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Picture $\mathcal{N}$-Sets and Applications in Semigroups

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Abstract. In this paper, we study picture $\mathcal{N}$-structures and apply it to semigroups. Moreover, we define picture $\mathcal{N}$-ideals in semigroups and investigate several properties of these ideals in semigroups.

1. Introduction

Fuzzy sets were introduced by Zadeh [8] in 1965 as an extension of the classical notion of sets. Fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the unit interval $[0, 1]$. Next, fuzzy sets were generalized to other concepts. Atanassov [1] generalized fuzzy sets into intuitionistic fuzzy sets in 1986 by considering for each element of the sets is a degree of membership and a degree of non-membership. The notion of the classical sets, fuzzy sets, and intuitionistic fuzzy sets were extended into neutrosophic sets which is the tool for dealing with incomplete, inconsistent, and indeterminate information by Smaradache [6]. In 2009, Jun et al. [4] gave the concept of a negative-valued function and constructed $\mathcal{N}$-structures. Later, Smaradache et al. [7] introduced the notion of neutrosophic $\mathcal{N}$-structures and applied it to semigroups in 2017. Next, Elavarasan et al. [3] introduced neutrosophic $\mathcal{N}$-ideals in semigroups and investigated its several properties. In 2014, the concept of picture fuzzy set was first introduced by Cuong [2] in 2014, which is a generalization of the concept of fuzzy sets and intuitionistic fuzzy sets. This concept is based on adequate in situations when we face human

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opinions involving more answers of types: yes, abstain, no, refusal. Picture fuzzy sets were focused on the degree of positive memberships, the degree of neutral memberships, and the degree of negative memberships. Picture fuzzy sets are generalizations of fuzzy sets and intuitionistic fuzzy sets. The concept of picture fuzzy sets will differ from the concept of neutrosophic sets. The applications of picture fuzzy sets in semigroups were studied by Yiarayong [5] in 2020. In this paper, we study picture $\mathcal{N}$-structures and apply it to semigroup. Moreover, we define picture $\mathcal{N}$-ideals in semigroups and investigate several properties of these ideals in semigroups.

2. Notations

In this section, we introduce the concept of picture $\mathcal{N}$-structures of sets.

Definition 2.1. A *picture $\mathcal{N}$-structure* over a set S defined to be the structure:

$$P_N := \left\{ \frac{x}{T_N(x), I_N(x), F_N(x)} \mid x \in S \right\}$$

where $T_N : S \to [-1, 0]$ is called the *negative positive membership function,*

$I_N : S \to [-1, 0]$ is called the *negative neutral membership function,* and

$F_N : S \to [-1, 0]$ is called the *negative false membership function*

with the condition $-1 \leq T_N(x) + I_N(x) + F_N(x) \leq 0$ for all $x \in S$. We denote this structure by $P_N = \frac{S}{(T_N, I_N, F_N)}$.

Definition 2.2. Let $P_N = \frac{S}{(T_N, I_N, F_N)}$ and $P_M = \frac{S}{(T_M, I_M, F_M)}$ be picture $\mathcal{N}$-structures over a set S. Then

(1) P_N is called a *picture $\mathcal{N}$-substructure* of P_M over S, denote by $P_N \subseteq P_M$, if

(a) $T_N(x) \geq T_M(x)$,

(b) $I_N(x) \leq I_M(x)$,

(c) $F_N(x) \leq F_M(x)$

for all $x \in S$.

(2) The *union* of P_N and P_M is defined to be a picture $\mathcal{N}$-structure over S

$$P_{N\cup M} = (S; T_{N\cup M}, I_{N\cup M}, F_{N\cup M})$$

where $T_{N\cup M}(x) = \min\{T_N(x), T_M(x)\}$,

$I_{N\cup M}(x) = \max\{I_N(x), I_M(x)\}$,

$F_{N\cup M}(x) = \max\{F_N(x), F_M(x)\}$

for all $x \in S$.

(3) The *intersection* of P_N and P_M is defined to be a picture $\mathcal{N}$-structure over S

$$P_{N\cap M} = (S; T_{N\cap M}, I_{N\cap M}, F_{N\cap M})$$

where $T_{N\cap M}(x) = \max\{T_N(x), T_M(x)\}$,

$I_{N\cap M}(x) = \min\{I_N(x), I_M(x)\}$,

$F_{N\cap M}(x) = \min\{F_N(x), F_M(x)\}$

for all $x \in S$.

