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# Exact Solutions and Stability of Fourth Order Systems of Difference Equations Using Padovan Numbers

**Marwa M. Alzubaidi***

*Department of Mathematics, College of Duba, University of Tabuk, Saudi Arabia*

**Corresponding author:* mmialzubaidi@hotmail.com

Abstract. Difference equations are widely utilized to describe some phenomena arising in nonlinear sciences. In particular, systems of difference equations play an important role in investigating most nonlinear applications. Future behaviors of such phenomena can be sometimes known and understood by using exact solutions of systems of difference equations. Therefore, this article investigates the exact solutions of fourth order systems of difference equations. We use successive iterations and Padovan numbers to obtain the exact solutions in the form of rational functions. The stability of the considered systems are analyzed using Jacobian matrix. Real equilibrium points are found saddle. Under some selected parameters, we plot some 2D figures to show the behavior of the obtained solutions. The used methods can be successfully applied for high order systems of difference equations.

## 1. Introduction

Differential equations have been widely used to model various economical, physical, biological and artificial phenomena. These equations describe populations or objects in which time is continuous. In contrast, difference equations have been extensively used to describe populations or objects that evolve discrete time. The study of the theory of difference equations has strongly become an active topic for some researchers. The main reason behind that is that difference equations are used to model and describe most natural and non-natural phenomena such as those occurred in physics, biology, chemistry, economy, engineering, etc. Difference equations appear as discrete mathematical models of these phenomena. Recursive equations model wide spectrum of biomedical phenomena such as cell proliferation, cancer growth and genetics [1]. Discrete dynamical systems are also used to describe

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various applications in different sciences. Moreover, difference equations play a key role in investigating the solutions of various differential equations.

The solutions and the stability of many systems of difference equations have been recently discussed. For instance, Tollu et al. [2] used Fibonacci numbers to investigate the solutions of the difference equations

$$x_{n+1} = \frac{1}{1+x_n}, \quad y_{n+1} = \frac{1}{-1+y_n}.$$

Yazlik et al. [3] obtained the solutions of the following systems

$$x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}}, \quad y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}}. \tag{1.1}$$

The author in [4] investigated the solutions of the the system

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1}\left(-1 - x_{n-1} y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1}\left(\pm 1 \pm y_{n-1} x_{n-3}\right)}, \quad n = 0, 1, \ldots,$$

Moreover, Almatrafi and Alzubaidi [5] investigated the stability, periodicity and the solutions of the difference equation

$$x_{n+1} = c_1 x_{n-3} + \frac{c_2 x_{n-3}}{c_3 x_{n-3} - c_4 x_{n-7}}.$$

Cinar [6] obtained the positive solutions of the following system

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{y_{n-1} x_{n-1}}.$$

In [7], Elsayed extracted the solutions of the following system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + x_{n-1} y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\mp 1 + y_{n-1} x_n}.$$

Alayachi et al. [8] discussed the solutions of nonlinear rational systems of difference equations of third order given by

$$x_{n+1} = \frac{y_n z_{n-1}}{y_n \pm x_{n-2}}, \quad y_{n+1} = \frac{z_n x_{n-1}}{z_n \pm y_{n-2}}, \quad z_{n+1} = \frac{x_n y_{n-1}}{x_n \pm z_{n-2}}.$$

Alayachi et al. [9] obtained the solutions of the following systems

$$x_{n+1} = \frac{x_{n-3} y_{n-4}}{y_n(1 + x_{n-1} y_{n-2} x_{n-3} y_{n-4})}, \quad y_{n+1} = \frac{y_{n-3} x_{n-4}}{x_n(\pm 1 \pm y_{n-1} x_{n-2} y_{n-3} x_{n-4})}.$$

More information about systems of difference equations and their stability can be obtained in [10–16]. The motivation of this paper arises from the investigation of system (1.1). Hence, the main purpose of this paper is to investigate the solutions of the following system using Padovan numbers:

$$\Phi_{n+1} = \frac{\Phi_{n-3} + 1}{\Psi_{n-1} \Phi_{n-3}}, \qquad \Psi_{n+1} = \frac{\Psi_{n-3} + 1}{\Phi_{n-1} \Psi_{n-3}}, \tag{1.2}$$

$$\Phi_{n+1} = \frac{\Phi_{n-3} - 1}{\Psi_{n-1} \Phi_{n-3}}, \qquad \Psi_{n+1} = \frac{\Psi_{n-3} - 1}{\Phi_{n-1} \Psi_{n-3}}, \tag{1.3}$$

where the initial conditions are real numbers. We also present the future pattern of the considered systems under some selected values for the initial conditions. The Padovan sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is defined by $\rho_{n+1} = \rho_{n-1} + \rho_{n-2}$, where $\rho_{-2} = 0$, $\rho_{-1} = 0$, $\rho_0 = 1$.

This paper is organized as follows. Section 4 presents the solutions of system (1.2) while Section 3 gives the solutions of system (1.3). Section (4) analyzes the stability of the equilibrium points. The numerical solutions of the considered systems are shown in Section 5. Finally, we conclude this article in Section 6.

## 2. Solutions of System (1.2)

This section is devoted to give explicit forms for the solutions of system (1.2) using Padovan numbers.

**Theorem 2.1.** *Let $\{\Phi_n, \Psi_n\}_{n=-3}^{\infty}$ be the solutions of system (1.2) and assume that $\Phi_{-3} = \alpha$, $\Phi_{-2} = \beta$, $\Phi_{-1} = \gamma$, $\Phi_0 = \delta$, $\Psi_{-3} = \epsilon$, $\Psi_{-2} = \zeta$, $\Psi_{-1} = \eta$, and $\Psi_0 = \kappa$. Then, for $n = 0, 1, ...,$ we have*

$$\begin{aligned}
\Phi_{4n-3} &= \frac{\alpha\ (\rho_{2n-1}\eta + \rho_{2n}) + \rho_{2n-2}}{\alpha\ (\rho_{2n-2}\eta + \rho_{2n-1}) + \rho_{2n-3}},\\
\Phi_{4n-2} &= \frac{\beta(\rho_{2n-1}\kappa + \rho_{2n}) + \rho_{2n-2}}{\beta(\rho_{2n-2}\kappa + \rho_{2n-1}) + \rho_{2n-3}},\\
\Phi_{4n-1} &= \frac{\epsilon(\rho_{2n}\gamma + \rho_{2n+1}) + \rho_{2n-1}}{\epsilon(\rho_{2n-1}\gamma + \rho_{2n}) + \rho_{2n-2}},\\
\Phi_{4n} &= \frac{\zeta\ (\rho_{2n}\delta + \rho_{2n+1}) + \rho_{2n-1}}{\zeta\ (\rho_{2n-1}\delta + \rho_{2n}) + \rho_{2n-2}}.
\end{aligned}$$

And

$$\begin{aligned}
\Psi_{4n-3} &= \frac{\epsilon\ (\rho_{2n-1}\gamma + \rho_{2n}) + \rho_{2n-2}}{\epsilon\ (\rho_{2n-2}\gamma + \rho_{2n-1}) + \rho_{2n-3}},\\
\Psi_{4n-2} &= \frac{\zeta\ (\rho_{2n-1}\delta + \rho_{2n}) + \rho_{2n-2}}{\zeta\ (\rho_{2n-2}\delta + \rho_{2n-1}) + \rho_{2n-3}},\\
\Psi_{4n-1} &= \frac{\alpha(\rho_{2n}\eta + \rho_{2n+1}) + \rho_{2n-1}}{\alpha(\rho_{2n-1}\eta + \rho_{2n}) + \rho_{2n-2}},\\
\Psi_{4n} &= \frac{\beta(\rho_{2n}\kappa + \rho_{2n+1}) + \rho_{2n-1}}{\beta(\rho_{2n-1}\kappa + \rho_{2n}) + \rho_{2n-2}},
\end{aligned}$$

where $\rho_{n+1} = \rho_{n-1} + \rho_{n-2}\ \forall\, n \in \mathbb{N}$ is the Padovan sequence.

