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EXISTENCE AND CONVERGENCE OF BEST PROXIMITY POINTS FOR SEMI CYCLIC CONTRACTION PAIRS

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Abstract. In this article, we introduce the notion of a semi cyclic φ-contraction pair of mappings, which contains semi cyclic contraction pairs as a subclass. Existence and convergence results of best proximity points for semi cyclic φ-contraction pair of mappings are obtained.

1. Introduction

As it is well known, fixed point theory is an indispensable tool for solving various equations involving self-mappings, defined on subsets of a metric space or a normed linear space. Nevertheless, when the mapping T (say) is a non-self one, then it is possible that the equation $Tx = x$ has no solution and, in this case, we have to focus the study on the problem of finding an element x which is in the closest proximity to Tx in some sense; in such circumstances, it may be speculated to determine an element x for which the "distance error" $d(x, Tx)$ is minimum.

Let A and B be nonempty subsets of a metric space (X, d) and T a mapping from A to B. Since $d(x, Tx)$ is greater than or equals to the distance between A and B for all x in A, a best proximity theorem offers sufficient conditions for the existence of an element x, called a *best proximity point* of the mapping T, satisfying the condition that $d(x, Tx) = \text{dist}(A, B)$. Also, it is interesting to see that best proximity point theorems emerge as a natural generalization of fixed point theorems, because a best proximity point reduces to a fixed point if the mapping under consideration turns out to be a self-mapping.

Now, let us consider S and T be given non-self mappings from A to B, where A and B are nonempty subsets of a metric space. As S and T are non-self mappings, the equations $Sx = x$ and $Tx = x$ do not necessarily have a common solution, called a *common fixed point* of the mappings S and T. Therefore, in such cases of non-existence of a common fixed points, it is attempted to find a point x that is closest to both Sx and Tx in some sense. Common best proximity theorems, explore the existence of such optimal solutions, known as best proximity point. In view of the fact that, for any element x in A, the distance between x and Sx, and the distance between x and Tx are at least the distance between the sets A and B, a common best proximity theorem states that, under certain conditions, there exists a point x satisfying $d(x, Sx) = d(x, Tx) = \text{dist}(A, B)$.

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In their elegant paper [1], Kirk et al. introduced the notion of cyclical contractive mapping, and proved fixed point results for this class of mappings, while in [2], Eldred and Veeramani studied existence and convergence results of best proximity points in a more general case. In [3], Thagafi and Shahzad introduced a new class of mappings known as cyclic φ-contraction, and proved their convergence and existence results for best proximity points. Recently, Gabeleh and Abkar [4], introduced results on best proximity points for semi-cylic contractive pairs in Banach spaces. For other recent results on this topic, see Chandok and Postolache [5], Shatanawi and Postolache [6].

This paper aims to develop further these results. In this respect, we introduce the new notion of a semi cyclic φ-contraction pair of mappings, which contains semi cyclic contraction pairs as a subclass. Existence and convergence results of best proximity points for semi cyclic φ-contraction pair of mappings, in the framework of a uniformly convex Banach space [7], are obtained. Our results are extensions of several results as in relevant items from the reference section of this paper, as well as in the literature in general. In particular, our results reduce to those of Gabeleh and Abkar [4].

2. Preliminaries

Let A and B be nonempty subsets of a metric space (X, d) and let $T\colon A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The mapping T is said to be *cyclic contraction* [1] if for some $k \in (0, 1)$ we have

$$d(Tx, Ty) \leq k\, d(x, y), \quad x \in A, \quad y \in B.$$

Kirk et al. [1] proved that if $A \cap B \neq \emptyset$ then the cyclic mapping T has a unique fixed point in $A \cap B$. But what happens when $A \cap B$ is not necessarily nonempty. In this situation, a mapping T is said to be cyclic contraction [2] if for some $k \in (0, 1)$ we have

$$d(Tx, Ty) \leq k\, d(x, y) + (1 - k)\mathrm{dist}(A, B),$$

for all $x \in A$ and $y \in B$. Eldred and Veeramani [2] proved existence, uniqueness and convergence for best proximity points of cyclic contraction mapping T. Thagafi and Shahzad [3] introduced the following new class of mapping known as cyclic φ-contraction:

The mapping T is said to be *cyclic φ-contraction* if $\varphi\colon [0, +\infty) \to [0, +\infty)$ is strictly increasing mapping and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(\mathrm{dist}(A, B)),$$

for all $x \in A$ and $y \in B$.

We can see from the following example [3] that a cyclic φ-contraction mapping need not be a cyclic contraction.

Example 2.1. Let $X = \mathbb{R}$ with usual metric. For $A = B = [0, 1]$, define $T\colon A \cup B \to A \cup B$ by the formula $Tx = \frac{x}{1+x}$. If $\varphi(t) = \frac{t^2}{1+t}$ for $t \geq 0$, then T is cyclic φ-contraction mapping which is not a cyclic contraction.

