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Modified Emden Type Oscillator Equations with Exact Harmonic Solutions

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Abstract. This paper is devoted to investigating the existence of exact harmonic solutions and limit cycles of certain modified Emden-type equations. The exact and general solutions obtained are in opposition to the predictions of classic existence theorems.

1. Introduction

The conservative Lienard equation

$$\ddot{x} + u(x) = 0, \tag{1.1}$$

where the restoring term $u(x)$ is a function of x and is presumed to have periodic solutions. When a frictional term is included in Equation (1.1), such as

$$\ddot{x} + \alpha x\dot{x} + u(x) = 0, \tag{1.2}$$

where α is an arbitrary parameter, and the periodic behavior of the Lienard equation (1.2) is no longer assured. In this way, the modified Emden equation [1–3] is as follows:

$$\ddot{x} + \alpha x\dot{x} + \lambda x^3 = 0, \tag{1.3}$$

where λ is a constant and $u(x) = \lambda x^3$ has been the subject of an intensive mathematical investigation in the literature using different approaches. While [[4], p.393] asked to show that the form

$$\ddot{x} + x\dot{x} + x^3 = 0, \tag{1.4}$$

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has a center at the origin, as an exercise, the exact and general solutions calculated in [2, 5] are nonperiodic formulas. The existence of a center at the origin of Equation (1.4) is in fact established based on existence theorems [4, 6, 7], such as Theorem 11.3 in [[4], p.390], using the phase plane. According to these theorems, an equation of type (1.2) has a center at the origin when the function $u(x)$ is continuous and odd, and $u(x) \succ 0$ for $x \succ 0$, which involves $u(0) = 0$, to ensure the existence of a single equilibrium point at the origin. Another interesting equation of type (1.2) can read as follows:

$$\ddot{x} + \alpha x\dot{x} + \beta x = 0, \tag{1.5}$$

where β is an arbitrary constant. This type of equation is known as a reversible system with a nonlinear center at the origin of the $(x, y = \dot{x})$ phase plane (see Theorem 6.6.1 of [[8], p.164]).The reversible system (1.5) can also exhibit both positive and negative damping. The exact and general solution of Equation (1.5) is not well documented in the literature. Equation (1.5) in the form

$$\ddot{x} - \sigma x\dot{x} + x = 0, \tag{1.6}$$

where $\sigma \succ 0$ is a constant, is given in [[4], p.137] as an exercise in which the authors [4] asked to show that the origin is a center. Equation (1.6) also satisfies the conditions required (see Theorem 11.3 of [4]) for the existence of a center. Despite this, according to the nonperiodic solutions obtained in [2, 5] for the modified Emden equation (1.3), also called the second-order Riccati equation or Painlevé-Ince equation, it is reasonable to suspect that the exact and explicit general solution of Equation (1.6) can be nonperiodic. Therefore, consider the following dissipative Lienard-type equation:

$$\ddot{x} + \alpha x\dot{x} + \beta x + \gamma x(a^2 - cx^2)^q = 0, \tag{1.7}$$

where γ, a, c, and q are arbitrary constants. Equation (1.7) includes the previous equations as special cases. When $q = 1$, $a = \beta = 0$, Equation (1.7) reduces to the modified Emden Equation (1.3). When $q = 1$ and $a = 0$, Equation (1.7) becomes

$$\ddot{x} + \alpha x\dot{x} + \beta x - \gamma cx^3 = 0, \tag{1.8}$$

which has been widely investigated in the literature [3, 9]. In [3], the authors claimed to present an unusual Lienard nonlinear oscillator with exact harmonic periodic solutions. However, Doutetien et al. [9] succeeded in showing that Equation (1.8) can exhibit an exact unbounded periodic solution. When $\gamma = 0$, Equation (1.7) takes the expression of Equation (1.5). Equation (1.7) is of type (1.2) such that the function $u(x) = \beta x + \gamma x(a^2 - cx^2)^q$ is odd and $u(0) = 0$. However, Equation (1.7) can have several equilibrium points whose origins in the $(x, y = \dot{x})$ phase plane and $u(x)$ can be positive or negative for $x \succ 0$. The above shows that Equation (1.7) is worth investigating from different points of view. Thus, it is mathematically important to study its integrability conditions and implications. In this paper, we assume that Equation (1.7) may or may not satisfy the classic theorems for the existence of a center at the origin. Hence, the question whether Equation (1.7) can experience exact harmonic and isochronous periodic solutions. This question will permit us to show that the classic existence theorems exclude several Lienard type equations and to highlight the existence of Lienard equations that can be used as conservative nonlinear oscillators and self-sustained oscillators. Another question is whether the equation