**Proof.** The forms of the solutions are true for $n = 0$. We assume that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
\Phi_{4n-7} &= \frac{\alpha\ (\rho_{2n-3}\eta + \rho_{2n-2}) + \rho_{2n-4}}{\alpha\ (\rho_{2n-4}\eta + \rho_{2n-3}) + \rho_{2n-5}},\\
\Phi_{4n-6} &= \frac{\beta(\rho_{2n-3}\kappa + \rho_{2n-2}) + \rho_{2n-4}}{\beta(\rho_{2n-4}\kappa + \rho_{2n-3}) + \rho_{2n-5}},\\
\Phi_{4n-5} &= \frac{\epsilon(\rho_{2n-2}\gamma + \rho_{2n-1}) + \rho_{2n-3}}{\epsilon(\rho_{2n-3}\gamma + \rho_{2n-2}) + \rho_{2n-4}},
\end{aligned}$$

$$\Phi_{4n-4} = \frac{\zeta(\rho_{2n-2}\delta + \rho_{2n-1}) + \rho_{2n-3}}{\zeta(\rho_{2n-3}\delta + \rho_{2n-2}) + \rho_{2n-4}},$$

and

$$\Psi_{4n-7} = \frac{\epsilon\ (\rho_{2n-3}\gamma + \rho_{2n-2}) + \rho_{2n-4}}{\epsilon\ (\rho_{2n-4}\gamma + \rho_{2n-3}) + \rho_{2n-5}},$$

$$\Psi_{4n-6} = \frac{\zeta\ (\rho_{2n-3}\delta + \rho_{2n-2}) + \rho_{2n-4}}{\zeta\ (\rho_{2n-4}\delta + \rho_{2n-3}) + \rho_{2n-5}},$$

$$\Psi_{4n-5} = \frac{\alpha(\rho_{2n-2}\eta + \rho_{2n-1}) + \rho_{2n-3}}{\alpha(\rho_{2n-3}\eta + \rho_{2n-2}) + \rho_{2n-4}},$$

$$\Psi_{4n-4} = \frac{\beta(\rho_{2n-2}\kappa + \rho_{2n-1}) + \rho_{2n-3}}{\beta(\rho_{2n-3}\kappa + \rho_{2n-2}) + \rho_{2n-4}}.$$

Next, it can be easily observed from system (1.2) that

$$\begin{aligned}
\Phi_{4n-3} &= \frac{\Phi_{4n-7} + 1}{\Psi_{4n-5}\Phi_{4n-7}} \\
&= \frac{\frac{\alpha\ (\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}{\alpha\ (\rho_{2n-4}\eta+\rho_{2n-3})+\rho_{2n-5}} + 1}{\left(\frac{\alpha(\rho_{2n-2}\eta+\rho_{2n-1})+\rho_{2n-3}}{\alpha(\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}\right)\left(\frac{\alpha\ (\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}{\alpha\ (\rho_{2n-4}\eta+\rho_{2n-3})+\rho_{2n-5}}\right)} \\
&= \frac{\frac{\alpha\ (\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}+\alpha\ (\rho_{2n-4}\eta+\rho_{2n-3})+\rho_{2n-5}}{\alpha\ (\rho_{2n-4}\eta+\rho_{2n-3})+\rho_{2n-5}}}{\frac{\alpha(\rho_{2n-2}\eta+\rho_{2n-1})+\rho_{2n-3}}{\alpha\ (\rho_{2n-4}\eta+\rho_{2n-3})+\rho_{2n-5}}} \\
&= \frac{\alpha\ (\rho_{2n-3}\eta + \rho_{2n-2}) + \rho_{2n-4} + \alpha\ (\rho_{2n-4}\eta + \rho_{2n-3}) + \rho_{2n-5}}{\alpha(\rho_{2n-2}\eta + \rho_{2n-1}) + \rho_{2n-3}} \\
&= \frac{\alpha\ \rho_{2n-3}\eta + \alpha\rho_{2n-2} + \rho_{2n-4} + \alpha\ \rho_{2n-4}\eta + \alpha\rho_{2n-3} + \rho_{2n-5}}{\alpha(\rho_{2n-2}\eta + \rho_{2n-1}) + \rho_{2n-3}} \\
&= \frac{\alpha\ \eta\,(\rho_{2n-3} + \ \rho_{2n-4}) + \alpha\,(\rho_{2n-2} + \rho_{2n-3}) + (\rho_{2n-4} + \rho_{2n-5})}{\alpha(\rho_{2n-2}\eta + \rho_{2n-1}) + \rho_{2n-3}}.
\end{aligned}$$

Since $\rho_{n+1} = \rho_{n-1} + \rho_{n-2}$, we have

$$\Phi_{4n-3} = \frac{\alpha\ \ (\rho_{2n-1}\eta + \ \rho_{2n}) + \rho_{2n-2}}{\alpha(\rho_{2n-2}\eta + \rho_{2n-1}) + \rho_{2n-3}}.$$

$$\begin{aligned}
\Psi_{4n-3} &= \frac{\Psi_{4n-7} + 1}{\Phi_{4n-5}\Psi_{4n-7}} \\
&= \frac{\frac{\epsilon\ (\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}{\epsilon\ (\rho_{2n-4}\gamma+\rho_{2n-3})+\rho_{2n-5}} + 1}{\left(\frac{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}}{\epsilon(\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}\right)\left(\frac{\epsilon\ (\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}{\epsilon\ (\rho_{2n-4}\gamma+\rho_{2n-3})+\rho_{2n-5}}\right)} \\
&= \frac{\frac{\epsilon\ (\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}+\epsilon\ (\rho_{2n-4}\gamma+\rho_{2n-3})+\rho_{2n-5}}{\epsilon\ (\rho_{2n-4}\gamma+\rho_{2n-3})+\rho_{2n-5}}}{\frac{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}}{\epsilon\ (\rho_{2n-4}\gamma+\rho_{2n-3})+\rho_{2n-5}}} \\
&= \frac{\epsilon\ (\rho_{2n-3}\gamma + \rho_{2n-2}) + \rho_{2n-4} + \epsilon\ (\rho_{2n-4}\gamma + \rho_{2n-3}) + \rho_{2n-5}}{\epsilon(\rho_{2n-2}\gamma + \rho_{2n-1}) + \rho_{2n-3}} \\
&= \frac{\epsilon\rho_{2n-3}\gamma + \epsilon\rho_{2n-2} + \rho_{2n-4} + \ \epsilon\rho_{2n-4}\gamma + \epsilon\rho_{2n-3} + \rho_{2n-5}}{\epsilon(\rho_{2n-2}\gamma + \rho_{2n-1}) + \rho_{2n-3}}
\end{aligned}$$