Recently, Gabeleh and Akbar [4] introduced the concept of a semi-cyclic contraction pair:

Definition 2.1. Let S, T be two self mappings on $A \cup B$. The pair (S, T) is called a *semi-cyclic contraction* if the following conditions hold:

(i) $S(A) \subseteq B$, $T(B) \subseteq A$;
(ii) $\exists \alpha \in (0,1)$, such that $d(Sx, Ty) \leq \alpha d(x,y) + (1-\alpha)\text{dist}(A,B)$, $x \in A$, $y \in B$.

Clearly when $S = T$, a semi-cyclic contraction pair reduces to a cyclic contraction mapping, already studied by Eldred and Veeramani [2]. But there exist a semi-cyclic contraction pair which is not cyclic, the following example [4] illustrates it.

Example 2.2. Let $X = \mathbb{R}^2$ and for all $(x,y) \in \mathbb{R}^2$ define $\|(x,y)\| = \max\{|x|,|y|\}$. Let $A = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} \leq x \leq 1,\ y = 0\}$, $B = \{(x,y) \in \mathbb{R}^2 : x = 0,\ 1 \leq y \leq 2\}$. Then A and B are closed and $\text{dist}(A,B) = 1$. Define $S, T\colon A \cup B \to A \cup B$ by

$$S(x,y) = \begin{cases} (0,1),\ (x,y) \in A \\ (x,y),\ (x,y) \in B, \end{cases} \qquad T(x,y) = \begin{cases} (x,y),\ (x,y) \in A \\ (y/2,0),\ (x,y) \in B. \end{cases}$$

Here $S(A) \subseteq B$ and $T(B) \subseteq A$ but $S(B) \nsubseteq A$ and $T(A) \nsubseteq B$, hence neither S nor T is cyclic.

On the other hand, if $b = (0,y) \in B$ and $a = (x,0) \in A$, then

$$\|Tb - Sa\| = \|T(0,y) - S(x,0)\| = \|(y/2,-1)\| = 1.$$

Similarly $\|a - b\| = \|(x,-y)\| = \max\{x, |-y|\} = y$. Therefore

$$\|Tb - Sa\| = 1 \leq \frac{1}{2}|y| + \frac{1}{2} < \frac{1}{2}\|b - a\| + \frac{1}{2}\text{dist}(A,B).$$

Hence (S,T) is a semi-cyclic contraction pair.

We now introduce the following new class of semi-cyclic contraction pair.

Definition 2.2. Let A and B be nonempty subsets of a metric space (X,d) and let $S, T\colon A \cup B \to A \cup B$ such that $S(A) \subseteq B$ and $T(B) \subseteq A$. Then (S,T) is said to be a *semi-cyclic φ-contraction* if $\varphi\colon [0,+\infty) \to [0,+\infty)$ is strictly increasing mapping and

$$d(Sx, Ty) \leq d(x,y) - \varphi(d(x,y)) + \varphi(\text{dist}(A,B)), \tag{1}$$

for all $x \in A$ and $y \in B$.

A semi-cyclic contraction pair is semi-cyclic φ-contraction pair with $\varphi(t) = (1-\alpha)t$ for $t \geq 0$ and $0 < \alpha < 1$.

We now give an example to illustrate that a semi-cyclic φ-contraction pair need not necessary a semi-cyclic contraction pair

Example 2.3. Let $X = \mathbb{R}$ with the usual metric. Let $A = [1,2]$, $B = [-2,-1]$, then $\text{dist}(A,B) = 2$. Define $T, S\colon A \cup B \to A \cup B$ by

$$S(x) = \begin{cases} \dfrac{-1-x}{2}, & x \in A \\ \dfrac{-1+x}{2}, & x \in B, \end{cases} \qquad T(x) = \begin{cases} \dfrac{1+x}{2}, & x \in A \\ \dfrac{1-x}{2}, & x \in B. \end{cases}$$

Clearly $S(A) \subseteq B$ and $T(B) \subseteq A$. Take $a \in A$, $b \in B$ and $\varphi(t) = \frac{t^2}{1+8t}$ for $t \geq 0$, then (S,T) is a semi-cyclic φ-contraction pair. On the other hand, if $a = 2 \in A$, $b = -2 \in B$ and $\alpha \in (0, \frac{1}{2})$, then

$$d(Tb, Sa) = 3 > \alpha \cdot 4 + (1-\alpha) \cdot 2 = \alpha\, \text{dist}(A,B) + (1-\alpha)\text{dist}(A,B).$$

Hence (S,T) is not a semi-cyclic contraction pair.

