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Better Uniform Approximation by New Bivariate Bernstein Operators

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Abstract. In this paper we introduce new bivariate Bernstein type operators $B_n^{M,i}(f;x,y)$, $i=1,2,3$. The rates of approximation by these operators are calculated and it is shown that the errors are significantly smaller than those of ordinary bivariate Bernstein operators for sufficiently smooth functions.

1. Introduction

In approximation theory the Bernstein operators are the most studied linear positive operators. Let, $B[0,1]$ denote the class of bounded functions and $f \in B[0,1]$, then the Bernstein operator $B_n : B[0,1] \to C^\infty[0,1]$ of degree n with respect to f is defined by

$$B_n f(x) = \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k}{n}\right)$$

where

$$P_{n,k}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k} & \text{if } k = 0,1,...,n, \\ 0 & \text{if } k < 0 \text{ or } k > n. \end{cases}$$

Here $C^\infty[0,1]$ is the class of infinitely differentiable functions on $[0,1]$.

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The operators $B_nf(x)$ exhibit interesting approximation properties. We mention a few of them in the following.

(1) $B_nf(0) = f(0)$ and $B_nf(1) = f(1)$ i.e. $B_nf(x)$ has end point interpolation property,
(2) The sequence $B_nf(x)$ converges uniformly to $f(x)$ whenever f is continuous on $[0, 1]$,
(3) $B_nf(x) \geq 0$ for a positive function $f(x)$,
(4) The polynomials $B_nf(x)$ are convex for a convex function $f(x)$,
(5) The Bernstein operators demonstrate simultaneous approximation property i.e. the derivative $\frac{d^r}{dt^r}\Big|_{t=x} B_nf(t)$ converges to the corresponding derivative $f^{(r)}(x)$ whenever the later derivative exists.
(6) The polynomials $B_nf(x)$ provide a quadrature rule(see [1]) i.e. for $f \in C[0, 1]$, $n \in \mathbb{N}$

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f\left(\frac{k}{n}\right).$$

For other interesting properties we refer to the monograph [2]. Inspite of extensive research work of more than one century, new and interesting results have been continuously appearing on the properties and applications of these polynomials, see [2] and the references therein.

Although the operators $B_nf(x)$ are easy to work with and show remarkable properties, they lack rapid convergence. In fact, the identity $B_n(t^2; x) = x^2 + \frac{x(1-x)}{n}$ implies that the order of approximation for the function, $f(t) = t^2$ is not less than $O(n^{-1})$ while the function is sufficiently smooth. In fact, it is known that the order of approximation is $O(n^{-1/2})$ for the operators $B_n(f; x)$ and can not be improved as such. This fact is indicated by the following result due to Voronovskaya(see, [3])

Theorem 1.1. *Let $f(x)$ be bounded in $[0, 1]$ and suppose that the second derivative $f''(x)$ exists at a certain point x of $[0, 1]$, then*

$$B_nf(x) - f(x) = \frac{1}{n}\left(\frac{x(1-x)}{2} f''(x)\right) + o\left(\frac{1}{n}\right).$$

For the details see [4], pp. 102. In order to improve the degree of approximation by the Bernstein operators Arab et al. [5] have introduced kernel decomposition method recently. This method, then have been applied by Maria et al. in [6] and Gairola et al. in [7] for other sequence of operators. The aim of this paper is to extend this technique to modify the bivariate Bernstein operators and apply the modified operators for surface plotting. We show that the new bivariate operator approximate a surface better than the classical bivariate Bernstein operator (1.1).

Let D denote the open square $(0, 1) \times (0, 1)$, and the closed square

$$\overline{D} := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

The class $B(\overline{D})$ consists of the functions $f(x,y)$ defined and bounded on $\overline{D}$. Then the bivariate Bernstein operators of degrees n,m with respect to $f\in B(\overline{D})$ are defined by (cf. [8])

$$B_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)f\left(\frac{k}{n},\frac{j}{m}\right), \tag{1.1}$$

where $P_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}, k\in\overline{0,n}$ and $P_{m,j}(y)=\binom{m}{j}y^j(1-y)^{m-j}, j\in\overline{0,m}$. We will need the class $C^s(\overline{D}), s\in N_0$ in order to establish estimates for the smooth functions.

$$C^s(\overline{D}):=\left\{f\in C(\overline{D}):\frac{\partial^r f}{\partial^i\partial^{r-i}}\in C(\overline{D}), i\in\overline{0,r}, r\in\overline{0,s}\right\}.$$

Here, $N_0:=\mathbb{N}\cup\{0\}$.

Approximation properties of multivariate Bernstein operators were studied by a number of researchers(see [8]- [16]) and they have established noteworthy results on degree of approximation, convergence properties etc. It is worth to mention the relevant work in [17]- [22]. In 1953 P.L. Butzer [23] discussed the simultaneous approximation properties of the operator 1.1. Moreover, Pop [24] obtained the rate of convergence in terms of Voronovskaja type asymptotic theorem and modulus of continuity for bivariate Bernstein operator $B_{m,n}(f;x,y)$ on the square $\overline{D}$. In 1960 D.D. Stancu [25] defined another bivariate Bernstein operators on the triangle and derived the rate of convergence in terms of complete modulus of continuity. Subsequently, in 1992 Zhou [26] defined the two dimensional Bernstein-Durrmeyer operators. In [28], N. Deo introduced a certain modification of Bernstein operator and obtained direct results. The bivariate Bernstein operators have been also studied on a simplex by several authors(see [26]- [30]).

By straightforward calculations it follows that $B_{n,m}(x^2+y^2;x,y)=x^2+y^2+\frac{x(1-x)}{n}+\frac{y(1-y)}{m}$, so that as in the case of the univariate Berntsein operator $B_n(f;x)$, the degree of approximation is at most $O\left(\frac{1}{n},\frac{1}{m}\right)$. Thus, this estimate can not be improved as such even for the sufficiently smooth function x^2+y^2 on $\overline{D}$. Although, these operators are easy to compute and demonstrate nice approximation properties, the slow rate of convergence has been a matter of concern. In order to achieve higher order of approximation we propose three bivariate Bernstein operators $B^{M,r}_{n,m}(f;x,y), r=1,2,3$ as follows.

2. The first order operator $B^{M,1}_{n,m}(f;x,y)$

Let $B(\overline{D})$ and $C(\overline{D})$ denote the space of bounded functions and space of continuous functions $f(x,y)$ on $\overline{D}$ with the defining norm $\|f\|:=\sup_{(x,y)\in\overline{D}}|f(x,y)|$. For $f\in B(\overline{D})$, the operator $B^{M,1}_{nm}(f;x,y)$ is defined by

$$B^{M,1}_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}P^{M,1}_{n,m;k,j}(x,y)f\left(\frac{k}{n},\frac{j}{m}\right), \tag{2.1}$$

where

$$P^{M,1}_{n,m;k,j}(x,y)=P^{M,1}_{n,k}(x)P^{M,1}_{m,j}(y),$$
$$P^{M,1}_{n,k}(x)=a(x,n)P_{n-1,k}(x)+a(1-x,n)P_{n-1,k-1}(x),$$

$$P_{m,j}^{M,1}(y) = b(y,m)P_{m-1,j}(y) + b(1-y,m)P_{m-1,j-1}(y)$$

and $a(x,n) = a_1(n)x + a_0(n)$, $b(x,n) = b_1(m)y + b_0(m)$. Here $a_i(n), b_i(m)$ $n,m = 0,1,2,$ are unknown sequences which are to be determined by imposing suitable convergence conditions. For $a_1(n) = b_1(m) = -1$, $a_0(n) = b_0(m) = 1$ the operator (2.1) reduces to the ordinary bivariate operator (1.1).