$$= \frac{\epsilon\gamma\left(\rho_{2n-3}+\rho_{2n-4}\right)+\epsilon\left(\rho_{2n-2}+\rho_{2n-3}\right)+\rho_{2n-4}+\rho_{2n-5}}{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}}$$
$$= \frac{\epsilon\gamma\rho_{2n-1}+\epsilon\rho_{2n}+\rho_{2n-2}}{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}} = \frac{\epsilon(\rho_{2n-1}\gamma+\rho_{2n})+\rho_{2n-2}}{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}}.$$

Also, system (1.2) gives

$$\begin{aligned}
\Phi_{4n-2} &= \frac{\Phi_{4n-6}+1}{\Psi_{4n-4}\Phi_{4n-6}} \\
&= \frac{\frac{\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}}{\beta(\rho_{2n-4}\kappa+\rho_{2n-3})+\rho_{2n-5}}+1}{\left(\frac{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}}{\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}}\right)\left(\frac{\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}}{\beta(\rho_{2n-4}\kappa+\rho_{2n-3})+\rho_{2n-5}}\right)} \\
&= \frac{\frac{\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}+\beta(\rho_{2n-4}\kappa+\rho_{2n-3})+\rho_{2n-5}}{\beta(\rho_{2n-4}\kappa+\rho_{2n-3})+\rho_{2n-5}}}{\frac{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}}{\beta(\rho_{2n-4}\kappa+\rho_{2n-3})+\rho_{2n-5}}} \\
&= \frac{\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}+\beta(\rho_{2n-4}\kappa+\rho_{2n-3})+\rho_{2n-5}}{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}} \\
&= \frac{\beta\rho_{2n-3}\kappa+\beta\rho_{2n-2}+\rho_{2n-4}+\beta\rho_{2n-4}\kappa+\beta\rho_{2n-3}+\rho_{2n-5}}{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}} \\
&= \frac{\beta\kappa\left(\rho_{2n-3}+\rho_{2n-4}\right)+\beta\left(\rho_{2n-2}+\rho_{2n-3}\right)+\rho_{2n-4}+\rho_{2n-5}}{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}} \\
&= \frac{\beta\kappa\rho_{2n-1}+\beta\rho_{2n}+\rho_{2n-2}}{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}} = \frac{\beta\left(\rho_{2n-1}\kappa+\rho_{2n}\right)+\rho_{2n-2}}{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}}.
\end{aligned}$$

$$\begin{aligned}
\Psi_{4n-2} &= \frac{\Psi_{4n-6}+1}{\Phi_{4n-4}\Psi_{4n-6}} \\
&= \frac{\frac{\zeta\ (\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}}{\zeta\ (\rho_{2n-4}\delta+\rho_{2n-3})+\rho_{2n-5}}+1}{\left(\frac{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}}{\zeta(\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}}\right)\left(\frac{\zeta\ (\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}}{\zeta\ (\rho_{2n-4}\delta+\rho_{2n-3})+\rho_{2n-5}}\right)} \\
&= \frac{\frac{\zeta\ (\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}+\zeta\ (\rho_{2n-4}\delta+\rho_{2n-3})+\rho_{2n-5}}{\zeta\ (\rho_{2n-4}\delta+\rho_{2n-3})+\rho_{2n-5}}}{\frac{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}}{\zeta\ (\rho_{2n-4}\delta+\rho_{2n-3})+\rho_{2n-5}}} \\
&= \frac{\zeta\ (\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}+\zeta\ (\rho_{2n-4}\delta+\rho_{2n-3})+\rho_{2n-5}}{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}} \\
&= \frac{\zeta\ \rho_{2n-3}\delta+\zeta\ \rho_{2n-2}+\rho_{2n-4}+\zeta\ \rho_{2n-4}\delta+\zeta\ \rho_{2n-3}+\rho_{2n-5}}{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}} \\
&= \frac{\zeta\delta(\rho_{2n-3}+\rho_{2n-4})+\zeta\left(\rho_{2n-2}+\rho_{2n-3}\right)+\rho_{2n-4}+\rho_{2n-5}}{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}} \\
&= \frac{\zeta\delta\rho_{2n-1}+\zeta\rho_{2n}+\rho_{2n-2}}{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}} = \frac{\zeta\left(\rho_{2n-1}\delta+\rho_{2n}\right)+\rho_{2n-2}}{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}}.
\end{aligned}$$

Furthermore, from system (1.2), we have

$$
\begin{aligned}
\Phi_{4n-1} &= \frac{\Phi_{4n-5}+1}{\Psi_{4n-3}\Phi_{4n-5}} \\
&= \frac{\frac{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}}{\epsilon(\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}+1}{\left(\frac{\epsilon\ (\rho_{2n-1}\gamma+\rho_{2n})+\rho_{2n-2}}{\epsilon\ (\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}}\right)\left(\frac{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}}{\epsilon(\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}\right)} \\
&= \frac{\frac{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}+\epsilon(\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}{\epsilon(\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}}{\frac{\epsilon\ (\rho_{2n-1}\gamma+\rho_{2n})+\rho_{2n-2}}{\epsilon(\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}} \\
&= \frac{\epsilon(\rho_{2n-2}\gamma+\rho_{2n-1})+\rho_{2n-3}+\epsilon(\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}{\epsilon\ (\rho_{2n-1}\gamma+\rho_{2n})+\rho_{2n-2}} \\
&= \frac{\epsilon(\rho_{2n}\gamma+\rho_{2n+1})+\rho_{2n-1}}{\epsilon\ (\rho_{2n-1}\gamma+\rho_{2n})+\rho_{2n-2}}.
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{4n-1} &= \frac{\Psi_{4n-5}+1}{\Phi_{4n-3}\Psi_{4n-5}} \\
&= \frac{\frac{\alpha(\rho_{2n-2}\eta+\rho_{2n-1})+\rho_{2n-3}}{\alpha(\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}+1}{\left(\frac{\alpha\ (\rho_{2n-1}\eta+\rho_{2n})+\rho_{2n-2}}{\alpha\ (\rho_{2n-2}\eta+\rho_{2n-1})+\rho_{2n-3}}\right)\left(\frac{\alpha(\rho_{2n-2}\eta+\rho_{2n-1})+\rho_{2n-3}}{\alpha(\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}\right)} \\
&= \frac{\frac{\alpha(\rho_{2n-2}\eta+\rho_{2n-1})+\rho_{2n-3}+\alpha(\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}{\alpha(\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}}{\frac{\alpha\ (\rho_{2n-1}\eta+\rho_{2n})+\rho_{2n-2}}{\alpha(\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}} \\
&= \frac{\alpha(\rho_{2n}\eta+\rho_{2n+1})+\rho_{2n-1}}{\alpha\ (\rho_{2n-1}\eta+\rho_{2n})+\rho_{2n-2}}.
\end{aligned}
$$