3. Main results

Consider $x_0 \in A$, then $Sx_0 \in B$, so there exists $y_0 \in B$ such that $y_0 = Sx_0$. Now $Ty_0 \in A$, so there exists $x_1 \in A$ such that $x_1 = Ty_0$. Inductively, we define sequences $\{x_n\}$ and $\{y_n\}$ in A and B, respectively by

$$x_{n+1} = Ty_n, \quad y_n \ = Sx_n. \tag{2}$$

For all $x \in A$ and $y \in B$, we have $\text{dist}(A, B) \leq d(x, y)$. Since φ is a strictly increasing function, we deduce that $\varphi(\text{dist}(A, B)) \leq \varphi(d(x, y))$. Also (S, T) is semi cyclic φ-contraction pair, hence

$$d(Sx, Ty) \leq d(x, y) - \varphi\left(d(x, y)\right) + \varphi\left(\text{dist}(A, B)\right) \leq d(x, y).$$

By (2), we have

$$d(x_n, Sx_n) = d(Ty_{n-1}, Sx_n) \leq d(y_{n-1}, x_n) = d(x_{n-1}, Sx_{n-1}),$$

and

$$d(x_{n+1}, y_n) = d(Ty_n, Sx_n) \leq d(y_n, x_n) = d(y_n, Ty_{n-1}).$$

Also,

$$d(y_{n+1}, Ty_n) = d(Sx_{n+1}, Ty_n) \leq d(x_{n+1}, y_n) = d(Ty_n, Sx_n) \leq d(y_n, x_n) = d(y_n, Ty_{n-1}).$$

We summarize these results in:

Lemma 3.1. *Let (X, d) be a metric space and let A, B be nonempty subsets of X. Let $S, T \colon A \cup B \to A \cup B$ such that the pair (S, T) is semi-cyclic $\varphi-$contraction. For $x_0 \in A \cup B$ the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (2). Then for all $x \in A$, $y \in B$, and $n \geq 1$, we have*

(i) $-\varphi\left(d(x, y)\right) + \varphi\left(\text{dist}(A, B)\right) \leq 0,$
(ii) $d(Sx, Ty) \leq d(x, y),$
(iii) $d\left(x_n, Sx_n\right) \leq d\left(x_{n-1}, Sx_{n-1}\right),$
(iv) $d\left(x_{n+1}, y_n\right) \leq d\left(y_n, Ty_{n-1}\right),$
(v) $d\left(y_{n+1}, Ty_n\right) \leq d\left(y_n, Ty_{n-1}\right).$

We now state and prove the following result which will be needed in what follows.

Theorem 3.1. *Let (X, d) be a metric space and let A, B be nonempty subsets of X. Let $S, T \colon A \cup B \to A \cup B$ such that the pair (S, T) is semi-cyclic φ-contraction. For $x_0 \in A \cup B$ the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (2). Then*

$$d(x_n, Sx_n) \to \text{dist}(A, B) \quad \textit{and} \quad d(y_n, Ty_{n-1}) \to \text{dist}(A, B).$$

Proof. Let $d_n = d(x_n, Sx_n)$. It follows from Lemma 3.1(iii), that $\{d_n\}$ is decreasing and bounded, so $\lim_{n\to\infty} d_n = t_0$, for some $t_0 \geq \text{dist}(A, B)$.

If $t_0 = \text{dist}(A, B)$ there is nothing to prove, so assume $t_0 > \text{dist}(A, B)$. Since

$$d(Sx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(\text{dist}(A, B)), \text{ for all } x \in A, \quad y \in B,$$

we have

$$\begin{aligned} d_{n+1} &= d\left(x_{n+1}, Sx_{n+1}\right) \\ &= d\left(Ty_n, Sx_{n+1}\right) \\ &\leq d\left(y_n, x_{n+1}\right) \\ &= d\left(Sx_n, Ty_n\right) \\ &\leq d\left(x_n, y_n\right) - \varphi\left(d\left(x_n, y_n\right)\right) + \varphi\left(\operatorname{dist}(A,B)\right) \\ &= d\left(x_n, Sx_n\right) - \varphi\left(d\left(x_n, Sx_n\right)\right) + \varphi\left(\operatorname{dist}(A,B)\right) \\ &= d_n - \varphi\left(d_n\right) + \varphi\left(\operatorname{dist}(A,B)\right). \end{aligned}$$

Hence,

$$\begin{aligned} \varphi\left(\operatorname{dist}(A,B)\right) &\leq \varphi(d_n) \\ &= d_n - d_{n+1} + \varphi\left(\operatorname{dist}(A,B)\right). \end{aligned}$$

Thus

$$\varphi\left(\operatorname{dist}(A,B)\right) \leq \lim_{n\to\infty} \varphi\left(d_n\right) \leq \varphi\left(\operatorname{dist}(A,B)\right),$$

which shows that

$$\lim_{n\to\infty} \varphi\left(d_n\right) = \varphi(t_0) = \varphi\left(\operatorname{dist}(A,B)\right). \tag{3}$$

On the other hand, since φ is strictly increasing and $d_n \geq t_0 > \operatorname{dist}(A,B)$, we have

$$\lim_{n\to\infty} \varphi\left(d_n\right) \geq \varphi(t_0) > \varphi\left(\operatorname{dist}(A,B)\right),$$

which contradicts to (3). Consequently, $t_0 = \operatorname{dist}(A,B)$.

