\partial\lambda}f\right|^m$ is $pq-convex$ on $[a,b]\times[c,d]$ for $m\geq 1$, then the following inequality for fractional integrals holds:

$$
\begin{aligned}
& \left|\frac{1}{4}\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\right. \\
& +\frac{p^{\alpha}q^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b^p-a^p)^{\alpha}(d^q-c^q)^{\beta}}\left[{}^{p,q}I^{\alpha,\beta}_{a^+,c^+}f(b,d)+{}^{p,q}I^{\alpha,\beta}_{a^+,d^-}f(b,c)+{}^{p,q}I^{\alpha,\beta}_{b^-,c^+}f(a,d)+{}^{p,q}I^{\alpha,\beta}_{b^-,d^-}f(a,c)\right] \\
& -\frac{p^{\alpha}\Gamma(\alpha+1)}{(b^p-a^p)^{\alpha}}\left[{}^pI^{\alpha}_{a^+}f(x,c)+{}^pI^{\alpha}_{b^-}f(x,c)+{}^pI^{\alpha}_{a^+}f(x,d)+{}^pI^{\alpha}_{b^-}f(x,d)\right] \\
& \left.-\frac{q^{\beta}\Gamma(\beta+1)}{(d^q-c^q)^{\beta}}\left[{}^qI^{\beta}_{c^+}f(a,y)+{}^qI^{\beta}_{d^-}f(a,y)+{}^qI^{\beta}_{c^+}f(b,y)+{}^qI^{\beta}_{d^-}f(b,y)\right]\right| \\
\supseteq\ & \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left\{M_6^{\frac{1}{n}}(\alpha,\beta)+M_7^{\frac{1}{n}}(\alpha,\beta)+M_8^{\frac{1}{n}}(\alpha,\beta)+M_9^{\frac{1}{n}}(\alpha,\beta)\right\} \\
& \times\left\{\frac{1}{4}\left[\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,c)\right|^m+\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,d)\right|^m+\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,c)\right|^m+\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,d)\right|^m\right]\right\}^{\frac{1}{m}}
\end{aligned}
$$

and $\frac{1}{m}+\frac{1}{n}=1$.

Proof :

From Lemma by using the property of the modulus, the Hölder inequality and the pq-convexity of $\left|\frac{\partial^2}{\partial t\partial\lambda}f\right|^m$, then we have

$$
\begin{aligned}
& \left|\frac{(b^p-a^p)(d^q-c^q)}{4pq}\left\{(ID)\int_0^1\int_0^1\frac{\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[(1-\lambda)^{\beta}-\lambda^{\beta}\right]\frac{\partial^2}{\partial t\partial\lambda}f\left(M_p(a,b,t),M_q(c,d,\lambda)\right)}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}dtd\lambda\right\}\right| \\
\supseteq\ & \frac{(b^p-a^p)(d^q-c^q)}{4pq}\times\left\{\left((ID)\int_0^1\int_0^1\frac{(1-t)^{\alpha n}(1-\lambda)^{\beta n}}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}dtd\lambda\right)^{\frac{1}{n}}\right. \\
& \times\left((ID)\int_0^1\int_0^1\left|\frac{\partial^2}{\partial t\partial\lambda}f\left(M_p(a,b,t),M_q(c,d,\lambda)\right)\right|^m dtd\lambda\right)^{\frac{1}{m}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left((ID)\int_0^1\int_0^1 \frac{t^{\alpha n}(1-\lambda)^{\beta n}}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}dtd\lambda\right)^{\frac{1}{n}} \\
&\qquad\qquad \times\left((ID)\int_0^1\int_0^1\left|\frac{\partial^2}{\partial t\partial\lambda}f\left(M_p(a,b,t),M_q(c,d,\lambda)\right)\right|^m dtd\lambda\right)^{\frac{1}{m}} \\
&\qquad +\left((ID)\int_0^1\int_0^1 \frac{(1-t)^{\alpha n}\lambda^{\beta n}}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}dtd\lambda\right)^{\frac{1}{n}} \\
&\qquad\qquad \times\left((ID)\int_0^1\int_0^1\left|\frac{\partial^2}{\partial t\partial\lambda}f\left(M_p(a,b,t),M_q(c,d,\lambda)\right)\right|^m dtd\lambda\right)^{\frac{1}{m}} \\
&\qquad +\left((ID)\int_0^1\int_0^1 \frac{t^{\alpha n}\lambda^{\beta n}}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}dtd\lambda\right)^{\frac{1}{n}} \\
&\qquad\qquad \times\left((ID)\int_0^1\int_0^1\left|\frac{\partial^2}{\partial t\partial\lambda}f\left(M_p(a,b,t),M_q(c,d,\lambda)\right)\right|^m dtd\lambda\right)^{\frac{1}{m}}\Bigg\} \\
\supseteq\ & \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left\{M_6^{\frac{1}{n}}(\alpha,\beta)+M_7^{\frac{1}{n}}(\alpha,\beta)+M_8^{\frac{1}{n}}(\alpha,\beta)+M_9^{\frac{1}{n}}(\alpha,\beta)\right\} \\
&\times\Bigg\{(ID)\int_0^1\int_0^1 \ t\lambda\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,c)\right|^m+t(1-\lambda)\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,d)\right|^m \\
&\qquad +(1-t)\lambda\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,c)\right|^m+(1-t)(1-\lambda)\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,d)\right|^m dtd\lambda\Bigg\}^{\frac{1}{m}} \\
=\ & \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left\{M_6^{\frac{1}{n}}(\alpha,\beta)+M_7^{\frac{1}{n}}(\alpha,\beta)+M_8^{\frac{1}{n}}(\alpha,\beta)+M_9^{\frac{1}{n}}(\alpha,\beta)\right\} \\
&\times\left\{\frac{1}{4}\left[\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,c)\right|^m+\left|\frac{\partial^2}{\partial t\partial\lambda}f(a,d)\right|^m+\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,c)\right|^m+\left|\frac{\partial^2}{\partial t\partial\lambda}f(b,d)\right|^m\right]\right\}^{\frac{1}{m}},
\end{aligned}
$$