Moreover, from system (1.2), we have

$$
\begin{aligned}
\Phi_{4n} &= \frac{{}_{4n-4}+1}{{}_{4n-2}\Phi_{4n-4}} \\
&= \frac{\frac{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}}{\zeta(\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}}+1}{\left(\frac{\zeta\ (\rho_{2n-1}\delta+\rho_{2n})+\rho_{2n-2}}{\zeta\ (\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}}\right)\left(\frac{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}}{\zeta(\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}}\right)} \\
&= \frac{\frac{\zeta(\rho_{2n-2}\delta+\rho_{2n-1})+\rho_{2n-3}+\zeta(\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}}{\epsilon(\rho_{2n-3}\gamma+\rho_{2n-2})+\rho_{2n-4}}}{\frac{\zeta\ (\rho_{2n-1}\delta+\rho_{2n})+\rho_{2n-2}}{\zeta(\rho_{2n-3}\delta+\rho_{2n-2})+\rho_{2n-4}}} \\
&= \frac{\zeta(\rho_{2n}\delta+\rho_{2n+1})+\rho_{2n-1}}{\zeta\ (\rho_{2n-1}\delta+\rho_{2n})+\rho_{2n-2}}.
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{4n} &= \frac{\Psi_{4n-4}+1}{\Phi_{4n-2}\Psi_{4n-4}} \\
&= \frac{\frac{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}}{\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}}+1}{\left(\frac{\beta(\rho_{2n-1}\kappa+\rho_{2n})+\rho_{2n-2}}{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}}\right)\left(\frac{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}}{\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}}\right)}
\end{aligned}
$$

$$= \frac{\frac{\beta(\rho_{2n-2}\kappa+\rho_{2n-1})+\rho_{2n-3}+\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}}{\alpha(\rho_{2n-3}\eta+\rho_{2n-2})+\rho_{2n-4}}}{\frac{\beta(\rho_{2n-1}\kappa+\rho_{2n})+\rho_{2n-2}}{\beta(\rho_{2n-3}\kappa+\rho_{2n-2})+\rho_{2n-4}}}$$
$$= \frac{\beta(\rho_{2n}\kappa+\rho_{2n+1})+\rho_{2n-1}}{\beta(\rho_{2n-1}\kappa+\rho_{2n})+\rho_{2n-2}}.$$

The proof has been completed.

## 3. Solutions of System (1.3)

In this section, we show analytic solutions for system (1.3) using Padovan numbers.

**Theorem 3.1.** *Let* $\{\Phi_n, \Psi_n\}_{n=-3}^{\infty}$ *be the solutions of system* (1.3) *and assume that* $\Phi_{-3} = \alpha$, $\Phi_{-2} = \beta$, $\Phi_{-1} = \gamma$, $\Phi_0 = \delta$, $\Psi_{-3} = \epsilon$, $\Psi_{-2} = \zeta$, $\Psi_{-1} = \eta$, *and* $\Psi_0 = \kappa$. *Then, for* $n = 0, 1, \ldots,$ *we have*

$$\Phi_{4n-3} = \frac{\alpha\,(\rho_{2n}-\rho_{2n-1}\eta)-\rho_{2n-2}}{\alpha\,(\rho_{2n-2}\eta-\rho_{2n-1})+\rho_{2n-3}},$$
$$\Phi_{4n-2} = \frac{\beta(\rho_{2n}-\rho_{2n-1}\kappa)-\rho_{2n-2}}{\beta(\rho_{2n-2}\kappa-\rho_{2n-1})+\rho_{2n-3}},$$
$$\Phi_{4n-1} = \frac{\epsilon(\rho_{2n+1}-\rho_{2n}\gamma)-\rho_{2n-1}}{\epsilon(\rho_{2n-1}\gamma-\rho_{2n})+\rho_{2n-2}},$$
$$\Phi_{4n} = \frac{\zeta\,(\rho_{2n+1}-\rho_{2n}\delta)-\rho_{2n-1}}{\zeta\,(\rho_{2n-1}\delta-\rho_{2n})+\rho_{2n-2}}.$$

And

$$\Psi_{4n-3} = \frac{\epsilon\,(\rho_{2n}-\rho_{2n-1}\gamma)-\rho_{2n-2}}{\epsilon\,(\rho_{2n-2}\gamma-\rho_{2n-1})+\rho_{2n-3}},$$
$$\Psi_{4n-2} = \frac{\zeta\,(\rho_{2n}-\rho_{2n-1}\delta)-\rho_{2n-2}}{\zeta\,(\rho_{2n-2}\delta-\rho_{2n-1})+\rho_{2n-3}},$$
$$\Psi_{4n-1} = \frac{\alpha(\rho_{2n+1}-\rho_{2n}\eta)-\rho_{2n-1}}{\alpha(\rho_{2n-1}\eta-\rho_{2n})+\rho_{2n-2}},$$
$$\Psi_{4n} = \frac{\beta(\rho_{2n+1}-\rho_{2n}\kappa)-\rho_{2n-1}}{\beta(\rho_{2n-1}\kappa-\rho_{2n})+\rho_{2n-2}},$$

where $\rho_{n+1} = \rho_{n-1} + \rho_{n-2}\ \forall\, n \in \mathbb{N}$ is the Padovan sequence.

**Proof.** The above solutions are true for $n = 0$. We suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$\Phi_{4n-7} = \frac{\alpha\,(\rho_{2n-2}-\rho_{2n-3}\eta)-\rho_{2n-4}}{\alpha\,(\rho_{2n-4}\eta-\rho_{2n-3})+\rho_{2n-5}},$$
$$\Phi_{4n-6} = \frac{\beta(\rho_{2n-2}-\rho_{2n-3}\kappa)-\rho_{2n-4}}{\beta(\rho_{2n-4}\kappa-\rho_{2n-3})+\rho_{2n-5}},$$
$$\Phi_{4n-5} = \frac{\epsilon(\rho_{2n-1}-\rho_{2n-2}\gamma)-\rho_{2n-3}}{\epsilon(\rho_{2n-3}\gamma-\rho_{2n-2})+\rho_{2n-4}},$$
$$\Phi_{4n-4} = \frac{\zeta(\rho_{2n-1}-\rho_{2n-2}\delta)-\rho_{2n-3}}{\zeta(\rho_{2n-3}\delta-\rho_{2n-2})+\rho_{2n-4}}.$$