Definition 2.3. For a subset A of a set S, consider the picture $\mathcal{N}$-structure

$$\chi_A(P_N) = \frac{S}{(\chi_A(T)_N, \chi_A(I)_N, \chi_A(F)_N)}$$

where $\chi_A(T)_N(x) = \begin{cases} -1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$

$\chi_A(I)_N(x) = \begin{cases} 0 & \text{if } x \in A, \\ -1 & \text{otherwise,} \end{cases}$

$\chi_A(F)_N(x) = \begin{cases} 0 & \text{if } x \in A, \\ -1 & \text{otherwise} \end{cases}$

for all $x \in S$. The picture $\mathcal{N}$-structure $\chi_A(P_N)$ is called the *characteristic picture $\mathcal{N}$-structure* of A over S.

Definition 2.4. Let P_N be a picture $\mathcal{N}$-structure over S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-1 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$T_N^\alpha = \{x \in S \mid T_N(x) \leq \alpha\},$$
$$I_N^\beta = \{x \in S \mid I_N(x) \geq \beta\},$$
$$F_N^\gamma = \{x \in S \mid F_N(x) \geq \gamma\}.$$

The set $P_N(\alpha, \beta, \gamma) := \{x \in S \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \geq \gamma\}$ is called an (α, β, γ)-*level set* of P_N.

Note that $P_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.

3. Applications of picture $\mathcal{N}$-sets in semigroups

The picture $\mathcal{N}$-product of P_N and P_M is defined to be a picture $\mathcal{N}$-structure over a semigroup S

$$P_N \odot P_M := \left\{ \frac{x}{T_{N\circ M}(x), I_{N\circ M}(x), F_{N\circ M}(x)} \mid x \in S \right\}$$

where $T_{N\circ M}(x) = \begin{cases} \inf_{x=ab} \max\{T_N(a), T_M(b)\} & \text{for } x = ab \text{ for some } a, b \in S, \\ 0 & \text{otherwise,} \end{cases}$

$I_{N\circ M}(x) = \begin{cases} \sup_{x=ab} \min\{I_N(a), I_M(b)\} & \text{for } x = ab \text{ for some } a, b \in S, \\ 0 & \text{otherwise,} \end{cases}$

$$F_{N\circ M}(x) = \begin{cases} \sup_{x=ab} \min\{F_N(a), F_M(b)\} & \text{for } x = ab \text{ for some } a, b \in S, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the picture $\mathcal{N}$-product of P_N and P_M by $P_N \odot P_M = \frac{S}{(T_{N\circ M}, I_{N\circ M}, F_{N\circ M})}$.

For $x \in S$, the element $\frac{x}{(T_{N\circ M}, I_{N\circ M}, F_{N\circ M})}$ is simply denoted by

$$(P_N \odot P_M)(x) = (T_{N\circ M}(x), I_{N\circ M}(x), F_{N\circ M}(x))$$

for the sake of convenience.

Definition 3.1. A picture $\mathcal{N}$-structure P_N over a semigroup S is called a *picture $\mathcal{N}$-subsemigroup* of S if it satisfies:

(1) $T_N(xy) \leq \max\{T_N(x), T_N(y)\}$,
(2) $I_N(xy) \geq \min\{I_N(x), I_N(y)\}$,
(3) $F_N(xy) \geq \min\{F_N(x), F_N(y)\}$

for all $x, y \in S$.

Definition 3.2. A picture $\mathcal{N}$-structure P_N over a semigroup S is called a *picture $\mathcal{N}$-left ideal* of S if it satisfies:

(1) $T_N(xy) \leq T_N(y)$,
(2) $I_N(xy) \geq I_N(y)$,
(3) $F_N(xy) \geq F_N(y)$

for all $x, y \in S$.

Definition 3.3. A picture $\mathcal{N}$-structure P_N over a semigroup S is called a *picture $\mathcal{N}$-right ideal* of S if it satisfies:

(1) $T_N(xy) \leq T_N(x)$,
(2) $I_N(xy) \geq I_N(x)$,
(3) $F_N(xy) \geq F_N(x)$

for all $x, y \in S$.