Similarly, using Lemma 3.1, it can be shown that $d(y_n, Ty_{n-1}) \to \operatorname{dist}(A,B)$. □

Remark 3.1. Proposition 3.1 of [4] is a special case of Theorem 3.1.

Theorem 3.2. *Let (X,d) be a metric space and let A, B be nonempty subsets of X. Let $S,T\colon A\cup B \to A\cup B$ such that the pair (S,T) is semi-cyclic φ-contraction. For $x_0 \in A\cup B$ the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (2). If both $\{x_n\}$ and $\{y_n\}$ have a convergent subsequence in A and B respectively, then there exists $x \in A$ and $y \in B$ such that*

$$d(x,Sx) = \operatorname{dist}(A,B) = d(y,Ty).$$

Proof. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_k} \to y$. Since

$$\operatorname{dist}(A,B) \leq d\left(Ty_{n_k}, y\right) \leq d\left(y_{n_k}, y\right) + d\left(y_{n_k}, Ty_{n_k}\right),$$

letting $k \to \infty$, by Theorem 3.1, we have

$$d\left(y, Ty_{n_k}\right) \to \operatorname{dist}(A,B).$$

Now, for each $k \geq 1$

$$\begin{aligned} \operatorname{dist}(A,B) &\leq d\left(Ty, y_{n_k}\right) \\ &= d\left(Ty, Sx_{n_k}\right) \\ &\leq d\left(y, x_{n_k}\right) \\ &\leq d\left(y, y_{n_k}\right) + d\left(y_{n_k}, x_{n_k}\right) \\ &= d\left(y, y_{n_k}\right) + d\left(Sx_{n_k}, x_{n_k}\right), \end{aligned}$$

i.e.,

$$\operatorname{dist}(A,B) \leq d\left(y, y_{n_k}\right) + d\left(Sx_{n_k}, x_{n_k}\right),$$

letting $k \to \infty$ we conclude that

$$d(Ty, y) = \operatorname{dist}(A,B).$$

Similarly, it can be proved that $d(x, Sx) = \operatorname{dist}(A,B)$. □

Remark 3.2. Proposition 3.2 of [4] is a special case of Theorem 3.2.

Theorem 3.3. *Let (X,d) be a metric space and let A, B be nonempty subsets of X. Let $S, T\colon A \cup B \to A \cup B$ such that the pair (S,T) is semi-cyclic φ-contraction. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (2) are bounded.*

Proof. By Theorem 3.1 we have that $d\left(x_n, Sx_n\right) \to \operatorname{dist}(A,B)$ as $n \to \infty$, so it is enough to prove that the sequence $\{Sx_n\}$ is bounded. If not, then for each $M > 0$ there exists $n \in \mathbb{N}$ such that

$$d(x_1, Sx_N) > M \quad \text{and} \quad d\left(x_1, Sx_{N-1}\right) \leq M.$$

We obtain

$$M < d\left(x_1, Sx_N\right) = d\left(Ty_0, Sx_N\right),$$

Furthermore, we have

$$\begin{aligned} M &< d\left(x_1, Sx_N\right) \\ &= d\left(Ty_0, Sx_N\right) \\ &\leq d\left(y_0, x_N\right) \\ &= d\left(Sx_0, Ty_{N-1}\right) \\ &\leq d\left(x_0, y_{N-1}\right) - \varphi\left(d\left(x_0, y_{N-1}\right)\right) + \varphi\left(\operatorname{dist}\left(A,B\right)\right) \\ &\leq d\left(x_0, x_1\right) + d\left(x_1, Sx_{N-1}\right) - \varphi\left(d\left(x_0, Sx_{N-1}\right)\right) + \varphi\left(\operatorname{dist}\left(A,B\right)\right), \end{aligned}$$

i.e.

$$M < d\left(x_0, x_1\right) + M - \varphi\left(d\left(x_0, Sx_{N-1}\right)\right) + \varphi\left(\operatorname{dist}\left(A,B\right)\right),$$

so,

$$\varphi\left(d\left(x_0, Sx_{N-1}\right)\right) < d\left(x_0, x_1\right) + \varphi\left(\operatorname{dist}\left(A,B\right)\right).$$

Since, φ is unbounded function, we can choose M such that

$$\varphi(M) > d(x_0, x_1) + \varphi\left(\operatorname{dist}(A,B)\right).$$

Now,

$$M < d\left(x_1, Sx_N\right) \leq d\left(y_0, x_N\right) = d\left(Sx_0, Ty_{N-1}\right) \leq d\left(x_0, y_{N-1}\right) = d\left(x_0, Sx_{N-1}\right).$$

We deduce that

$$\varphi(M) < \varphi\left(d\left(x_1, Sx_N\right)\right) \leq \varphi\left(d\left(x_0, Sx_{N-1}\right)\right) < d\left(x_0, x_1\right) + \varphi\left(\operatorname{dist}\left(A,B\right)\right),$$

a contradiction.