The total modulus of continuity $\omega(f;\delta_1,\delta_2)$ of $f \in C(\overline{D})$ is defined as

$$\omega(f;\delta_1,\delta_2) := \sup\left\{|f(s,t) - f(x,y)| : s,t \in \overline{D}, |s-x| \leq \delta_1, |t-y| \leq \delta_2\right\},$$

where $\delta_1, \delta_2 > 0$. The modulus $\omega(f;\delta_1,\delta_2)$ satisfies the following properties.

(i) $\omega(f;\delta_1,\delta_2) \to 0$, if $\delta_1 \to 0$, $\delta_2 \to 0$,
(ii) $|f(s,t) - f(x,y)| \leq \omega(f;\delta_1,\delta_2)\left(1 + \frac{|s-x|}{\delta_1}\right)\left(1 + \frac{|t-y|}{\delta_2}\right)$.

The full modulus of continuity, $\omega(f;\delta)$ for $f \in C(\overline{D})$ is defined as

$$\omega(f;\delta) := \max_{\substack{\sqrt{(s-x)^2+(t-y)^2} \leq \delta \\ (x,y) \in \overline{D}}} |f(s,t) - f(x,y)|.$$

Other tools used to measure smoothness of a functions in $C(\overline{D})$ are the partial modulus of continuity with respect to x and y and these are defined as

$$\omega^{(1)}(f;\delta) = \sup\{|f(x_1,y) - f(x_2,y)| : y \in [0,1], |x_1 - x_2| \leq \delta\},$$

$$\omega^{(2)}(f;\delta) = \sup\{|f(x,y_1) - f(x,y_2)| : x \in [0,1], |y_1 - y_2| \leq \delta\}.$$

3. Preliminaries

Let $e_{i,j}$ be the function $e_{i,j}(x,y) = x^i y^j$. For a sequence $L_{n,m}(f;x,y)$ of bivariate linear positive operators we have the following extension of the Korovkin theorem in the square $\overline{D}$.

Theorem 3.1. *[**?**] If $\{L_{n,m}\}$ is a sequence of linear positive operators satisfying the conditions*

$$\lim_{n,m\to\infty} \|L_{n,m}(e_{00};x,y) - 1\|_{C(\overline{D})} = 0,$$
$$\lim_{n,m\to\infty} \|L_{n,m}(e_{10};x,y) - x\|_{C(\overline{D})} = 0,$$
$$\lim_{n,m\to\infty} \|L_{n,m}(e_{01};x,y) - y\|_{C(\overline{D})} = 0,$$
$$\lim_{n,m\to\infty} \left\|L_{n,m}(e_{01}^2 + e_{10}^2;x,y) - (x^2+y^2)\right\|_{C(\overline{D})} = 0$$

then for any $f \in C(\overline{D})$,

$$\lim_{n,m\to\infty} \|L_{n,m}(f;x,y) - f(x,y)\|_{C(\overline{D})} = 0.$$

Lemma 3.1. *The operator $B_{n,m}^{M,1}$ verifies following identities:*

(1) $B_{n,m}^{M,1}(e_{00},x,y) = (2a_0(n) + a_1(n))(2b_0(m) + b_1(m))$,
(2) $B_{n,m}^{M,1}(e_{10},x,y) = \left[x(2a_0(n) + a_1(n)) + \frac{(1-2x)(a_0(n)+a_1(n))}{n}\right](2b_0(n) + b_1(m))$,

(3) $B_{n,m}^{M,1}(e_{01},x,y) = (2a_0(n)+a_1(n))\left[y(2b_0(m)+b_1(m)) + \frac{(1-2y)(b_0(m)+b_1(m))}{m}\right]$,

(4) $B_{n,m}^{M,1}(e_{11},x,y) = \left[x(2a_0(n)+a_1(n)) + \frac{(1-2x)(a_0(n)+a_1(n))}{n}\right]$
$\times \left[y(2b_0(m)+b_1(m)) + \frac{(1-2y)(b_0(m)+b_1(m))}{m}\right]$,

(5) $B_{n,m}^{M,1}(e_{20},x,y) = \left[x^2(2a_0(n)+a_1(n)) + \frac{(4x-6x^2)a_0(n)+(3x-5x^2)a_1(n)}{n} + \frac{(1-4x+4x^2)(a_1(n)+a_0(n))}{n^2}\right]$
$(2b_0(n)+b_1)$,

(6) $B_{n,m}^{M,1}(e_{02},x,y) = (2a_0(n)+a_1(n))\Big[y^2(2b_0(m)+b_1(m)) + \frac{(4y-6y^2)b_0(m)+(3y-5y^2)b_1(m)}{m}$
$+ \frac{(1-4y+4y^2)(b_1(m)+b_0(m))}{m^2}\Big]$,

(7) $B_{n,m}^{M,1}(e_{22},x,y) = \left[x^2(2a_0(n)+a_1(n)) + \frac{(4x-6x^2)a_0(n)+(3x-5x^2)a_1(n)}{n} + \frac{(1-4x+4x^2)(a_1(n)+a_0(n))}{n^2}\right]$
$\left[y^2(2b_0(m)+b_1(m)) + \frac{(4y-6y^2)b_0(m)+(3y-5y^2)b_1(m)}{m} + \frac{(1-4y+4y^2)(b_1(m)+b_0(m))}{m^2}\right]$,

(8) $B_{n,m}^{M,1}(e_{12},x,y) = \left[x(2a_0(n)+a_1(n)) + \frac{(1-2x)(a_0(n)+a_1(n))}{n}\right]$
$\left[y^2(2b_0(m)+b_1(m)) + \frac{(4y-6y^2)b_0(m)+(3y-5y^2)b_1(m)}{m} + \frac{(1-4y+4y^2)(b_1(m)+b_0(m))}{m^2}\right]$,

(9) $B_{n,m}^{M,1}(e_{21},x,y) = \left[x^2(2a_0(n)+a_1(n)) + \frac{(4x-6x^2)a_0(n)+(3x-5x^2)a_1(n)}{n} + \frac{(1-4x+4x^2)(a_1(n)+a_0(n))}{n^2}\right]$
$\times \left[x(2a_0(n)+a_1(n)) + \frac{(1-2x)(a_0(n)+a_1(n))}{n}\right]$.

Proof. The proof follows by straightforward calculations. □

4. Rate of Approximation

In the next theorem we show that the linear positive operator sequence $B_{n,m}^{M,1}$ defined by equation (2.1) converges uniformly to f with the help Theorem3.1 given by Volkov in [31].

Theorem 4.1. *Let $f \in B(\overline{D})$ and the sequences $a_0(n), a_1(n), b_0(m), b_1(m)$ be convergent. If $2a_0(n)+a_1(n) = 2b_0(m)+b_1(m) = 1$ and the operator (2.1) is positive, then*

$$\lim_{n,m\to\infty} B_{n,m}^{M,1}(f;x,y) = f(x,y).$$

Further, the convergence is uniform if $f \in C(\overline{D})$.