$$\ddot{x} - x\dot{x} + x = 0, \tag{1.9}$$

which satisfies the theorems for the existence of a center at the origin and is given as an exercise in [[4], p.404], for $|x| \prec 1$ (see [[10], p.490]), can admit nonperiodic solutions. The objective, in view of the above, is to show that Equation (1.7) can have exact harmonic and isochronous periodic solutions and can be modified to build oscillators with exact algebraic limit cycles. In addition we show that Equation (1.9) can possess nonoscillatory solutions. To that end, we solve (Section 2) Equation (1.9) and Equation (1.7) in an explicit way . The results are compared to numerical solutions obtained by using the fourth-order Runge-Kutta (RK4) method (Section 3). We additionally highlight the existence of modified Emden-type nonpolynomial differential equations with exact algebraic limit cycles (Section 4). Finally, we present a conclusion for the work.

2. Methods and solutions

2.1. **Method and solution of Equation (1.9).** To solve Equation (1.9), consider the general theory [11–13] of the following fundamental equation:

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x} + abx^{\ell}\frac{f'(x)}{g^2(x)} - a^2x^{2\ell}\frac{f'(x)f(x)}{g^2(x)} = 0, \tag{2.1}$$

where a, b and ℓ are arbitrary parameters, and $f(x)$ and $g(x)$ are functions of x and prime means derivative with respect to x, with the corresponding first integral

$$b = g(x)\dot{x} + af(x)x^{\ell}. \tag{2.2}$$

Substituting $\ell = 0$ and $g(x) = \exp(\gamma x^2)$ leads to

$$\ddot{x} + 2\gamma x\dot{x}^2 - a^2f'(x)f(x)\exp(-2\gamma x^2) + abf'(x)\exp(-2\gamma x^2) = 0, \tag{2.3}$$

where γ is an arbitrary constant. From $f(x) = g(x) = \exp(\gamma x^2)$ and b=0 one can obtain

$$\ddot{x} - 2a\gamma x\dot{x} - 2a^2\gamma x = 0, \tag{2.4}$$

such that Equation (2.2) becomes

$$\dot{x} = -a. \tag{2.5}$$

In this situation, the general solution of Equation (2.4) is immediately obtained as follows:

$$x = -at + k, \tag{2.6}$$

where k is an integration constant. Hence, the solution of Equation (1.9) takes following the form:

$$x = t + k, \tag{2.7}$$

where $a = -1$, and $\gamma = -\frac{1}{2}$. In this way, we have proven the following result.

Theorem 2.1. *Consider Equation (2.4). If $a = -1$, and $\gamma = -\frac{1}{2}$, then Equation (2.4) has the exact and general solution (2.7).*

Therefore, the general solution of Equation (1.9) is not oscillatory, that is, nonperiodic, contrary to the assertion of Jordan and Smith in their books [4, 10] on the basis of existence theorems.

2.2. **Method and solution of Equation (1.7).**

2.2.1. *Qualitative analysis.*

In this part, the purpose is to study Equation (1.7) considering the classic theorems for the existence of a center at the origin [4, 6–8]. Indeed, Equation (1.7) is equivalent to the following system:

$$\dot{x} = y, \quad \dot{y} = -\alpha x y - \beta x - \gamma x(a^2 - cx^2)^q, \tag{2.8}$$

where γ is an arbitrary constant. The equilibrium points are defined by

$$y = 0, \quad \beta x + \gamma x(a^2 - cx^2)^q = 0. \tag{2.9}$$

Hence, there is an equilibrium point at the origin $(0, 0)$ in the (x, y) phase plane, and the other is given by

$$x = \pm\left[\frac{a^2\gamma^{\frac{1}{q}} - (-\beta)^{\frac{1}{q}}}{c\gamma^{\frac{1}{q}}}\right]^{\frac{1}{2}}, \tag{2.10}$$

where $\gamma c \neq 0$. For simplicity, let $c = 1$. Then, Solution (2.10) becomes

$$x = \pm\left[a^2 - (\frac{-\beta}{\gamma})^{\frac{1}{q}}\right]^{\frac{1}{2}}. \tag{2.11}$$