And

$$\begin{aligned}
\Psi_{4n-7} &= \frac{\epsilon\ (\rho_{2n-2}-\rho_{2n-3}\gamma)-\rho_{2n-4}}{\epsilon\ (\rho_{2n-4}\gamma-\rho_{2n-3})+\rho_{2n-5}},\\
\Psi_{4n-6} &= \frac{\zeta\ (\rho_{2n-2}-\rho_{2n-3}\delta)-\rho_{2n-4}}{\zeta\ (\rho_{2n-4}\delta-\rho_{2n-3})+\rho_{2n-5}},\\
\Psi_{4n-5} &= \frac{\alpha(\rho_{2n-1}-\rho_{2n-2}\eta)-\rho_{2n-3}}{\alpha(\rho_{2n-3}\eta-\rho_{2n-2})+\rho_{2n-4}},\\
\Psi_{4n-4} &= \frac{\beta(\rho_{2n-1}-\rho_{2n-2}\kappa)-\rho_{2n-3}}{\beta(\rho_{2n-3}\kappa-\rho_{2n-2})+\rho_{2n-4}}.
\end{aligned}$$

System (1.3) leads to

$$\begin{aligned}
\Phi_{4n-3} &= \frac{\Phi_{4n-7}-1}{\Psi_{4n-5}\Phi_{4n-7}}\\
&= \frac{\frac{\alpha\ (\rho_{2n-2}-\rho_{2n-3}\eta)-\rho_{2n-4}}{\alpha\ (\rho_{2n-4}\eta-\rho_{2n-3})+\rho_{2n-5}}-1}{\left(\frac{\alpha(\rho_{2n-1}-\rho_{2n-2}\eta)-\rho_{2n-3}}{\alpha(\rho_{2n-3}\eta-\rho_{2n-2})+\rho_{2n-4}}\right)\left(\frac{\alpha(\rho_{2n-2}-\rho_{2n-3}\eta)-\rho_{2n-4}}{\alpha\ (\rho_{2n-4}\eta-\rho_{2n-3})+\rho_{2n-5}}\right)}\\
&= \frac{\frac{\alpha\ (\rho_{2n-2}-\rho_{2n-3}\eta)-\rho_{2n-4}-(\alpha\ (\rho_{2n-4}\eta-\rho_{2n-3})+\rho_{2n-5})}{\alpha\ (\rho_{2n-4}\eta-\rho_{2n-3})+\rho_{2n-5}}}{-\frac{\alpha(\rho_{2n-1}-\rho_{2n-2}\eta)-\rho_{2n-3}}{\alpha\ (\rho_{2n-4}\eta-\rho_{2n-3})+\rho_{2n-5}}}\\
&= \frac{\alpha\ (\rho_{2n-2}-\rho_{2n-3}\eta)-\rho_{2n-4}-(\alpha\ (\rho_{2n-4}\eta-\rho_{2n-3})+\rho_{2n-5})}{-\alpha(\rho_{2n-1}-\rho_{2n-2}\eta)-\rho_{2n-3}}\\
&= \frac{\alpha\,(\rho_{2n-2}+\rho_{2n-3})-\alpha\eta\,(\rho_{2n-3}+\rho_{2n-4})-(\rho_{2n-4}+\rho_{2n-5})}{\alpha(\rho_{2n-2}\eta-\rho_{2n-1})+\rho_{2n-3}}\\
&= \frac{\alpha\rho_{2n}-\alpha\eta\rho_{2n-1}-\rho_{2n-2}}{\alpha(\rho_{2n-2}\eta-\rho_{2n-1})+\rho_{2n-3}} = \frac{\alpha\,(\rho_{2n}-\rho_{2n-1}\eta)-\rho_{2n-2}}{\alpha(\rho_{2n-2}\eta-\rho_{2n-1})+\rho_{2n-3}}.
\end{aligned}$$

$$\begin{aligned}
\Psi_{4n-3} &= \frac{\Psi_{4n-7}-1}{\Phi_{4n-5}\Psi_{4n-7}}\\
&= \frac{\frac{\epsilon\ (\rho_{2n-2}-\rho_{2n-3}\gamma)-\rho_{2n-4}}{\epsilon\ (\rho_{2n-4}\gamma-\rho_{2n-3})+\rho_{2n-5}}-1}{\left(\frac{\epsilon(\rho_{2n-1}-\rho_{2n-2}\gamma)-\rho_{2n-3}}{\epsilon(\rho_{2n-3}\gamma-\rho_{2n-2})+\rho_{2n-4}}\right)\left(\frac{\epsilon\ (\rho_{2n-2}-\rho_{2n-3}\gamma)-\rho_{2n-4}}{\epsilon\ (\rho_{2n-4}\gamma-\rho_{2n-3})+\rho_{2n-5}}\right)}\\
&= \frac{\frac{\epsilon\ (\rho_{2n-2}-\rho_{2n-3}\gamma)-\rho_{2n-4}-(\epsilon\ (\rho_{2n-4}\gamma-\rho_{2n-3})+\rho_{2n-5})}{\epsilon\ (\rho_{2n-4}\gamma-\rho_{2n-3})+\rho_{2n-5}}}{-\frac{\epsilon(\rho_{2n-1}-\rho_{2n-2}\gamma)-\rho_{2n-3}}{\epsilon\ (\rho_{2n-4}\gamma-\rho_{2n-3})+\rho_{2n-5}}}\\
&= \frac{\epsilon\ (\rho_{2n-2}-\rho_{2n-3}\gamma)-\rho_{2n-4}-(\epsilon\ (\rho_{2n-4}\gamma-\rho_{2n-3})+\rho_{2n-5})}{-\epsilon(\rho_{2n-1}-\rho_{2n-2}\gamma)-\rho_{2n-3}}\\
&= \frac{\epsilon\,(\rho_{2n-2}+\rho_{2n-3})-\epsilon\gamma\,(\rho_{2n-3}+\rho_{2n-4})-(\rho_{2n-4}+\rho_{2n-5})}{\epsilon(\rho_{2n-2}\gamma-\rho_{2n-1})+\rho_{2n-3}}\\
&= \frac{\epsilon\rho_{2n}-\epsilon\gamma\rho_{2n-1}-\rho_{2n-2}}{\epsilon(\rho_{2n-2}\gamma-\rho_{2n-1})+\rho_{2n-3}} = \frac{\epsilon\,(\rho_{2n}-\rho_{2n-1}\gamma)-\rho_{2n-2}}{\epsilon(\rho_{2n-2}\gamma-\rho_{2n-1})+\rho_{2n-3}}.
\end{aligned}$$