Proof. By lemma 3.1, for each i,j with $0 \le i \le 2, 0 \le j \le 2$ $\lim_{n,m\to\infty} B_{n,m}^{M,1}(e_{ij},x,y) = e_{ij}$, uniformly on $\overline{D}$. Hence,

$$\lim_{n,m\to\infty} \left\|B_{n,m}^{M,1}(e_{00};x,y) - 1\right\|_{C(\overline{D})} = 0$$

$$\lim_{n,m\to\infty} \left\|B_{n,m}^{M,1}(e_{10};x,y) - x\right\|_{C(\overline{D})} = 0$$

$$\lim_{n,m\to\infty} \left\|B_{n,m}^{M,1}(e_{01};x,y) - y\right\|_{C(\overline{D})} = 0$$

$$\lim_{n,m\to\infty} \left\|B_{n,m}(e_{01}^2 + e_{10}^2;x,y) - (x^2+y^2)\right\|_{C(\overline{D})} = 0$$

$$\lim_{n,m\to\infty} \|B_{n,m}(f;x,y) - f(x,y)\|_{C(\overline{D})} = 0$$

Now the proof follows by the theorem 3.1.

□

Theorem 4.2. *(Voronovskaja Theorem)Let $f \in C(D)$, f_x, f_y, f_{xx} and f_{yy} exist at a certain point $(x,y) \in D$ and operator $B_{n,m}^{M,1}(f;x,y)$ be positive. If $a_1(n)+a_0(n) = b_1(m)+b_0(m) = 0$ and then we have,*

$$\begin{aligned}
&\lim_{n,m\to\infty} 2n\left(B_{n,m}^{M,1}(f,x,y)-f(x,y)\right)\\
&=\left((4x-6x^2)\lim_{n\to\infty} a_0(n)+(3x-5x^2)\lim_{n\to\infty} a_1(n)f_{xx}\right)\\
&+\left((4y-6y^2)\lim_{m\to\infty} b_0(m)+(3y-5y^2)\lim_{m\to\infty} b_1(m)f_{yy}\right),
\end{aligned}$$

where f_{xx} and f_{yy} are second order partial derivatives of f with respect to x and y respectively. Moreover, the above relation holds uniformly on $\overline{D}$ if $f_{xx}, f_{yy} \in C(D)$.

Proof. By Taylor's formula,

$$\begin{aligned}
f\left(\frac{k}{n},\frac{j}{m}\right) &= f(x,y)+\left(\left(\frac{k}{n}-x\right)\frac{\partial}{\partial x}+\left(\frac{j}{m}-y\right)\frac{\partial}{\partial y}\right)f(x,y)\\
&+\frac{1}{2!}\left(\left(\frac{k}{n}-x\right)\frac{\partial}{\partial x}+\left(\frac{j}{m}-y\right)\frac{\partial}{\partial y}\right)^2 f(x,y)\\
&+\varepsilon\left(\frac{k}{n},\frac{j}{m}\right)\left(\left(\frac{k}{n}-x\right)\frac{\partial}{\partial x}+\left(\frac{j}{m}-y\right)\frac{\partial}{\partial y}\right)^2 f(x,y).
\end{aligned}$$

The function $\varepsilon(s,t)$ is bounded on $\overline{D}$ and $\lim_{(s,t)\to(x,y))}\varepsilon(s,t)=0$. Multiplying the above equation by $P_{n,k}^{M,1}(x)P_{n,j}^{M,1}(y)$ and taking sum over i,j it follows that

$$\begin{aligned}
B_{n,m}^{M,1}(f;x,y) &= f(x,y)+B_{n,m}^{M,1}((e_{10}-x);x,y)f_x+B_{n,m}^{M,1}((e_{01}-y);x,y)f_y\\
&+\frac{1}{2}(B_{n,m}^{M,1}((e_{10}-x)^2;x,y)f_{xx}+B_{n,m}^{M,1}((e_{01}-y)^2;x,y)f_{yy}\\
&+B_{n,m}^{M,1}(((e_{10}-x);x,y)((e_{01}-y);x,y)f_xf_y))\\
&+B_{n,m}^{M,1}\left(\varepsilon\left(e_{10},e_{01}\right)\left(\left(\left(e_{10},x,y\right)-x\right)f_x+\left(\left(e_{01},x,y\right)-y\right)f_y\right)^2;x,y\right).
\end{aligned}$$

In view of the relation $2a_0(n)+a_1(n)=2b_0(m)+b_1(m)=1$ and lemma 1, it follows that

$$\begin{aligned}
B_{n,m}^{M,1}(f;x,y) &= f(x,y)+\left(\frac{1}{2}\frac{(4a_0(n)+3a_1(n))x-(6a_0(n)+5a_1(n))x^2}{n}\right)f_{xx}\\
&+\left(\frac{1}{2}\frac{(4b_0(m)+3b_1(m))y-(6b_0(m)+5b_1(m))y^2}{m}\right)f_{yy}\\
&+B_{n,n}^{M,1}\left(\varepsilon\left(e_{10},e_{01}\right)\left(\left(\left(e_{10},x,y\right)-x\right)f_x+\left(\left(e_{01},x,y\right)-y\right)f_y\right)^2;x,y\right).
\end{aligned} \tag{4.1}$$

Since $\varepsilon\left(\frac{k}{n},\frac{j}{m}\right)$ is bounded on $\overline{D}$ and $\lim_{(s,t)\to(x,y))}\varepsilon(s,t)=0$, proceeding in the standard manner, it follows that

$$\lim_{n,m\to\infty} B_{n,m}^{M,1}\left(\varepsilon\left(e_{10},e_{01}\right)\left(\left(\left(e_{10},x,y\right)-x\right)f_x+\left(\left(e_{01},x,y\right)-y\right)f_y\right)^2;x,y\right)=\mathcal{O}\left(\frac{1}{n},\frac{1}{m}\right). \tag{4.2}$$

Now from equation (4.1) and (4.2) we obtain the desired result. □

Theorem 4.3. *Let $B^{M,1}_{n,m}(f;x,y)$ be the operator (2.1). If $f \in C(\overline{D})$ and $a_0(n), a_1(n), b_0(m)$ and $b_1(m)$ are convergent sequences such that for each n and m, $2a_0(n)+a_1(n)=2b_0(m)+b_1(m)=1$ then,*

$$\left|B^{M,1}_{n,m}(f;x,y)-f(x,y)\right| \leq 2\Bigg((1+3|a_1(n)|)\,\omega^{(1)}\left(f,\frac{n-1}{2n\sqrt{n-2}}\right) + (1+3|b_1(m)|)\,\omega^{(2)}\left(f,\frac{m-1}{2m\sqrt{m-2}}\right)\Bigg).$$