Similarly, we can prove boundedness of $\{y_n\}$ in B. □

Remark 3.3. Proposition 3.3 of [4] is a special case of Theorem 3.3.

In 1936, Clarkson [7] introduced the notion of uniform convexity of norm in a Banach space.

Definition 3.1. A Banach space X is said to be *uniformly convex* if and only if given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{l}\|x\| \leq 1\\ \|y\| \leq 1\\ \|x-y\| \geq \varepsilon\end{array}\right\} \Rightarrow \left\|\frac{x+y}{2}\right\| \leq 1-\delta(\varepsilon), \tag{4}$$

where $\delta\colon [0,2] \to [0,1]$ given by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \,:\, \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

The function δ is known as modulus of convexity of a Banach space X.

The implication (4) has following more general form. For $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$

$$\left.\begin{array}{l}\|x-p\| \leq R\\ \|y-p\| \leq R\\ \|x-y\| \geq r\end{array}\right\} \Rightarrow \left\|p - \frac{x+y}{2}\right\| \leq \left(1-\delta\left(\frac{r}{R}\right)\right) R\,.$$

Now, define a sequence $\{z_n\}$ in $A \cup B$ in the following manner:

$$z_n = \begin{cases} Ty_k\,, & n = 2k\\ Sx_k\,, & n = 2k-1. \end{cases} \tag{5}$$

Lemma 3.2. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X and let $S, T\colon A\cup B \to A\cup B$ are semi-cyclic φ-contraction mappings such that $T(A) \subseteq B$ and $S(B) \subseteq A$. For $x_0 \in A \cup B$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by* (2)*. The sequences $\{z_n\}$ is generated by* (5)*, then $\|z_{2n+2} - z_{2n}\| \to 0$ and $\|z_{2n+3} - z_{2n+1}\| \to 0$ as $n \to \infty$.*

Proof. To show $\|z_{2n+2} - z_{2n}\| \to 0$ as $n \to \infty$, assume the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exists $n_k \geq k$ so that

$$\|z_{2n_k+2} - z_{2n_k}\| \geq \varepsilon_0\,. \tag{6}$$

Choose $\varepsilon > 0$ so that $\left(1 - \delta\left(\frac{\varepsilon_0}{\operatorname{dist}(A,B)+\varepsilon}\right)\right)(\operatorname{dist}(A,B)+\varepsilon) < \operatorname{dist}(A,B)$.
By Theorem 3.1, we have

$$\begin{aligned}\|z_{2n_k+2} - z_{2n_k+1}\| &= \|Ty_{n_k+1} - Sx_{n_k+1}\|\\ &\leq \|y_{n_k+1} - x_{n_k+1}\|\\ &= \|Sx_{n_k+1} - x_{n_k+1}\| \to \operatorname{dist}(A,B),\end{aligned}$$

hence, there exists N_1 such that

$$\|z_{2n_k+2} - z_{2n_k+1}\| \leq \operatorname{dist}(A,B) + \varepsilon\,, \quad \forall\, n_k \geq N_1\,. \tag{7}$$

Also,

$$\begin{aligned}\|z_{2n_k} - z_{2n_k+1}\| &= \|Ty_{n_k} - Sx_{n_k+1}\|\\ &\leq \|y_{n_k} - x_{n_k+1}\|\\ &= \|y_{n_k} - Ty_{n_k}\|\\ &= \|Sx_{n_k} - Ty_{n_k}\|\\ &\leq \|x_{n_k} - y_{n_k}\|\\ &= \|Ty_{n_k-1} - y_{n_k}\| \to \operatorname{dist}(A,B),\end{aligned}$$

so, there exists N_2 such that

$$\|z_{2n_k} - z_{2n_k+1}\| \le \operatorname{dist}(A,B) + \varepsilon\,, \;\; \forall\, n_k \ge N_2\,. \tag{8}$$

Let $N = \max\{N_1, N_2\}$. It follows from the uniform convexity of X and (6)-(8) that

$$\left\| \frac{z_{2n_k+2} + z_{2n_k}}{2} - z_{2n_k+1} \right\| \le \left(1 - \delta\left(\frac{\varepsilon_0}{\operatorname{dist}(A,B)+\varepsilon}\right)\right)(\operatorname{dist}(A,B)+\varepsilon)\,, \;\; \forall\, n_k \ge N.$$

Since A is convex $\frac{z_{2n_k+2}+z_{2n_k}}{2} \in A$, the choice of ε and the fact that δ is strictly increasing imply that

$$\left\| \frac{z_{2n_k+2} + z_{2n_k}}{2} - z_{2n_k+1} \right\| < \operatorname{dist}(A,B)\,, \;\; \forall\, n_k \ge N,$$

a contradiction.