Furthermore, we can obtain from system (1.3) that

$$
\begin{aligned}
\Phi_{4n-2} &= \frac{\Phi_{4n-6}-1}{\Psi_{4n-4}\Phi_{4n-6}} \\
&= \frac{\frac{\beta(\rho_{2n-2}-\rho_{2n-3}\kappa)-\rho_{2n-4}}{\beta(\rho_{2n-4}\kappa-\rho_{2n-3})+\rho_{2n-5}}-1}{\left(\frac{\beta(\rho_{2n-1}-\rho_{2n-2}\kappa)-\rho_{2n-3}}{\beta(\rho_{2n-3}\kappa-\rho_{2n-2})+\rho_{2n-4}}\right)\left(\frac{\beta(\rho_{2n-2}-\rho_{2n-3}\kappa)-\rho_{2n-4}}{\beta(\rho_{2n-4}\kappa-\rho_{2n-3})+\rho_{2n-5}}\right)} \\
&= \frac{\frac{\beta(\rho_{2n-2}-\rho_{2n-3}\kappa)-\rho_{2n-4}-(\beta(\rho_{2n-4}\kappa-\rho_{2n-3})+\rho_{2n-5})}{\beta(\rho_{2n-4}\kappa-\rho_{2n-3})+\rho_{2n-5}}}{-\frac{\beta(\rho_{2n-1}-\rho_{2n-2}\kappa)-\rho_{2n-3}}{\beta(\rho_{2n-4}\kappa-\rho_{2n-3})+\rho_{2n-5}}} \\
&= \frac{\beta(\rho_{2n-2}-\rho_{2n-3}\kappa)-\rho_{2n-4}-(\beta(\rho_{2n-4}\kappa-\rho_{2n-3})+\rho_{2n-5})}{-\beta(\rho_{2n-1}-\rho_{2n-2}\kappa)-\rho_{2n-3}} \\
&= \frac{\beta(\rho_{2n}-\rho_{2n-1}\kappa)-\rho_{2n-2}}{\beta(\rho_{2n-2}\kappa-\rho_{2n-1})+\rho_{2n-3}}.
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{4n-2} &= \frac{\Psi_{4n-6}-1}{\Phi_{4n-4}\Psi_{4n-6}} \\
&= \frac{\frac{\zeta\ (\rho_{2n-2}-\rho_{2n-3}\delta)-\rho_{2n-4}}{\zeta\ (\rho_{2n-4}\delta-\rho_{2n-3})+\rho_{2n-5}}-1}{\left(\frac{\zeta(\rho_{2n-1}-\rho_{2n-2}\delta)-\rho_{2n-3}}{\zeta(\rho_{2n-3}\delta-\rho_{2n-2})+\rho_{2n-4}}\right)\left(\frac{\zeta\ (\rho_{2n-2}-\rho_{2n-3}\delta)-\rho_{2n-4}}{\zeta\ (\rho_{2n-4}\delta-\rho_{2n-3})+\rho_{2n-5}}\right)} \\
&= \frac{\frac{\zeta\ (\rho_{2n-2}-\rho_{2n-3}\delta)-\rho_{2n-4}-(\zeta\ (\rho_{2n-4}\delta-\rho_{2n-3})+\rho_{2n-5})}{\zeta\ (\rho_{2n-4}\delta-\rho_{2n-3})+\rho_{2n-5}}}{-\frac{\zeta(\rho_{2n-1}-\rho_{2n-2}\delta)-\rho_{2n-3}}{\zeta\ (\rho_{2n-4}\delta-\rho_{2n-3})+\rho_{2n-5}}} \\
&= \frac{\zeta\ (\rho_{2n-2}-\rho_{2n-3}\delta)-\rho_{2n-4}-(\zeta\ (\rho_{2n-4}\delta-\rho_{2n-3})+\rho_{2n-5})}{-\zeta(\rho_{2n-1}-\rho_{2n-2}\delta)-\rho_{2n-3}} \\
&= \frac{\zeta\ (\rho_{2n}-\rho_{2n-1}\delta)-\rho_{2n-2}}{\zeta(\rho_{2n-2}\delta-\rho_{2n-1})+\rho_{2n-3}}.
\end{aligned}
$$

In addition, one can obtain from system (1.3) that

$$
\begin{aligned}
\Phi_{4n-1} &= \frac{\Phi_{4n-5}-1}{\Psi_{4n-3}\Phi_{4n-5}} \\
&= \frac{\frac{\epsilon(\rho_{2n-1}-\rho_{2n-2}\gamma)-\rho_{2n-3}}{\epsilon(\rho_{2n-3}\gamma-\rho_{2n-2})+\rho_{2n-4}}-1}{\left(\frac{\epsilon\ (\rho_{2n}-\rho_{2n-1}\gamma)-\rho_{2n-2}}{\epsilon\ (\rho_{2n-2}\gamma-\rho_{2n-1})+\rho_{2n-3}}\right)\left(\frac{\epsilon(\rho_{2n-1}-\rho_{2n-2}\gamma)-\rho_{2n-3}}{\epsilon(\rho_{2n-3}\gamma-\rho_{2n-2})+\rho_{2n-4}}\right)} \\
&= \frac{\frac{\epsilon(\rho_{2n-1}-\rho_{2n-2}\gamma)-\rho_{2n-3}-(\epsilon(\rho_{2n-3}\gamma-\rho_{2n-2})+\rho_{2n-4})}{\epsilon(\rho_{2n-3}\gamma-\rho_{2n-2})+\rho_{2n-4}}}{-\frac{\epsilon\ (\rho_{2n}-\rho_{2n-1}\gamma)-\rho_{2n-2}}{\epsilon(\rho_{2n-3}\gamma-\rho_{2n-2})+\rho_{2n-4}}} \\
&= \frac{\epsilon(\rho_{2n-1}-\rho_{2n-2}\gamma)-\rho_{2n-3}-(\epsilon(\rho_{2n-3}\gamma-\rho_{2n-2})+\rho_{2n-4})}{-\epsilon\ (\rho_{2n}-\rho_{2n-1}\gamma)-\rho_{2n-2}} \\
&= \frac{\epsilon(\rho_{2n+1}-\rho_{2n}\gamma)-\rho_{2n-1}}{\epsilon\ (\rho_{2n-1}\gamma-\rho_{2n})+\rho_{2n-2}}.
\end{aligned}
$$

$$\begin{aligned}
\Psi_{4n-1} &= \frac{\Psi_{4n-5}-1}{\Phi_{4n-3}\Psi_{4n-5}} \\
&= \frac{\frac{\alpha(\rho_{2n-1}-\rho_{2n-2}\eta)-\rho_{2n-3}}{\alpha(\rho_{2n-3}\eta-\rho_{2n-2})+\rho_{2n-4}}-1}{\left(\frac{\alpha\ (\rho_{2n}-\rho_{2n-1}\eta)-\rho_{2n-2}}{\alpha\ (\rho_{2n-2}\eta-\rho_{2n-1})+\rho_{2n-3}}\right)\left(\frac{\alpha(\rho_{2n-1}-\rho_{2n-2}\eta)-\rho_{2n-3}}{\alpha(\rho_{2n-3}\eta-\rho_{2n-2})+\rho_{2n-4}}\right)} \\
&= \frac{\frac{\alpha(\rho_{2n-1}-\rho_{2n-2}\eta)-\rho_{2n-3}-(\alpha(\rho_{2n-3}\eta-\rho_{2n-2})+\rho_{2n-4})}{\alpha(\rho_{2n-3}\eta-\rho_{2n-2})+\rho_{2n-4}}}{-\frac{\alpha\ (\rho_{2n}-\rho_{2n-1}\eta)-\rho_{2n-2}}{\alpha(\rho_{2n-3}\eta-\rho_{2n-2})+\rho_{2n-4}}} \\
&= \frac{\alpha(\rho_{2n-1}-\rho_{2n-2}\eta)-\rho_{2n-3}-(\alpha(\rho_{2n-3}\eta-\rho_{2n-2})+\rho_{2n-4})}{-\alpha\ (\rho_{2n}-\rho_{2n-1}\eta)-\rho_{2n-2}} \\
&= \frac{\alpha(\rho_{2n+1}-\rho_{2n}\eta)-\rho_{2n-1}}{\alpha\ (\rho_{2n-1}\eta-\rho_{2n})+\rho_{2n-2}}.
\end{aligned}$$