Proof. We have

$$\begin{aligned}
&\left|B^{M,1}_{n,m}(f;x,y)-f(x,y)\right| \\
&\leq \Big|\sum_{k=0}^{n}\sum_{j=0}^{m}((a_0(n)+a_1(n))P_{n-1,k}(x)+(a_0(n)+a_1(n)(1-x))P_{n-1,k-1}(x)) \\
&((b_0(m)+b_1(m))P_{m-1,j}(y)+(b_0(m)+b_1(m)(1-y))P_{m-1,j-1}(y))\times \\
&\times\left(f\left(\frac{k}{n},\frac{j}{m}\right)-f\left(x,\frac{j}{m}\right)\right)\Big| \\
&+\Big|\sum_{k=0}^{n}\sum_{j=0}^{m}((a_0(n)+a_1(n))P_{n-1,k}(x)+(a_0(n)+a_1(n)(1-x))P_{n-1,k-1}(x)) \\
&((b_0(m)+b_1(m))P_{m-1,j}(y)+(b_0(m)+b_1(m)(1-y))P_{m-1,j-1}(y))\times \\
&\times\left(f\left(\frac{k}{n},\frac{j}{m}\right)-f\left(\frac{k}{n},y\right)\right)\Big| \\
&\leq\left(|a_0(n)+a_1(n)|\sum_{k=0}^{n}P_{n-1,k}(x)+|(a_0(n)+a_1(n)(1-x))|\sum_{k=0}^{n}P_{n-1,k-1}(x)\right)\times \\
&\times\left|f\left(\frac{k}{n},\frac{j}{m}\right)-f\left(x,\frac{j}{m}\right)\right| \\
&+\left(|b_0(m)+b_1(m)|\sum_{j=0}^{m}P_{m-1,j}(y)+|(b_0(m)+b_1(m)(1-y))|\sum_{j=0}^{m}P_{m-1,j-1}(y)\right)\times \\
&\times\left|f\left(\frac{k}{n},\frac{j}{m}\right)-f\left(\frac{k}{n},y\right)\right| \\
&\leq|a_0(n)+a_1(n)|\sum_{k=0}^{n}P_{n-1,k}(x)\omega^{(1)}\left(\left|\frac{k}{n}-x\right|\right) \\
&+|a_0(n)+a_1(n)(1-x)|\sum_{k=0}^{n}P_{n-1,k-1}(x)\omega^{(1)}\left(\left|\frac{k}{n}-x\right|\right) \\
&+|b_0(m)+b_1(m)|\sum_{j=0}^{m}P_{m-1,j}(y)\omega^{(2)}\left(\left|\frac{j}{m}-y\right|\right) \\
&+|b_0(m)+b_1(m)(1-y)|\sum_{j=0}^{m}P_{m-1,j-1}(y)\omega^{(2)}\left(\left|\frac{j}{m}-y\right|\right).
\end{aligned}$$

Now using the property

$$\omega(f,\lambda\delta)\leq(\lambda+1)\omega(f,\delta),\quad \lambda\geq 0$$

of modulus of continuity, we get,

$$\omega^{(1)}\left(f;\left|\frac{k}{n}-x\right|\right)\leq\left(1+\frac{2n\sqrt{n-2}}{n-1}\left|\frac{k}{n}-x\right|\right)\omega^{(1)}\left(f,\frac{n-1}{2n\sqrt{n-2}}\right).$$

Similarly,

$$\omega^{(2)}\left(f;\left|\frac{j}{m}-y\right|\right)\leq\left(1+\frac{2m\sqrt{m-2}}{m-1}\left|\frac{j}{m}-y\right|\right)\omega^{(2)}\left(f,\frac{m-1}{2m\sqrt{m-2}}\right).$$

Therefore,

$$\begin{aligned}
&\left|B^{M,1}_{n,m}(f;x,y)-f(x,y)\right|\\
&|a_0(n)+a_1(n)|\,\omega^{(1)}\left(f,\frac{n-1}{2n\sqrt{n-2}}\right)\left(1+\frac{2n\sqrt{n-2}}{n-1}\sum_{k=0}^{n}P_{n-1,k}(x)\left|\frac{k}{n}-x\right|\right)\\
&+|a_0(n)+a_1(n)(1-x)|\,\omega^{(1)}\left(f,\frac{n-1}{2n\sqrt{n-2}}\right)\left(1+\frac{2n\sqrt{n-2}}{n-1}\sum_{k=0}^{n}P_{n-1,k-1}(x)\left|\frac{k}{n}-x\right|\right)\\
&+|b_0(m)+b_1(m)|\,\omega^{(2)}\left(f,\frac{m-1}{2m\sqrt{m-2}}\right)\left(1+\frac{2m\sqrt{m-2}}{m-1}\sum_{j=0}^{m}P_{m-1,j}(y)\left|\frac{j}{m}-x\right|\right)\\
&+|b_0(m)+b_1(m)(1-y)|\,\omega^{(2)}\left(f,\frac{m-1}{2m\sqrt{m-2}}\right)\left(1+\frac{2m\sqrt{m-2}}{m-1}\sum_{j=0}^{m}P_{m-1,j-1}(y)\left|\frac{j}{m}-y\right|\right).
\end{aligned}$$

Now an application of Schwarz inequality and Lemma 1 yields,

$$\begin{aligned}
\sum_{k=0}^{n}P_{n-1,k}(x)\left|\frac{k}{n}-x\right| &\leq \left(\sum_{k=0}^{n}P_{n-1,k}(x)\left(\frac{k}{n}-x\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}P_{n-1,k}(x)\right)^{\frac{1}{2}}\\
&=\left(\frac{(2-n)x^2}{n^2}+\frac{(n-1)x}{n^2}\right)^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n}P_{n-1,k-1}(x)\left|\frac{k}{n}-x\right| &\leq \left(\sum_{k=0}^{n}P_{n-1,k-1}(x)\left(\frac{k}{n}-x\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}P_{n-1,k-1}(x)\right)^{\frac{1}{2}}\\
&=\left(\frac{(2-n)x^2}{n^2}+\frac{(n-3)x}{n^2}+\frac{1}{n^2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Since,

$$\max_{x\in[0,1],n>2}\left(\frac{(2-n)x^2}{n^2}+\frac{(n-1)x}{n^2}\right)=\max_{x\in[0,1],n>2}\left(\frac{(2-n)x^2}{n^2}+\frac{(n-3)x}{n^2}+\frac{1}{n^2}\right)=\frac{(n-1)^2}{4n^2(n-2)}$$

therefore,

$$\sum_{k=0}^{n}P_{n-1,k}(x)\left|\frac{k}{n}-x\right|\leq\frac{(n-1)}{2n\sqrt{n-2}},$$

and similarly

$$\sum_{k=0}^{n}P_{n-1,k-1}(x)\left|\frac{k}{n}-x\right|\leq\frac{(n-1)}{2n\sqrt{n-2}}.$$

In the same manner,

$$\sum_{j=0}^{m} P_{m-1,j}(y)\left|\frac{j}{m}-y\right| \le \frac{(m-1)}{2m\sqrt{m-2}},$$

$$\sum_{j=0}^{m} P_{m-1,j-1}(y)\left|\frac{j}{m}-y\right| \le \frac{(m-1)}{2m\sqrt{m-2}}.$$

Finally, using $2a_0(n)+a_1(n)=2b_0(m)+b_1(m)=1$, we get

$$\begin{aligned}
&\left|B_{n,m}^{M,1}(f;x,y)-f(x,y)\right| \\
&\le 4\omega^{(1)}\left(f,\frac{n-1}{2n\sqrt{n-2}}\right)\left(\left|a_1(n)x+\frac{1-a_1(n)}{2}\right|+\left|a_1(n)(1-x)+\frac{1-a_1(n)}{2}\right|\right) \\
&+4\omega^{(2)}\left(f,\frac{m-1}{2m\sqrt{m-2}}\right)\left(\left|b_1(m)y+\frac{1-b_1(m)}{2}\right|+\left|b_1(m)(1-y)+\frac{1-b_1(m)}{2}\right|\right) \\
&\le 2\left[(1+3|a_1(n)|)\,\omega^{(1)}\left(f,\frac{n-1}{2n\sqrt{n-2}}\right)+(1+3|b_1(m)|)\,\omega^{(2)}\left(f,\frac{m-1}{2m\sqrt{m-2}}\right)\right].
\end{aligned}$$