By a similar argument, we can show that $\|z_{2n+3} - z_{2n+1}\| \to 0$, as $n \to \infty$. □

Theorem 3.4. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X and let $S, T\colon A \cup B \to A \cup B$ are semi-cyclic $\varphi-$contraction mappings such that $T(A) \subseteq B$ and $S(B) \subseteq A$. For $x_0 \in A$ the sequences $\{z_n\}$ is generated by* (5). *Then, for each $\varepsilon > 0$, there exists a positive integer N_0 such that for all $m \ge n \ge N_0$,*

$$\|z_{2m} - z_{2n+1}\| < \operatorname{dist}(A,B) + \varepsilon.$$

Proof. Suppose the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \ge 1$, there is $m_k > n_k \ge k$ satisfying

$$\|z_{2m_k} - z_{2n_k+1}\| \ge \operatorname{dist}(A,B) + \varepsilon_0 \tag{9}$$

and

$$\left\|z_{2(m_k-1)} - z_{2n_k+1}\right\| < \operatorname{dist}(A,B) + \varepsilon_0. \tag{10}$$

It follows from (9) and (10), that

$$\begin{aligned}
\operatorname{dist}(A,B) + \varepsilon_0 &\le \|z_{2m_k} - z_{2n_k+1}\| \\
&\le \left\|z_{2m_k} - z_{2(m_k-1)}\right\| + \left\|z_{2(m_k-1)} - z_{2n_k+1}\right\| \\
&< \left\|z_{2m_k} - z_{2(m_k-1)}\right\| + \operatorname{dist}(A,B) + \varepsilon_0,
\end{aligned}$$

i.e.

$$\operatorname{dist}(A,B) + \varepsilon_0 \le \|z_{2m_k} - z_{2n_k+1}\| < \left\|z_{2m_k} - z_{2(m_k-1)}\right\| + \operatorname{dist}(A,B) + \varepsilon_0,$$

letting $k \to \infty$, Lemma 3.2 implies that $\left\|z_{2m_k} - z_{2(m_k-1)}\right\| \to 0$, hence

$$\lim_{k\to\infty} \|z_{2m_k} - z_{2n_k+1}\| = \operatorname{dist}(A,B) + \varepsilon_0. \tag{11}$$

Since (S,T) is semi cyclic $\varphi-$contraction pair, by Lemma 3.1(i),(ii) we obtain

$$\begin{aligned}
\|z_{2m_k} - z_{2n_k+1}\| &\le \|z_{2m_k} - z_{2m_k+2}\| + \|z_{2m_k+2} - z_{2n_k+3}\| + \|z_{2n_k+3} - z_{2n_k+1}\| \\
&= \|z_{2m_k} - z_{2m_k+2}\| + \|Ty_{m_k+1} - Sx_{n_k+2}\| + \|z_{2n_k+3} - z_{2n_k+1}\| \\
&\le \|z_{2m_k} - z_{2m_k+2}\| + \|y_{m_k+1} - x_{n_k+2}\| + \|z_{2n_k+3} - z_{2n_k+1}\| \\
&= \|z_{2m_k} - z_{2m_k+2}\| + \|Sx_{m_k+1} - Ty_{n_k+1}\| + \|z_{2n_k+3} - z_{2n_k+1}\| \\
&\le \|z_{2m_k} - z_{2m_k+2}\| + \|x_{m_k+1} - y_{n_k+1}\| - \varphi\left(\|x_{m_k+1} - y_{n_k+1}\|\right) \\
&\quad + \varphi\left(\mathrm{dist}(A,B)\right) + \|z_{2n_k+3} - z_{2n_k+1}\| \\
&= \|z_{2m_k} - z_{2m_k+2}\| + \|Ty_{m_k} - Sx_{n_k+1}\| - \varphi\left(\|Ty_{m_k} - Sx_{n_k+1}\|\right) \\
&\quad + \varphi\left(\mathrm{dist}(A,B)\right) + \|z_{2n_k+3} - z_{2n_k+1}\| \\
&= \|z_{2m_k} - z_{2m_k+2}\| + \|z_{2m_k} - z_{2n_k+1}\| - \varphi\left(\|z_{2m_k} - z_{2n_k+1}\|\right) \\
&\quad + \varphi\left(\mathrm{dist}(A,B)\right) + \|z_{2n_k+3} - z_{2n_k+1}\|.
\end{aligned}$$

Letting $k \to \infty$, using Lemma 3.2 and (11), we get

$$\begin{aligned}
\mathrm{dist}(A,B) + \varepsilon_0 &\le \mathrm{dist}(A,B) + \varepsilon_0 - \lim_{k\to\infty} \varphi\left(\|z_{2m_k} - z_{2n_k+1}\|\right) + \varphi\left(\mathrm{dist}(A,B)\right) \\
&\le \mathrm{dist}(A,B) + \varepsilon_0.
\end{aligned}$$