We called P_N a *picture $\mathcal{N}$-ideal* if it is both a picture $\mathcal{N}$-left ideal and a picture $\mathcal{N}$-right ideal of S.

Theorem 3.1. *Let P_N be a picture $\mathcal{N}$-structure over a semigroup S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-1 \leq \alpha + \beta + \gamma \leq 0$. If P_N is a picture $\mathcal{N}$-left ideal of S, then (α, β, γ)-level set of P_N is a picture $\mathcal{N}$-left ideal of S whenever it is nonempty.*

Proof. Assume that $P_N(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-1 \leq \alpha + \beta + \gamma \leq 0$. Let P_N be a picture $\mathcal{N}$-left ideal of S, and let $x, y \in P_N(\alpha, \beta, \gamma)$. Then $T_N(xy) \leq T_N(y) \leq \alpha$, $I_N(xy) \geq I_N(y) \geq \beta$, and $F_N(xy) \geq F_N(y) \geq \gamma$ which imply $xy \in P_N(\alpha, \beta, \gamma)$. Hence $P_N(\alpha, \beta, \gamma)$ is a picture $\mathcal{N}$-left ideal of S. $\square$

Theorem 3.2. *Let P_N be a picture $\mathcal{N}$-structure over a semigroup S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-1 \leq \alpha + \beta + \gamma \leq 0$. If P_N is a picture $\mathcal{N}$-right ideal of S, then (α, β, γ)-level set of P_N is a picture $\mathcal{N}$-right ideal of S whenever it is nonempty.*

Proof. It is similar to Theorem 3.1. □

Theorem 3.3. *Let P_N be a picture $\mathcal{N}$-structure over a semigroup S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-1 \leq \alpha + \beta + \gamma \leq 0$. If P_N is a picture $\mathcal{N}$-ideal of S, then (α, β, γ)-level set of P_N is a picture $\mathcal{N}$-ideal of S whenever it is nonempty.*

Proof. It follows from Theorem 3.1 and 3.2. □

Theorem 3.4. *Let P_N be a picture $\mathcal{N}$-structure over a semigroup S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-1 \leq \alpha + \beta + \gamma \leq 0$. If T_N^{α}, I_N^{β}, and F_N^{γ} are left ideals of S, then P_N is a picture $\mathcal{N}$-left ideal of S whenever it is nonempty.*

Proof. Let $a, b \in S$ such that $T_N(ab) > T_N(b)$. Then $T_N(ab) > t_\alpha \geq T_N(b)$ for some $t_\alpha \in [-1, 0)$. Thus $b \in T_N^{t_\alpha}(b)$ but $ab \notin T_N^{t_\alpha}(b)$, this is a contradiction. So $T_N(ab) \leq T_N(b)$. Similarly way we can get $I_N(ab) \geq I_N(b)$ and $F_N(ab) \geq F_N(b)$. Therefore P_N is a picture $\mathcal{N}$-left ideal of S. □

Theorem 3.5. *Let P_N be a picture $\mathcal{N}$-structure over a semigroup S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-1 \leq \alpha + \beta + \gamma \leq 0$. If T_N^{α}, I_N^{β}, and F_N^{γ} are right ideals of S, then P_N is a picture $\mathcal{N}$-right ideal of S whenever it is nonempty.*

Proof. It is similar to Theorem 3.4. □

Theorem 3.6. *Let P_N be a picture $\mathcal{N}$-structure over a semigroup S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-1 \leq \alpha + \beta + \gamma \leq 0$. If T_N^{α}, I_N^{β}, and F_N^{γ} are ideals of S, then P_N is a picture $\mathcal{N}$-ideal of S whenever it is nonempty.*

Proof. It follows from Theorem 3.4 and 3.5. □

Theorem 3.7. *Let S be a semigroup. Then intersection of two picture $\mathcal{N}$-left ideals of S is also a picture $\mathcal{N}$-left ideal of S.*

Proof. Let $P_N := \frac{S}{(T_N, I_N, F_N)}$ and $P_M := \frac{S}{(T_M, I_M, F_M)}$ be picture $\mathcal{N}$-left ideals of S. Then

$$T_{N\cap M}(xy) = \max\{T_N(xy), T_M(xy)\} \leq \max\{T_N(y), T_M(y)\} = T_{N\cap M}(y),$$

$$I_{N\cap M}(xy) = \min\{I_N(xy), I_M(xy)\} \geq \min\{I_N(y), I_M(y)\} = I_{N\cap M}(y),$$

$$F_{N\cap M}(xy) = \min\{F_N(xy), F_M(xy)\} \geq \min\{F_N(y), F_M(y)\} = F_{N\cap M}(y)$$

for all $x, y \in S$. Then $P_{N\cap M}$ is a picture $\mathcal{N}$-left ideal of S. □