The proof is now completed. □

Remark 4.1. *For a Lipschitz class we can established the degree of approximation by bivariate operator* $B_{n,m}^{M,1}(f;x,y)$. *For* $0<\theta_1\le 1$ *and* $0<\theta_2\le 1$ *and* $f\in B(\overline{D})$ *the Lipschitz class* $Lip_M(\theta_1,\theta_2)$ *is defined as follows*

$$Lip_M(\theta_1,\theta_2) := \left\{f : |f(x_1,x_2)-f(y_1,y_2)| \le M\,|x_1-y_1|^{\theta_1}\,|x_2-y_2|^{\theta_2}\right\},$$

where $M>0$. *So, if* $f(x,y)$ *belongs to the class* $Lip_M(\theta_1,\theta_2)$ *then, there holds the estimate*

$$\left|B_{n,m}^{M,1}(f;x,y)-f(x,y)\right| \le M(n,m)n^{\frac{\theta_1}{2}}m^{\frac{\theta_2}{2}}$$

in view of the upper bounds $\omega^{(1)}\left(f,\frac{n-1}{2n\sqrt{n-2}}\right)\le Cn^{\theta_1/2}$ *and* $\omega^{(2)}\left(f,\frac{m-1}{2m\sqrt{m-2}}\right)\le Cm^{\theta_2/2}$. *Here, the constant* $M(n,m)=2\max\left\{(1+3\,|a_1(n)|),(1+3\,|b_1(n)|)\right\}$.

5. The Second order operator $B_{n,m}^{M,2}(f;x,y)$

We extend our results to achieve 2nd order approximation on two variable. The proposed operator $B_n^{M,2}(f;x,y)$ is defined as

$$B_{n,m}^{M,2}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}^{M,2}(x)P_{m,j}^{M,2}(y)f\left(\frac{k}{n},\frac{j}{m}\right), \tag{5.1}$$

where

$$P_{n,k}^{M,2}(x)=c(x,n)P_{n-2,k}(x)+d(x,n)P_{n-2,k-1}(x)+c(1-x,n)P_{n-2,k-2}(x),$$

$$P_{m,j}^{M,2}(y)=e(y,m)P_{m-2}(j)(y)+f(y,m)P_{m-2,j-1}(y)+e(1-y,m)P_{m-2}(j-2)(y)$$

and

$$c(x,n)=c_2(n)x^2+c_1(n)x+c_0(n), \quad d(x,n)=d_0(n)x(1-x),$$

$$e(y,m)=e_2(m)y^2+e_1(m)y+e_0, \quad f(x,n)=f_0(m)y(1-y).$$

As in the first order case, $c_i(n), e_i(m), i = 0, 1, 2, d_0(n), f_0(m)$ are the unknown sequences.

Remark 5.1. *If we put $c_2(n) = e_2(m) = 1, c_1(n) = e_1(m) = -2, c_0(n) = e_0(m) = 1$ and $d_0 = f_0 = 2$ then the operator (5.1) becomes ordinary bivariate Bernstein operator (1.1).*

By straightforward calculation we obtain values of unknown sequences as follows.

$$c_2 = \frac{n}{2}, \quad c_1 = \frac{-2-n}{2}, \quad c_0 = 1, \quad d_0 = n \tag{5.2}$$

$$e_2 = \frac{m}{2}, \quad e_1 = \frac{-2-m}{2}, \quad e_0 = 1, \quad f_0 = m. \tag{5.3}$$

Using these sequences we prove the following lemma.

Lemma 5.1. *For the operator $B^{M,2}_{n,m}(f, x, y)$ with the sequences chosen in (5.2)-(5.3), there holds the estimates*

(1) $B^{M,2}_{n,m}(e_{00}, x, y) = 1,$
(2) $B^{M,2}_{n,m}(e_{10}, x, y) = x,$
(3) $B^{M,2}_{n,m}(e_{01}, x, y) = y,$
(4) $B^{M,2}_{n,m}(e_{20}, x, y) = x^2 + \frac{2x(1-x)}{n^2},$
(5) $B^{M,2}_{n,m}(e_{02}, x, y) = y^2 + \frac{2y(1-y)}{m^2},$
(6) $B^{M,2}_{n,m}(e_{30}, x, y) = x^3 + \frac{2x(1-x)(5x-1)}{n^2)} - \frac{6x(1-x)(2x-1)}{n^3},$
(7) $B^{M,2}_{n,m}(e_{03}, x, y) = y^3 + \frac{2y(1-y)(5y-1)}{m^2} - \frac{6y(1-y)(2y-1)}{m^3}.$

Proof. The proof follows by straightforward calculations. □

Remark 5.2. *By the lemma 5.1 we get improved rate of approximation for the functions e_{ij}, $i, j = \overline{0(2)}$ i.e. $B^{M,2}_{n,m}((e_{01} - y), x, y) = 0, B^{M,2}_{n,m}((e_{10} - x), x, y) = 0, B^{M,2}_{n,m}((e_{20} - x^2), x, y) = \frac{2x(1-x)}{n^2}$ and $B^{M,2}_{n,m}((e_{02} - y^2), x, y) = \frac{2y(1-y)}{m^2}$.*

For twice differentiable function we obtain the following theorem.

Theorem 5.1. *Let $B^{M,2}_{n,m}$ be the operator, $f \in C^3(\overline{D}), n = m$ and let the sequences $c_2, c_1, c_0, e_2, e_1, e_0, d_0, f_0$ be bounded. Then,*

$$\lim_{n\to\infty} n^2(B^{M,2}_{n,n}(f, x, y) - f(x, y)) = x(1-x)\left(f_{xx} + \frac{(2x-1)}{3} f_{xxx}\right) + y(1-y)\left(f_{yy} + \frac{(2y-1)}{3} f_{yyy}\right).$$

Proof. We prove theorem for the cases when the operator $B^{M,2}_{n,n}(f, x, y)$ is positive. Since $f \in C^3(\overline{D})$, we can write

$$\begin{aligned} f\left(\frac{k}{n},\frac{j}{m}\right) &= f(x,y) + \left(\left(\frac{k}{n}-x\right)\frac{\partial}{\partial x} + \left(\frac{j}{m}-y\right)\frac{\partial}{\partial y}\right) f(x,y) \\ &+ \frac{1}{2!}\left(\left(\frac{k}{n}-x\right)\frac{\partial}{\partial x} + \left(\frac{j}{m}-y\right)\frac{\partial}{\partial y}\right)^2 f(x,y) \\ &+ \frac{1}{3!}\left(\left(\frac{k}{n}-x\right)\frac{\partial}{\partial x} + \left(\frac{j}{m}-y\right)\frac{\partial}{\partial y}\right)^3 f(x,y) \\ &+ \epsilon\left(\frac{k}{n},\frac{j}{m}\right)\left(\left(\frac{k}{n}-x\right)\frac{\partial}{\partial x} + \left(\frac{j}{m}-y\right)\frac{\partial}{\partial y}\right)^3 f(x,y). \end{aligned} \tag{5.4}$$