and by calculations of integrals, we obtain,

$$
\begin{aligned}
M_6(\alpha,\beta) &= \int_0^1\int_0^1 \frac{(1-t)^{\alpha n}(1-\lambda)^{\beta n}}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}dtd\lambda \\
&= \left(\int_0^1 \frac{(1-t)^{\alpha n}}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}}dt\right)\left(\int_0^1 \frac{(1-\lambda)^{\beta n}}{\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}d\lambda\right) \\
&= \frac{b^{(1-p)n}}{\alpha p+1}\ {}_2F_1\left(n-\frac{n}{p},1;\alpha n+2,1-\frac{a^p}{b^p}\right)\cdot\frac{d^{(1-q)n}}{\beta q+1}\ {}_2F_1\left(n-\frac{n}{q},1;\beta n+2,1-\frac{c^q}{d^q}\right) \\
M_7(\alpha,\beta) &= \int_0^1\int_0^1 \frac{t^{\alpha n}(1-\lambda)^{\beta n}}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}dtd\lambda \\
&= \left(\int_0^1 \frac{t^{\alpha n}}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}}dt\right)\left(\int_0^1 \frac{(1-\lambda)^{\beta n}}{\left[\lambda c^q+(1-\lambda)d^q\right]^{1-\frac{1}{q}}}d\lambda\right) \\
&= \frac{b^{(1-p)n}}{\alpha p+1}\ {}_2F_1\left(n-\frac{n}{p},\alpha n+1;\alpha n+2,1-\frac{a^p}{b^p}\right)\cdot\frac{d^{(1-q)n}}{\beta q+1}\ {}_2F_1\left(n-\frac{n}{q},1;\beta n+2,1-\frac{c^q}{d^q}\right)
\end{aligned}
$$

$$\begin{aligned}
M_8(\alpha,\beta) &= \int_0^1\int_0^1 \frac{(1-t)^{\alpha n}\lambda^{\beta n}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}dtd\lambda \\
&= \left(\int_0^1 \frac{(1-t)^{\alpha n}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)\left(\int_0^1 \frac{\lambda^{\beta n}}{[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}d\lambda\right) \\
&= \frac{b^{(1-p)n}}{\alpha p+1}\,{}_2F_1\left(n-\frac{n}{p},\alpha n+1;\alpha n+2,1-\frac{a^p}{b^p}\right)\cdot\frac{d^{(1-q)n}}{\beta q+1}\,{}_2F_1\left(n-\frac{n}{q},\beta n+1;\beta n+2,1-\frac{c^q}{d^q}\right)
\end{aligned}$$

$$\begin{aligned}
M_9(\alpha,\beta) &= \int_0^1\int_0^1 \frac{t^{\alpha n}\lambda^{\beta n}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}dtd\lambda \\
&= \left(\int_0^1 \frac{t^{\alpha n}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)\left(\int_0^1 \frac{\lambda^{\beta n}}{[\lambda c^q+(1-\lambda)d^q]^{1-\frac{1}{q}}}d\lambda\right) \\
&= \frac{b^{(1-p)n}}{\alpha p+1}\,{}_2F_1\left(n-\frac{n}{p},\alpha n+1;\alpha n+2,1-\frac{a^p}{b^p}\right)\cdot\frac{d^{(1-q)n}}{\beta q+1}\,{}_2F_1\left(n-\frac{n}{q},\beta n+1;\beta n+2,1-\frac{c^q}{d^q}\right)
\end{aligned}$$

Conclusion

In this work, the author established Hermite-Hadamard type inequalities via Katugampola fractional integral. Furthermore, the author extend the ineqalities on interval-valued coordinated. It is an interesting issue, and many researchers work to generalize the Ostrowski' inequalities, Chebyshev type inequalities and Opial-type inequalities on fuzzy interval-valued set. We hope to establish the general fractional integrals in their future research.

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