Theorem 3.8. *Let S be a semigroup. Then intersection of two picture $\mathcal{N}$-right ideals of S is also a picture $\mathcal{N}$-right ideal of S.*

Proof. It is similar to Theorem 3.7. □

Theorem 3.9. *Let S be a semigroup. Then intersection of two picture $\mathcal{N}$-ideals of S is also a picture $\mathcal{N}$-ideal of S.*

Proof. It follows from Theorem 3.7 and 3.8. □

Theorem 3.10. *For any nonempty subset A of a semigroup S, the following conditions are equivalent:*

(1) *A is a left ideal of S.*

(2) *The characteristic picture $\mathcal{N}$-structure $\chi_A(P_N)$ over S is a picture $\mathcal{N}$-left ideal of S.*

Proof. Assume that A is a left ideal of S. Let $x, y \in S$.

If $y \notin A$, then

$$\chi_A(T)_N(xy) \leq 0 = \chi_A(T)_N(y),$$

$$\chi_A(I)_N(xy) \geq -1 = \chi_A(I)_N(y),$$

$$\chi_A(F)_N(xy) \geq -1 = \chi_A(F)_N(y).$$

Otherwise $y \in A$. Then $xy \in A$, we have

$$\chi_A(T)_N(xy) = -1 = \chi_A(T)_N(y),$$

$$\chi_A(I)_N(xy) = 0 = \chi_A(I)_N(y),$$

$$\chi_A(F)_N(xy) = 0 = \chi_A(F)_N(y).$$

Therefore $\chi_A(P_N)$ is a picture $\mathcal{N}$-left ideal of S.

Conversely, suppose that $\chi_A(P_N)$ is a picture $\mathcal{N}$-left ideal of S. Let $y \in A$ and $x \in S$. Then

$$\chi_A(T)_N(xy) \leq \chi_A(T)_N(y) = -1,$$

$$\chi_A(I)_N(xy) \geq \chi_A(I)_N(y) = 0,$$

$$\chi_A(F)_N(xy) \geq \chi_A(F)_N(y) = 0.$$

Hence $xy \in A$. Therefore A is a left ideal of S. □

Theorem 3.11. *For any nonempty subset A of a semigroup S, the following conditions are equivalent:*

(1) *A is a right ideal of S.*

(2) *The characteristic picture $\mathcal{N}$-structure $\chi_A(P_N)$ over S is a picture $\mathcal{N}$-right ideal of S.*

Proof. It is similar to Theorem 3.10. □

Theorem 3.12. *For any nonempty subset A of a semigroup S, the following conditions are equivalent:*

(1) *A is an ideal of S.*
(2) *The characteristic picture $\mathcal{N}$-structure $\chi_A(P_N)$ over S is a picture $\mathcal{N}$-ideal of S.*

Proof. It follows from Theorem 3.11 and 3.12. □

Theorem 3.13. *Let $\chi_A(P_N)$ and $\chi_B(P_N)$ be characteristic picture $\mathcal{N}$-structures over a semigroup S for subsets A and B of S. Then*

(1) $\chi_A(P_N) \cap \chi_B(P_N) = \chi_{A\cap B}(P_N)$.
(2) $\chi_A(P_N) \odot \chi_B(P_N) = \chi_{AB}(P_N)$.