Multiplying equation (5.4) by $P_{n,k}^{M,1}(x)P_{n,j}^{M,1}(y)$ and taking the sum over i, j we obtain

$$\begin{aligned} B_{n,n}^{M,2}(f;x,y) &= f(x,y) + B_{n,n}^{M,2}((e_{10}-x);x,y)f_x + B_{n,n}^{M,2}((e_{01}-y);x,y)f_y \\ &+ \frac{1}{2}(B_{n,n}^{M,2}((e_{10}-x)^2;x,y)f_{xx} + B_{n,n}^{M,2}((e_{01}-y)^2;x,y)f_{yy} \\ &+ B_{n,n}^{M,2}(((e_{10}-x);x,y)((e_{01}-y);x,y)f_xf_y) \\ &+ \frac{1}{6}\left(B_{n,n}^{M,2}((e_{10}-x);x,y)^3f_{xxx} + B_{n,n}^{M,2}((e_{01}-y)^3;x,y)f_{yyy}\right) \\ &+ 3B_{n,n}^{M,2}((e_{10}-x);x,y)(e_{01}-y)^2;x,y)f_xf_{yy} \\ &+ 3B_{n,n}^{M,2}((e_{10}-x)^2;x,y)(e_{01}-y);x,y)f_{xx}f_y) \\ &+ B_{n,n}^{M,2}\left(\varepsilon\left(e_{10},e_{01}\right)\left(\left(\left(e_{10},x,y\right)-x\right)f_x + \left(\left(e_{01},x,y\right)-y\right)f_y\right)^3;x,y\right) \end{aligned}$$

here,

$$B_{n,n}^{M,2}\left(\varepsilon\left(e_{10},e_{01}\right)\left(\left(\left(e_{10},x,y\right)-x\right)f_x + \left(\left(e_{01},x,y\right)-y\right)f_y\right)^3;x,y\right) \le \epsilon$$

ϵ is the error term which we have proved earlier for $B_{n,m}^{M,1}$, Now by lemma 5.1 we have,

$$\begin{aligned} B_{n,n}^{M,2}(f;x,y) - f(x,y) &= \frac{1}{2}\left(\frac{2x(1-x)}{n^2}f_{xx} + \frac{2y(1-y)}{n^2}f_{yy}\right) \\ &+ \frac{1}{6}\left(\frac{2x(1-x)(2x-1)}{n^2}f_{xxx} + \frac{2y(1-y)(2y-1)}{n^2}f_{yyy}\right) + \epsilon \end{aligned}$$

Finally, by standard computations, we have

$$\lim_{n\to\infty} n^2(B_{n,n}^{M,2}(f,x,y) - f(x,y)) = x(1-x)\left(f_{xx} + \frac{(2x-1)}{3}f_{xxx}\right) + y(1-y)\left(f_{yy} + \frac{(2x-1)}{3}f_{yyy}\right).$$

□

Remark 5.3. *By theorem 5.1, for thrice differentiable functions, the degree of approximation is proved to be $O(n^{-2})$ which is higher than the degree $O(n^{-1})$ obtained by the ordinary operator $B_{n,n}(f;x,y)$.*

6. The Third Order operator $B^{M,3}_{n,m}(f;x,y)$

The third order operator $B^{M,3}_{n,m}(f;x,y)$ on $\overline{D}$ is defined by

$$B^{M,3}_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{k}{n},\frac{j}{m}\right)P^{M,3}_{n,k}(x)P^{M,3}_{m,j}(y) \tag{6.1}$$

where,

$$\begin{aligned} P^{M,3}_{n,k}(x) &= g(x,n)P_{n-4,k}(x)+r(x,n)P_{n-4,k-1}(x)+s(x,n)P_{n-4,k-2}(x)\\ &\quad +r(1-x,n)P_{n-4,k-3}(x)+g(1-x,n)P_{n-4,k-4}(x), \end{aligned}$$

$$\begin{aligned} P^{M,3}_{m,j}(y) &= h(y,m)P_{m-4,j}(y)+t(y,m)P_{m-4,j-1}(y)+w(y,m)P_{m-4,j-2}(y)\\ &\quad +t(1-y,m)P_{m-4,j-3}(y)+g(1-y,n)P_{m-4,j-4}(y). \end{aligned}$$

and

$g(x,n)=g_4(n)x^4+g_3(n)x^3+g_2(n)x^2+g_1(n)x+g_0(n)$,
$r(x,n)=r_4(n)x^4+r_3(n)x^3+r_2(n)x^2+r_1(n)x+r_0(n)$,
$s(x,n)=s_0(n)(x(1-x)^2)^2$,
$h(y,m)=h_4(m)y^4+h_3(m)y^3+h_2(m)y^2+h_1(m)y+h_0(m)$,
$t(y,m)=t^4(m)y^4+t_3(m)y^3+t_2(m)y^2+t_1(m)y+t_0(m)$,
$w(y,m)=w_0(m)(y(1-y)^2)^2$.

Note 1. *If we put* $g_4(n)=h_4(m)=1, g_3(n)=h_3(m)=-4, g_2(n)=h_2(m)=6, g_1(n)=h_1(m)=-4, g_0(n)=h_0(m)=0, r_4(n)=t_4(m)=-4, r_3(n)=t_3(m)=12, r_2(n)=t_2(m)=-12, r_0(n)=t_0(m)=0$ *and* $s_0(n)=w_0(m)=6$,*then the operator become the Bernstein operator (1.1) of two variable.*

By straightforward calculations we have the following values of the unknown sequences $g_i(n), h_i(m), r_i(n), t_i(m), i=0,1,2,3,4.$ $s_0(n)$ and $w_0(m)$ for which the operator $B^{M,3}_{n,m}(f;x,y)$ attains the order $\mathcal{O}\left(\frac{1}{n^2},\frac{1}{m^2}\right)$. $g_4(n)=1+\frac{23}{12}n+\frac{1}{8}n^2$, $g_3(n)=-4-\frac{14}{3}n-\frac{1}{4}n^2$, $g_2(n)=6+\frac{10}{3}n+\frac{1}{8}n^2$,
$g_1(n)=-4-\frac{7}{12}n$, $g_0(n)=1$
$r_4(n)=-4-\frac{23}{3}n-\frac{1}{2}n^2$, $r_3(n)=12+17n+n^2$, $r_2(n)=-12-\frac{31}{3}n-\frac{1}{2}n^2$
$r_1(n)=4+n$, $r_0(n)=0$,
$s_0(n)=6+\frac{23}{2}n+\frac{3}{4}n^2$
$h_4(m)=1+\frac{23}{12}m+\frac{1}{8}m^2$, $h_3(m)=-4-\frac{14}{3}m-\frac{1}{4}m^2$, $h_2(m)=6+\frac{10}{3}m+\frac{1}{8}m^2$,
$h_1(m)=-4-\frac{7}{12}m$, $h_0(m)=1$
$t_4(m)=-4-\frac{23}{3}m-\frac{1}{2}m^2$, $t_3(m)=12+17m+m^2$, $t_2(m)=-12-\frac{31}{3}m-\frac{1}{2}m^2$
$t_1(m)=4+m$, $t_0(m)=0$,
$w_0(m)=6+\frac{23}{2}m+\frac{3}{4}m^2$.

Lemma 6.1. *The operator* $B_{n,m}^{M,3}(f, x, y)$ *together with the above values* $g_i(n), h_i(m), r_i(n), t_i(m)$, $s_0(n)$ *and* $w_0(m)$, $i = \overline{0(4)}$ *verifies*

(1) $B_{n,m}^{M,3}(e_{00}, x, y) = 1$
(2) $B_{n,m}^{M,3}(e_{10}, x, y) = x$
(3) $B_{n,m}^{M,3}(e_{01}, x, y) = y$
(4) $B_{n,m}^{M,3}(e_{20}, x, y) = x^2$
(5) $B_{n,m}^{M,3}(e_{02}, x, y) = y^2$
(6) $B_{n,m}^{M,3}(e_{30}, x, y) = x^3$
(7) $B_{n,m}^{M,3}(e_{03}, x, y) = y^3$
(8) $B_{n,m}^{M,3}(e_{10}^2 + e_{01}^2; x, y) = x^2 + y^2$.