Finally, system (1.3) leads to

$$\begin{aligned}
\Phi_{4n} &= \frac{\Phi_{4n-4}-1}{\Psi_{4n-2}\Phi_{4n-4}} \\
&= \frac{\frac{\zeta(\rho_{2n-1}-\rho_{2n-2}\delta)-\rho_{2n-3}}{\zeta(\rho_{2n-3}\delta-\rho_{2n-2})+\rho_{2n-4}}-1}{\left(\frac{\zeta\ (\rho_{2n}-\rho_{2n-1}\delta)-\rho_{2n-2}}{\zeta\ (\rho_{2n-2}\delta-\rho_{2n-1})+\rho_{2n-3}}\right)\left(\frac{\zeta(\rho_{2n-1}-\rho_{2n-2}\delta)-\rho_{2n-3}}{\zeta(\rho_{2n-3}\delta-\rho_{2n-2})+\rho_{2n-4}}\right)} \\
&= \frac{\frac{\zeta(\rho_{2n-1}-\rho_{2n-2}\delta)-\rho_{2n-3}-(\zeta(\rho_{2n-3}\delta-\rho_{2n-2})+\rho_{2n-4})}{\zeta(\rho_{2n-3}\delta-\rho_{2n-2})+\rho_{2n-4}}}{-\frac{\zeta\ (\rho_{2n}-\rho_{2n-1}\delta)-\rho_{2n-2}}{\zeta(\rho_{2n-3}\delta-\rho_{2n-2})+\rho_{2n-4}}} \\
&= \frac{\zeta(\rho_{2n-1}-\rho_{2n-2}\delta)-\rho_{2n-3}-(\zeta(\rho_{2n-3}\delta-\rho_{2n-2})+\rho_{2n-4})}{-\zeta\ (\rho_{2n}-\rho_{2n-1}\delta)-\rho_{2n-2}} \\
&= \frac{\zeta(\rho_{2n+1}-\rho_{2n}\delta)-\rho_{2n-1}}{\zeta\ (\rho_{2n-1}\delta\rho_{2n})+\rho_{2n-2}}.
\end{aligned}$$

$$\begin{aligned}
\Psi_{4n} &= \frac{\Psi_{4n-4}-1}{\Phi_{4n-2}\Psi_{4n-4}} \\
&= \frac{\frac{\beta(\rho_{2n-1}-\rho_{2n-2}\kappa)-\rho_{2n-3}}{\beta(\rho_{2n-3}\kappa-\rho_{2n-2})+\rho_{2n-4}}-1}{\left(\frac{\beta(\rho_{2n}-\rho_{2n-1}\kappa)-\rho_{2n-2}}{\beta(\rho_{2n-2}\kappa-\rho_{2n-1})+\rho_{2n-3}}\right)\left(\frac{\beta(\rho_{2n-1}-\rho_{2n-2}\kappa)-\rho_{2n-3}}{\beta(\rho_{2n-3}\kappa-\rho_{2n-2})+\rho_{2n-4}}\right)} \\
&= \frac{\frac{\beta(\rho_{2n-1}-\rho_{2n-2}\kappa)-\rho_{2n-3}-(\beta(\rho_{2n-3}\kappa-\rho_{2n-2})+\rho_{2n-4})}{\beta(\rho_{2n-3}\kappa-\rho_{2n-2})+\rho_{2n-4}}}{-\frac{\beta(\rho_{2n}-\rho_{2n-1}\kappa)-\rho_{2n-2}}{\beta(\rho_{2n-3}\kappa-\rho_{2n-2})+\rho_{2n-4}}} \\
&= \frac{\beta(\rho_{2n-1}-\rho_{2n-2}\kappa)-\rho_{2n-3}-(\beta(\rho_{2n-3}\kappa-\rho_{2n-2})+\rho_{2n-4})}{-\beta(\rho_{2n}-\rho_{2n-1}\kappa)-\rho_{2n-2}} \\
&= \frac{\beta(\rho_{2n+1}-\rho_{2n}\kappa)-\rho_{2n-1}}{\beta(\rho_{2n-1}\kappa-\rho_{2n})+\rho_{2n-2}}.
\end{aligned}$$

Hence, the proof is completed.

## 4. Stability of the equilibrium points

This section discusses the stability of the equilibrium points of the considered systems.

**Theorem 4.1.** *System (1.2) has a unique real equilibrium point $(q_1, q_1)$ which is a saddle point.*

**Proof.** The equilibrium point of system (1.2) is given by

$$\begin{aligned} \Psi^* &= \frac{\Psi^*+1}{\Phi^*\Psi^*}, \\ \Phi^* &= \frac{\Phi^*+1}{\Phi^*\Psi^*}. \end{aligned} \tag{4.1}$$

Subtract the second equation of system (4.1) from the first equation to have

$$\Phi^{*^2}\,\Psi^* - \Phi^* - 1 - \Phi^*\,\Psi^{*^2} + \Psi^* + 1 = 0,$$

or,

$$(\Phi^*\,\Psi^* - 1)\,(\Phi^* - \Psi^*) = 0.$$

Hence,

$$\Phi^*\,\Psi^* - 1 = 0\,, \qquad \Phi^*\,\Psi^* = 1\,, \tag{4.2}$$

$$\Phi^* - \Psi^* = 0\,, \qquad \Psi^* = \Phi^*\,. \tag{4.3}$$

Equation. (4.2) dose not satisfy system (4.1). Therefore, the only real equilibrium point is obtained from substituting equation. (4.3) into any equation of system (4.1). This gives

$$\Phi^{*^3} - \Phi^* - 1 = 0. \tag{4.4}$$

Solving equation. (4.4) gives

$$\begin{aligned} q_1 &= \frac{\alpha^2+12}{6\alpha}, \\ q_{2,3} &= -\frac{\alpha^2+12}{6\alpha} \pm \frac{\sqrt{3}}{2}\left(\frac{\alpha}{6} - \frac{2}{3\alpha}\right) i, \end{aligned}$$

where, $\alpha = \sqrt[3]{108+12\sqrt{69}}$. Hence, the only real equilibrium point is $(q_1, q_1)$. Now, we find the Jacobian matrix. Let $F(u,v) = (f(u,v), g(u,v))$, where

$$f(u,v) = \frac{u+1}{uv}, \qquad g(u,v) = \frac{v+1}{uv}.$$

Then,

$$J_F = \begin{pmatrix} \frac{-1}{u^2v} & -\frac{u+1}{uv^2} \\ & \\ -\frac{v+1}{u^2v} & \frac{-1}{uv^2} \end{pmatrix}.$$