Proof. (1) Let $s \in S$. If $s \in A \cap B$, then $s \in A$ and $s \in B$. Thus

$$(\chi_A(T)_N \cap \chi_B(T)_N)(s) = \max\{\chi_A(T)_N(s), \chi_B(T)_N(s)\} = -1 = \chi_{A\cap B}(T)_N(s),$$
$$(\chi_A(I)_N \cap \chi_B(I)_N)(s) = \min\{\chi_A(I)_N(s), \chi_B(I)_N(s)\} = 0 = \chi_{A\cap B}(I)_N(s),$$
$$(\chi_A(F)_N \cap \chi_B(F)_N)(s) = \min\{\chi_A(F)_N(s), \chi_B(F)_N(s)\} = 0 = \chi_{A\cap B}(F)_N(s).$$

Hence $\chi_A(P_N) \cap \chi_B(P_N) = \chi_{A\cap B}(P_N)$.
If $s \notin A \cap B$, then $s \notin A$ or $s \notin B$. Thus

$$(\chi_A(T)_N \cap \chi_B(T)_N)(s) = \max\{\chi_A(T)_N(s), \chi_B(T)_N(s)\} = 0 = \chi_{A\cap B}(T)_N(s),$$
$$(\chi_A(I)_N \cap \chi_B(I)_N)(s) = \min\{\chi_A(I)_N(s), \chi_B(I)_N(s)\} = -1 = \chi_{A\cap B}(I)_N(s),$$
$$(\chi_A(F)_N \cap \chi_B(F)_N)(s) = \min\{\chi_A(F)_N(s), \chi_B(F)_N(s)\} = -1 = \chi_{A\cap B}(F)_N(s).$$

Hence $\chi_A(P_N) \cap \chi_B(P_N) = \chi_{A\cap B}(P_N)$.

(2) Let $x \in S$. If $x \notin AB$, then

$$(\chi_A(T)_N \circ \chi_B(T)_N)(x) = 0 = \chi_{AB}(T)_N(x),$$
$$(\chi_A(I)_N \circ \chi_B(I)_N)(x) = 0 = \chi_{AB}(I)_N(x),$$
$$(\chi_A(F)_N \circ \chi_B(F)_N)(x) = 0 = \chi_{AB}(F)_N(x).$$

If $x \in AB$, then $x = ab$ for some $a \in A$ and $b \in B$. We have

$$\begin{aligned}(\chi_A(T)_N \circ \chi_B(T)_N)(x) &= \inf_{x=ab} \max\{\chi_A(T)_N(a), \chi_B(T)_N(b)\}\\ &\leq \max\{\chi_A(T)_N(a), \chi_B(T)_N(b)\}\\ &= -1\\ &= \chi_{AB}(T)_N(x),\end{aligned}$$

$$
\begin{aligned}
(\chi_A(I)_N \circ \chi_B(I)_N)(x) &= \sup_{x=ab} \min\{\chi_A(I)_N(a), \chi_B(I)_N(b)\} \\
&\geq \min\{\chi_A(I)_N(a), \chi_B(I)_N(b)\} \\
&= 0 \\
&= \chi_{AB}(I)_N(x), \\
(\chi_A(F)_N \circ \chi_B(F)_N)(x) &= \sup_{x=ab} \min\{\chi_A(F)_N(a), \chi_B(F)_N(b)\} \\
&\geq \min\{\chi_A(F)_N(a), \chi_B(F)_N(b)\} \\
&= 0 \\
&= \chi_{AB}(F)_N(x).
\end{aligned}
$$

Therefore $\chi_A(P_N) \odot \chi_B(P_N) = \chi_{AB}(P_N)$.

□

Theorem 3.14. *Let P_M be a picture $\mathcal{N}$-structure over a semigroup S. Then P_M is a picture $\mathcal{N}$-left ideal of S if and only if $P_N \odot P_M \subseteq P_M$ for any picture $\mathcal{N}$-structure P_N over S.*

Proof. Assume that P_M is a picture $\mathcal{N}$-left ideal of S and let $s, t, u \in S$. If $s = tu$, then we have

(i) $T_M(s) = T_M(tu) \leq T_M(u) \leq \max\{T_M(t), T_M(u)\}$ which implies $T_M(s) \leq T_{N\circ M}(s)$. Otherwise $s \neq tu$. Then $T_M(s) \leq 0 = T_{N\circ M}(s)$.

(ii) $I_M(s) = I_M(tu) \geq I_M(u) \geq \min\{I_M(t), I_M(u)\}$ which implies $I_M(s) \geq I_{N\circ M}(s)$. Otherwise $s \neq tu$. Then $I_M(s) \geq -1 = I_{N\circ M}(s)$.

(iii) $F_M(s) = F_M(tu) \geq F_M(u) \geq \min\{F_M(t), F_M(u)\}$ which implies $F_M(s) \geq F_{N\circ M}(s)$. Otherwise $s \neq tu$. Then $F_M(s) \geq -1 = F_{N\circ M}(s)$.