Proof. The proof follows by straightforward computations. □

Proceeding along the lines of theorem 5.1, we obtain following theorem.

Theorem 6.1. *Let* $f \in C^4(\overline{D})$ *and together the values of* $g_i, h_i, r_i, t_i, s_i, w_i$, *we have*

$$\lim_{n,m\to\infty}(B_{n,n}^{M,2}(f, x, y) - f(x, y)) = \mathcal{O}\left(\frac{1}{n^3}, \frac{1}{m^3}\right).$$

7. Numerical Verification

To verify the order of approximation by operator $B_{n,m}^{M,3}(f; x, y)$ we choose the function $\sin \pi x \cos \pi y$ and for its images under the operator $B_{n,m}^{M,3}$ for $n = 5, 10$ and $m = 5, 10$.

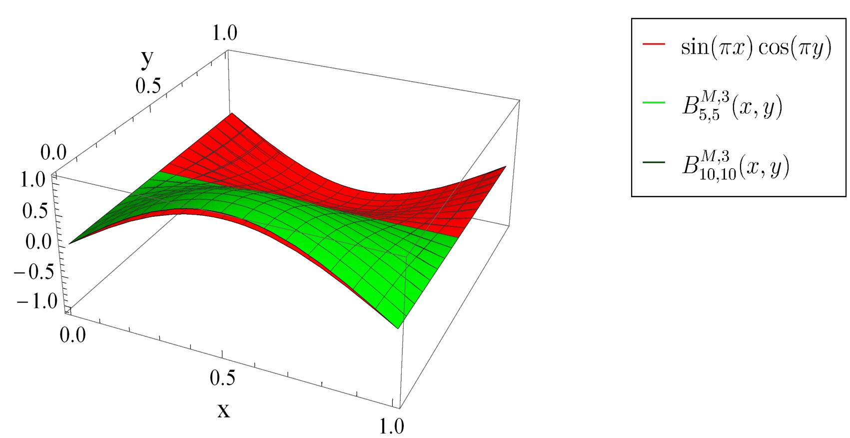

Figure 1. Comparison of the function $\sin \pi x \cos \pi y$ with $B_{n,m}^{M,3}(f, x, y)$ for $n = m = 5$, $n = m = 10$.

Table 1 provides values of the function $\sin \pi x \cos \pi y$ at equidistant nodes (x, y) with step size 0.2 and Tables 2, 3 provide values of the functions $B^{M,3}_{5,5}(f; x, y)$ and $B^{M,3}_{10,10}(f; x, y)$ at the corresponding nodes.

Table 1. $f(x, y) = \sin \pi x \cos \pi y$,
(starred * rows are of order 10^{-17})

	0	0.2	0.4	0.6	0.8	1
0.	0.	0.	0.	0.	0.	0.
0.2	0.587785	0.475528	0.181636	-0.181636	-0.475528	-0.587785
0.4	0.951057	0.769421	0.293893	-0.293893	-0.769421	-0.951057
0.6	0.951057	0.769421	0.293893	-0.293893	-0.769421	-0.951057
0.8	0.587785	0.475528	0.181636	-0.181636	-0.475528	-0.587785
*1	12.2465	9.9076	3.78437	-3.78437	-9.9076	-12.2465

Table 2. $B^{M,3}_{5,5}(f; x, y)$

	0	0.2	0.4	0.6	0.8	1
0.	0.	0.	0.	0.	0.	0.
0.2	0.643288	0.527466	0.200931	-0.200931	-0.527466	-0.643288
0.4	1.03431	0.848086	0.323067	-0.323067	-0.848086	-1.03431
0.6	1.03431	0.848086	0.323067	-0.323067	-0.848086	-1.03431
0.8	0.643288	0.527466	0.200931	-0.200931	-0.527466	-0.643288
1.	0.	0.	0.	0.	0.	0.

Table 3. $B^{M,3}_{10,10}(f; x, y)$,
(starred * rows are of order 10^{-12})

	0	0.2	0.4	0.6	0.8	1
0.	0.	0.	0.	0.	0.	0.
0.2	0.594148	0.482834	0.184553	-0.184553	-0.482834	-0.594148
0.4	0.961538	0.781392	0.298671	-0.298671	-0.781392	-0.961538
0.6	0.961538	0.781392	0.298671	-0.298671	-0.781392	-0.961538
0.8	0.594148	0.482834	0.184553	-0.184553	-0.482834	-0.594148
*1	5.38325	4.37469	1.67213	-1.67213	-4.37469	-5.38325

It follows that the maximum absolute errors by $B^{M,3}_{n,n}(f; x, y)$ for $n = 5, 10$ are found to be 0.0943346 at $(0.5, 0.919599)$ and 0.0134438 at $(0.5, 0.879425)$ respectively. Hence, in the figure 2 we compare $f(x, y)$ with $B^{M,3}_{n,n}(f; x, y)$ for $n = 5$ and $n = 10$ in the neighbourhood of the point $(0.5, 0.8)$.

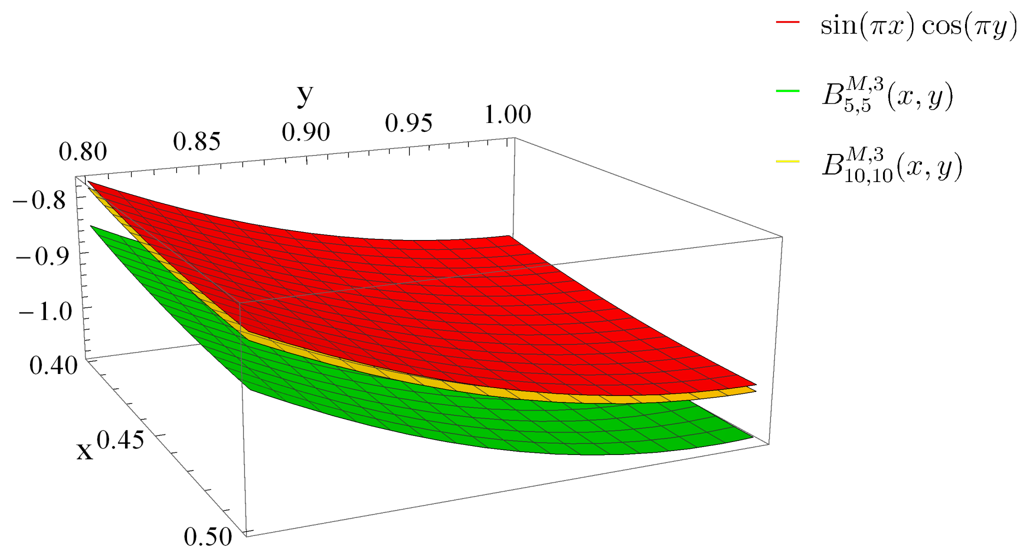

Figure 2. Comparison of the function $\sin \pi x \cos \pi y$ with $B^{M,3}_{n,m}(f,x,y)$ for $n = m = 5$, $n = m = 10$.