Evaluating the Jacobian matrix about $(q_1, q_1)$ gives,

$$J_F(q_1, q_1) = \begin{pmatrix} \frac{-1}{q_1^3} & -\frac{q_1+1}{q_1^3} \\ \\ -\frac{q_1+1}{q_1^3} & \frac{-1}{q_1^3} \end{pmatrix}.$$

Thus, the characteristic equation of this matrix is given by

$$\left(\lambda + \frac{1}{q_1^3}\right)\left(\lambda + \frac{1}{q_1^3}\right) - \frac{(q_1+1)^2}{q_1^6} = 0.$$

Or,

$$\left(\lambda + \frac{1}{q_1^3}\right)^2 - \frac{(q_1+1)^2}{q_1^6} = 0.$$

Note that $\frac{(q_1+1)^2}{q_1^6} = 1$. Hence,

$$\left(\lambda + \frac{1}{q_1^3}\right)^2 - 1 = 0.$$

Then,

$$\lambda + \frac{1}{q_1^3} = \pm 1.$$

For $\lambda + \frac{1}{q_1^3} = 1$, we have $\lambda_1 = |1 - \frac{1}{q_1^3}| < 1$. For $\lambda + \frac{1}{q_1^3} = -1$, we have $\lambda_2 = |-1 - \frac{1}{q_1^3}| = 1 + \frac{1}{q_1^3} > 1$. Since $\lambda_1 < 1$ and $\lambda_2 > 1$, the point $(q_1, q_1)$ is a saddle point.

**Theorem 4.2.** *System (1.3) has a unique real equilibrium point $(-q_1, -q_1)$ which is a saddle point.*

**Proof.** The proof is similar to the proof of Theorem 4.1.

## 5. Behavior of the solutions

This section presents the future pattern of the considered systems under specific initial conditions. We selected some random values for the initial conditions to illustrate the long behavior solutions.

**Example 1.** Figure 1 (left) presents the dynamical behavior of the solutions of system (1.2) under the selected values $\Phi_{-3} = 7$, $\Phi_{-2} = 3$, $\Phi_{-1} = 4$, $\Phi_0 = 6$, $\Psi_{-3} = 5$, $\Psi_{-2} = 3$, $\Psi_{-1} = 5$, and $\Psi_0 = 6$.

**Example 2.** The exact solutions of system (1.2) are also plotted in Figure 1 (right) under the initial conditions $\Phi_{-3} = 1$, $\Phi_{-2} = 3$, $\Phi_{-1} = 1$, $\Phi_0 = 2$, $\Psi_{-3} = 1$, $\Psi_{-2} = 4$, $\Psi_{-1} = 2$, and $\Psi_0 = 4$.

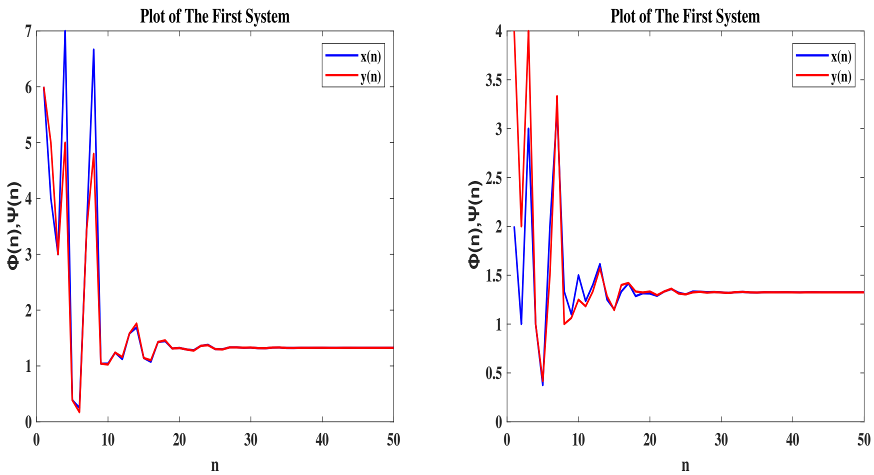


Figure 1. The dynamical behavior of the solutions of system (1.2).

**Example 3.** In Figure 2 (left), we plot the dynamical behavior of the solutions of system (1.3) under the selected values $\Phi_{-3} = 7$, $\Phi_{-2} = 2.5$, $\Phi_{-1} = 6$, $\Phi_0 = 4.4$, $\Psi_{-3} = 3.8$, $\Psi_{-2} = 2$, $\Psi_{-1} = 5$, and $\Psi_0 = 3.3$.

**Example 4.** This example illustrates the behavior of the exact solutions of system (1.3) when we use the following initial values: $\Phi_{-3} = 7$, $\Phi_{-2} = 3$, $\Phi_{-1} = 4$, $\Phi_0 = 6$, $\Psi_{-3} = 5$, $\Psi_{-2} = 3$, $\Psi_{-1} = 5$, and $\Psi_0 = 6$. See Figure 2 (right).

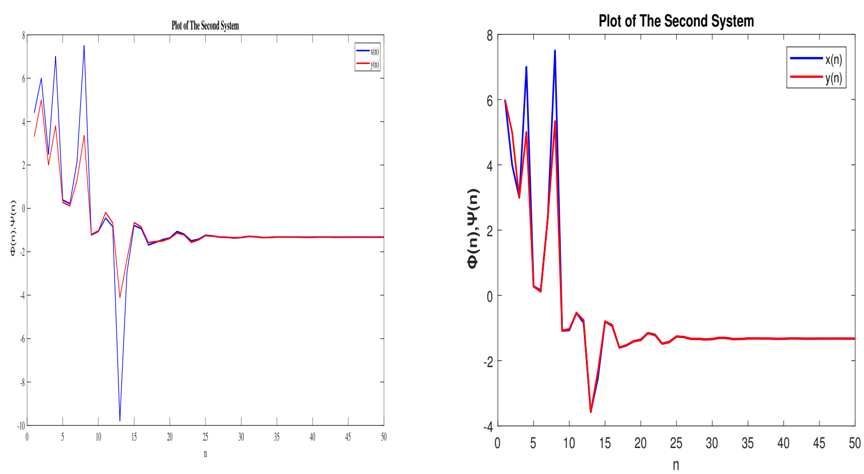


Figure 2. The dynamical behavior of the solutions of system (1.3).

## 6. Conclusion

In brief, we have discussed the solutions of the considered systems using Padovan numbers. Successive iterations have been successfully used in extracting the exact solutions. Theorem 2.1 presents the solutions of system 1.2 while Theorem 3.1 gives the solutions of system 1.3. The obtained solutions are presented in the form of rational relations. Jacobian matrix has been successfully used to examine the stability of the real equilibrium points. The equilibrium points are saddle. In Section 5, we depict the behavior of the solutions for some random initial conditions. The constructed exact solutions are stable. The used approaches can be utilized to deal with high order systems of difference equations.

**Conflicts of Interest:** The author declares that there are no conflicts of interest regarding the publication of this paper.

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