Conversely, assume that P_M is a picture $\mathcal{N}$-structure over S such that $P_N \odot P_M \subseteq P_M$ for any picture $\mathcal{N}$-structure P_N over S. Let $x, y \in S$. If $a = xy$, then

$$
\begin{aligned}
T_M(xy) &= T_M(a) \\
&\leq (\chi_X(T)_N \circ T_M)(a) \\
&= \inf_{a=st} \max\{\chi_X(T)_N(s), T_M(t)\} \\
&\leq \max\{\chi_X(T)_N(x), T_M(y)\} \\
&= T_M(y), \\
I_M(xy) &= I_M(a) \\
&\geq (\chi_X(I)_N \circ I_M)(a) \\
&= \sup_{a=st} \min\{\chi_X(I)_N(s), I_M(t)\}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \min\{\chi_X(I)_N(x), I_M(y)\}\\
&= I_M(y),\\
F_M(xy) &= F_M(a)\\
&\geq (\chi_X(F)_N \circ F_M)(a)\\
&= \sup_{a=st} \min\{\chi_X(F)_N(s), F_M(t)\}\\
&\geq \max\{\chi_X(F)_N(x), F_M(y)\}\\
&= F_M(y).
\end{aligned}
$$

Therefore P_M is a picture $\mathcal{N}$-left ideal of S. $\square$

Theorem 3.15. *Let P_M be a picture $\mathcal{N}$-structure over a semigroup S. Then P_M is a picture $\mathcal{N}$-right ideal of S if and only if $P_M \odot P_N \subseteq P_M$ for any picture $\mathcal{N}$-structure P_N over S.*

Proof. It is similar to Theorem 3.14. $\square$

Theorem 3.16. *Let P_M be a picture $\mathcal{N}$-structure over S. Then P_M is a picture $\mathcal{N}$-ideal of S if and only if $P_M \odot P_N \subseteq P_M$ for any picture $\mathcal{N}$-structure P_N over S.*

Proof. It follows from Theorem 3.14 and 3.15. $\square$

Theorem 3.17. *Let P_M and P_N be picture $\mathcal{N}$-structure over S. If P_M is a picture $\mathcal{N}$-left ideal of S, then so is $P_M \odot P_N$.*

Proof. Assume that P_M is a picture $\mathcal{N}$-left ideal of S, and let $x, y \in S$. If there exist $a, b \in S$ such that $y = ab$, then $xy = x(ab) = (xa)b$. We have

$$
\begin{aligned}
(T_N \circ T_M)(y) &= \inf_{y=ab} \max\{T_N(a), T_M(b)\}\\
&\leq \inf_{xy=(xa)b} \max\{T_N(xa), T_M(b)\}\\
&= \inf_{xy=cb} \max\{T_N(c), T_M(b)\}\\
&= (T_N \circ T_M)(xy),\\
(I_N \circ I_M)(y) &= \sup_{y=ab} \min\{I_N(a), I_M(b)\}\\
&\geq \sup_{xy=(xa)b} \min\{I_N(xa), I_M(b)\}\\
&= \sup_{xy=cb} \min\{I_N(c), I_M(b)\}\\
&= (I_N \circ I_M)(xy),
\end{aligned}
$$

$$\begin{aligned}(F_N \circ F_M)(y) &= \sup_{y=ab} \min\{F_N(a), F_M(b)\}\\ &\geq \sup_{xy=(xa)b} \min\{F_N(xa), F_M(b)\}\\ &= \sup_{xy=cb} \min\{F_N(c), F_M(b)\}\\ &= (F_N \circ F_M)(xy).\end{aligned}$$

Therefore $P_M \odot P_N$ is a picture $\mathcal{N}$-left ideal of S. □

Theorem 3.18. *Let P_M and P_N be picture $\mathcal{N}$-structure over a semigroup S. If P_M is a picture $\mathcal{N}$-right ideal of S, then so is $P_M \odot P_N$.*

Proof. It is similar to Theorem 3.17. □

Theorem 3.19. *Let P_M and P_N be picture $\mathcal{N}$-structure over a semigroup S. If P_M is a picture $\mathcal{N}$-ideal of S, then so is $P_M \odot P_N$.*

Proof. It follows from Theorem 3.17 and 3.18. □

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