The next example compares $B^{M,2}_{n,n}(f;x,y)$ with $B^{M,3}_{n,n}(f;x,y)$ for $n = 5$ (See Fig. 3.) corresponding to the sufficiently smooth function e^{-x-y}. It is observed that the accuracy increases as we move from $B^{M,2}_{n,n}(f;x,y)$ to $B^{M,3}_{n,n}(f;x,y)$ for a fixed number of nodes.

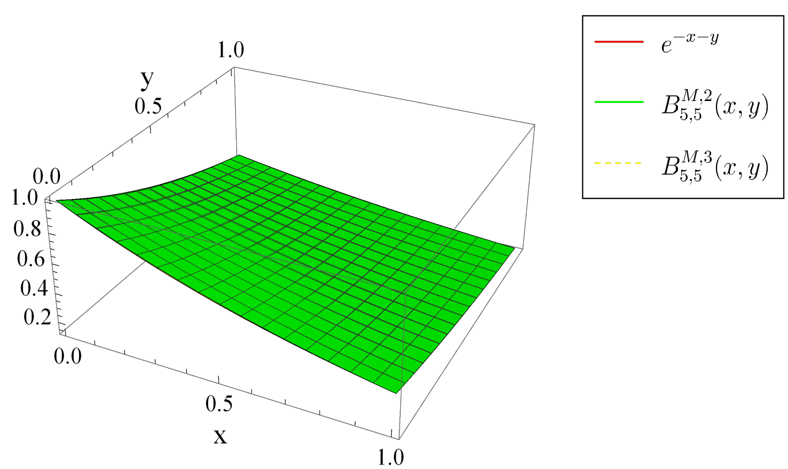

Figure 3. Comparison of the function e^{-x-y} with $B^{M,2}_{n,m}(f,x,y)$, $B^{M,3}_{n,m}(f,x,y)$ for $n = 5$, $m = 5$.

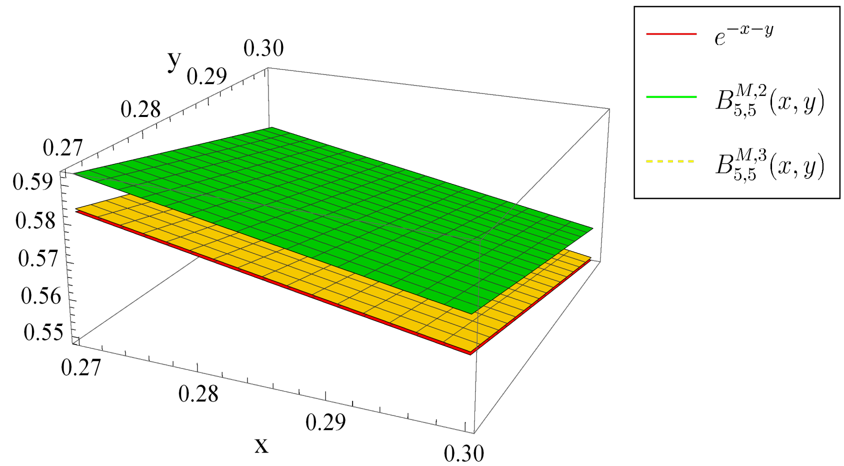

Figure 4. Comparison of the function e^{-x-y} with $B_{n,m}^{M,2}(f,x,y)$, $B_{n,m}^{M,3}(f,x,y)$ for $n = 5, m = 5$ in small rectangle

The following tables show the values of $f(x,y)$ and corresponding values under the operator $B_{n,m}^{M,3}$ for $n = 5, 10, 15, 20 m = 5, 10, 15, 20$ at different nodes with steps size 0.1.

Table 4. $f(x,y) = e^{-(x+y)}$

	0	0.2	0.4	0.6	0.8	1
0	1.	0.818731	0.67032	0.548812	0.449329	0.367879
0.2	0.818731	0.67032	0.548812	0.449329	0.367879	0.301194
0.4	0.67032	0.548812	0.449329	0.367879	0.301194	0.246597
0.6	0.548812	0.449329	0.367879	0.301194	0.246597	0.201897
0.8	0.449329	0.367879	0.301194	0.246597	0.201897	0.165299
1	0.367879	0.301194	0.246597	0.201897	0.165299	0.135335

Table 5. $B_{5,5}^{M,3}(f;x,y)$

	0	0.2	0.4	0.6	0.8	1
0	1.	0.819147	0.670916	0.549393	0.449698	0.367879
0.2	0.819147	0.671002	0.549579	0.450034	0.368369	0.301347
0.4	0.670916	0.549579	0.450129	0.368597	0.30171	0.246816
0.6	0.549393	0.450034	0.368597	0.301833	0.247061	0.202111
0.8	0.449698	0.368369	0.30171	0.247061	0.202228	0.165435
1	0.367879	0.301347	0.246816	0.202111	0.165435	0.135335

Table 6. $B_{10,10}^{M,3}(f;x,y)$

	0	0.2	0.4	0.6	0.8	1
0	1.	0.818788	0.670397	0.548882	0.44937	0.367879
0.2	0.818788	0.670414	0.548913	0.449418	0.367939	0.301215
0.4	0.670397	0.548913	0.449432	0.367969	0.301257	0.246625
0.6	0.548882	0.449418	0.367969	0.301271	0.246651	0.201922
0.8	0.44937	0.367939	0.301257	0.246651	0.201934	0.165314
1	0.367879	0.301215	0.246625	0.201922	0.165314	0.135335

Table 7. $B_{20,20}^{M,3}(f;x,y)$

	0	0.2	0.4	0.6	0.8	1
0	1.	0.818738	0.67033	0.54882	0.449334	0.367879
0.2	0.818738	0.670332	0.548825	0.44934	0.367887	0.301197
0.4	0.67033	0.548825	0.449342	0.367891	0.301202	0.246601
0.6	0.54882	0.44934	0.367891	0.301204	0.246604	0.2019
0.8	0.449334	0.367887	0.301202	0.246604	0.201901	0.165301
1	0.367879	0.301197	0.246601	0.2019	0.165301	0.135335

The global errors of approximation of the function by $B_{5,5}^{M,2}(f,x,y)$ and $B_{5,5}^{M,3}(f,x,y)$ are 0.00975603 and 0.000803997 which are obtained at $(0.278459, 0.278459)$ and $(0.365884, 0.365884)$ respectively. Here, again we choose small intervals for comparison in Fig 4 to show the three surfaces sufficiently separated.

8. Conclusion and Future Scope

The surface plotting of a smooth function by a bivariate linear operators is constructed using the m, n nodes $k/n, j/m$ $k = \overline{0,n}, j = \overline{0,m}$. The operator $B_{n,m}^{M,j}(f;x,y), j = 1,2,3$ approximate such surfaces with the degree of convergence $O(n^{-j}), j = 1,2,3$ which is significantly high in comparison of approximation by regular bivariate operator $B_{n,m}(f;x,y)$. We observe that the global absolute error decreases with the order $O(n^{-1})$ from $B_{n,n}^{M,2}(f;x,y)$ to $B_{n,n}^{M,3}(f;x,y)$ while keeping the degree of the polynomials same. On the other hand choosing large number of nodes can also improve the degree of approximation however, for a given fixed number of values it is better to apply the methods $B_{n,m}^{M,j}(f;x,y)$ for $j = 2,3,\ldots$ It is of interest to determine exact class of the functions for which the methods $B_{n,m}^{M,j}(f;x,y)$ provide optimal degree of approximation. This can be considered as an open problem.

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