Hence, we obtain

$$\lim_{k\to\infty} \varphi\left(\|z_{2m_k} - z_{2n_k+1}\|\right) = \varphi\left(\mathrm{dist}(A,B)\right). \tag{12}$$

Since φ is strictly increasing, from (9) and (12), it follows that

$$\begin{aligned}
\varphi\left(\mathrm{dist}(A,B) + \varepsilon_0\right) &\le \lim_{k\to\infty} \varphi\left(\|z_{2m_k} - z_{2n_k+1}\|\right) \\
&= \varphi\left(\mathrm{dist}(A,B)\right) \\
&< \varphi\left(\mathrm{dist}(A,B) + \varepsilon_0\right),
\end{aligned}$$

a contradiction. □

Theorem 3.5. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and let $S,T\colon A\cup B \to A\cup B$ are semi-cyclic φ-contraction mappings such that $T(A) \subseteq B$ and $S(B) \subseteq A$. For $x_0 \in A$ the sequences $\{z_n\}$ is generated by* (5)*. If* $\mathrm{dist}(A,B) = 0$ *then* (S,T) *have a unique common fixed point in $A \cap B$.*

Proof. Let $\varepsilon > 0$ be given. By Theorem 3.1, we have

$$\begin{aligned}
\|z_{2n} - z_{2n+1}\| &= \|Ty_n - Sx_{n+1}\| \\
&\le \|y_n - x_{n+1}\| \\
&\le \|x_n - y_n\| \\
&= \|x_n - Sx_n\| \\
&\to \mathrm{dist}(A,B) = 0.
\end{aligned}$$

Hence, for given $\varepsilon > 0$, there exists a positive integer N_1 such that

$$\|z_{2n} - z_{2n+1}\| < \varepsilon,$$

for all $n \ge N_1$. By Theorem 3.4, there exists a positive integer N_2 such that

$$\|z_{2m} - z_{2n+1}\| < \varepsilon$$

for all $m > n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Then, for all $m > n \geq N_2$, we have

$$\|z_{2m} - z_{2n}\| \leq \|z_{2m} - z_{2n+1}\| + \|z_{2n+1} - z_{2n}\| < 2\varepsilon.$$

Thus $\{z_{2n}\}$ is a Cauchy sequence in A, since A is closed subset of a complete space X then there exists $z \in A$ such that $z_{2n} \to z$ as $n \to \infty$. Since $\{z_{2n-1}\} \subseteq B$, and B is closed, it follows that $z \in B$, and finally $z \in A \cap B$. It follows from Theorem 3.2 that

$$\|z - Tz\| = \text{dist}(A, B) = 0 = \|z - Sz\|.$$

So, z is a common fixed point of S and T and hence $z \in F(T \cap S) \subseteq A \cap B$.

We claim that the fixed point z is unique.

In fact, if $Tw = w = Sw$ for some $w \in A \cap B$, $z \neq w$, then

$$\|z - w\| = \|Tz - Sw\| \leq \|z - w\| - \varphi(\|z - w\|) + \varphi(0),$$

it follows that

$$\varphi(0) < \varphi(\|z - w\|) \leq \varphi(0),$$

a contradiction.

Let $r_n = \|z_n - z\|$ for each $n \geq 0$. As the sequence $\{r_n\}$ is bounded and decreasing and $r_{2n} \to 0$ as $n \to \infty$, we conclude that $r_n \to 0$ as $n \to \infty$, Thus $z_n \to z$ as $n \to \infty$. □

Theorem 3.6. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and let $S, T\colon A \cup B \to A \cup B$ are semi-cyclic φ-contraction mappings such that $T(A) \subseteq B$ and $S(B) \subseteq A$. For $x_0 \in A$ the sequences $\{z_n\}$ is generated by* (5). *Then $\{z_{2n}\}$ and $\{z_{2n+1}\}$ are Cauchy sequences.*

Proof. If $\text{dist}(A, B) = 0$ then the result follows from Theorem 3.5. So assume that $\text{dist}(A, B) > 0$.

To show that $\{z_{2n}\}$ is a Cauchy sequence in A we assume the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exists $m_k > n_k \geq k$ so that

$$\|z_{2m_k} - z_{2n_k}\| \geq \varepsilon_0. \tag{13}$$

Choose $\varepsilon > 0$ so that $\left(1 - \delta\left(\frac{\varepsilon_0}{\text{dist}(A,B)+\varepsilon}\right)\right)(\text{dist}(A, B) + \varepsilon) < \text{dist}(A, B)$.
By Theorem 3.1, we have

$$\begin{aligned}\|z_{2n_k} - z_{2n_k+1}\| &= \|Ty_{n_k} - Sx_{n_k+1}\| \\ &\leq \|y_{n_k} - x_{n_k+1}\| \\ &= \|Sx_{n_k} - Ty_{n_k}\| \\ &\leq \|x_{n_k} - y_{n_k}\| \\ &= \|x_{n_k} - Sx_{n_k}\| \to \text{dist}(A, B),\end{aligned}$$

hence, there exists a positive integer N_1 such that

$$\|z_{2n_k} - z_{2n_k+1}\| \leq \text{dist}(A, B) + \varepsilon, \quad \forall\, n_k \geq N_1. \tag{14}$$

Also, by Theorem 3.4, there exists a positive integer N_2 such that

$$\|z_{2m_k} - z_{2n_k+1}\| \leq \text{dist}(A, B) + \varepsilon, \quad \forall\, m_k > n_k \geq N_2. \tag{15}$$

Let $N = \max\{N_1, N_2\}$. It follows from the uniform convexity of X and (13)-(15) that

$$\left\| \frac{z_{2m_k} + z_{2n_k}}{2} - z_{2n_k+1} \right\| \leq \left(1 - \delta\left(\frac{\varepsilon_0}{\operatorname{dist}(A,B)+\varepsilon}\right)\right)(\operatorname{dist}(A,B)+\varepsilon), \ \forall m_k > n_k \geq N.$$

Since A is convex $\frac{z_{2m_k}+z_{2n_k}}{2} \in A$, the choice of ε and the fact that δ is strictly increasing imply that

$$\left\| \frac{z_{2m_k} + z_{2n_k}}{2} - z_{2n_k+1} \right\| < \operatorname{dist}(A,B), \ \forall m_k > n_k \geq N,$$

a contradiction. Thus $\{z_{2n}\}$ is a Cauchy sequence in A.

By a similar argument we can show that $\{z_{2n+1}\}$ is a Cauchy sequence in B. □

Theorem 3.7. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and let $S, T\colon A \cup B \to A \cup B$ are semi-cyclic φ-contraction mappings such that $T(A) \subseteq B$ and $S(B) \subseteq A$. For $x_0 \in A$, the sequences $\{z_n\}$ is generated by* (5)*. Then there exists unique x in A and y in B such that $z_{2n} \to y$, $z_{2n+1} \to x$ and*

$$\|x - Sx\| = \operatorname{dist}(A,B) = \|y - Ty\|.$$

Proof. Since $\{z_{2n}\}$ is a Cauchy sequence, we can find a $y \in B$ such that $\{z_{2n}\}$ converges to y. It follows from Theorem 3.2 that $\|y - Ty\| = \operatorname{dist}(A,B)$. Similarly we can show that the sequence $\{z_{2n+1}\}$ is convergent to some $x \in A$ and $\|x - Sx\| = \operatorname{dist}(A,B)$.

To prove uniqueness, assume that there is another $w \in A$ such that $\|w - Sw\| = \operatorname{dist}(A,B)$. Since

$$\begin{aligned}
\operatorname{dist}(A,B) \leq \|TSx - Sx\| &\leq \|Sx - x\| - \varphi(\|Sx - x\|) + \varphi(\operatorname{dist}(A,B)) \\
&= \operatorname{dist}(A,B) - \varphi(\operatorname{dist}(A,B)) + \varphi(\operatorname{dist}(A,B)) \\
&= \operatorname{dist}(A,B),
\end{aligned}$$

it follows that $\|TSx - Sx\| = \|x - Sx\|$, this in turn gives $TSx = x$.

Similarly, we can see that $TSw = w$.

Now if $w \neq x$, then $\|x - Sw\| > \operatorname{dist}(A,B)$, from which we get

$$\begin{aligned}
\|Sx - w\| = \|Sx - TSw\| &\leq \|x - Sw\| - \varphi(\|x - Sw\|) + \varphi(\operatorname{dist}(A,B)) \\
&< \|x - Sw\| - \varphi(\operatorname{dist}(A,B)) + \varphi(\operatorname{dist}(A,B)) \\
&= \|x - Sw\| \\
&= \|TSx - Sw\| \\
&\leq \|Sx - w\| - \varphi(\|Sx - w\|) + \varphi(\operatorname{dist}(A,B)) \\
&\leq \|Sx - w\|,
\end{aligned}$$

which is a contraction. Thus $x = w$.

Similarly, we can see the uniqueness of $y \in B$.

This completes the proof. □

4. Conclusion

In this paper, the new notion of a semi cyclic φ-contraction pair of mappings, which contains semi cyclic contraction pairs as a subclass, is introduced. Existence and convergence results of best proximity points for semi cyclic φ-contraction pair of mappings are obtained. Our results are extensions of several results as in relevant items from the reference section of this paper, as well as in the literature in general. In particular, our results reduce to those of Gabeleh and Abkar [4].

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