Whether the value of Solutions (2.11) is real or complex depends on the values of the parameters q, a, β and γ. When $q = \frac{1}{2}$, for example, Equation (2.11) takes the form

$$x = \pm\left[a^2 - \frac{\beta^2}{\gamma^2}\right]^{\frac{1}{2}}. \tag{2.12}$$

This clearly shows that there are values of β, a, and γ for which Solution (2.12) is real. In other words, if $a^2 \succ \frac{\beta^2}{\gamma^2}$, then Solution (2.12) is real. In this case, there is not a single equilibrium point but several fixed points, so according to the above, Equation (1.7) of interest does not satisfy the theorem for the existence of an isochronous center at the origin. In contrast, if $a^2 \prec \frac{\beta^2}{\gamma^2}$, then Solution (2.12) is complex-valued, so the origin is a single equilibrium point, and Equation (1.7) satisfies the conditions for the existence of a center when $\beta \succ 0$ and $\gamma \succ 0$. Now the problem is to find the conditions that ensure the exact harmonic solution of Equation (1.7) for any arbitrary value of $\gamma \neq 0$. This means that Equation (1.7) may or may not satisfy the conditions for the existence of a single equilibrium point depicted previously by the qualitative analysis. Figures 1 and 2 exhibit the phase portraits and vector field of Equation (1.7) or an equivalent planar dynamical system (2.8) for $\alpha = 2$, $\beta = a = c = 1$,

$\gamma = 2$, and $q = \frac{1}{2}$, and $\alpha = 5$, $\beta = a = c = 1$, $\gamma = 5$, and $q = \frac{1}{2}$, respectively, showing the existence of a center at the origin.

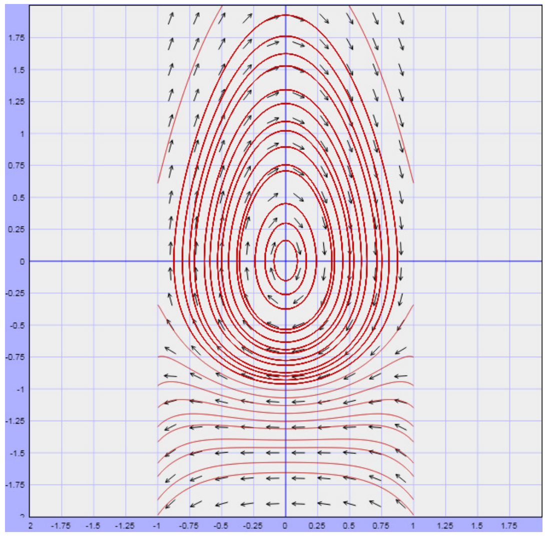

Figure 1. Phase portrait and vector field of Equation (1.7) for $\alpha = 2$, $\beta = a = c = 1$, $\gamma = 2$ and $q = 0.5$

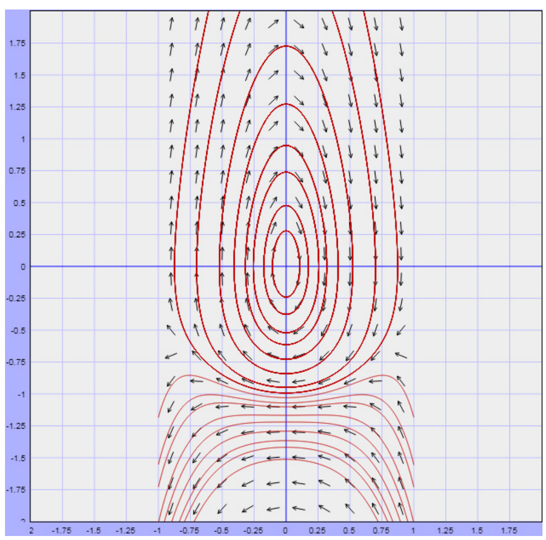

Figure 2. Phase portrait and vector field of Equation (1.7) for $\alpha = 5$, $\beta = a = c = 1$, $\gamma = 5$ and $q = 0.5$.

2.2.2. *Exact harmonic and isochronous solution.*

This section is devoted to the integrability of Equation (1.7) in terms of exact harmonic and isochronous periodic solutions. In this context, consider the general solution of Equation (1.7) in the form

$$x(t) = A\cos(\omega t + \phi), \tag{2.13}$$

where A, ω, and ϕ are parameters to be determined. From Equation (2.13)

$$\dot{x}(t) = -\omega A\sin(\omega t + \phi), \tag{2.14}$$

and

$$\ddot{x}(t) = -\omega^2 A\cos(\omega t + \phi). \tag{2.15}$$

Substituting Equations (2.13), (2.14) and (2.15) into Equation (1.7) leads to

$$\begin{aligned} &-\omega^2 A\cos(\omega t + \phi) - \omega A^2\alpha\cos(\omega t + \phi)\sin(\omega t + \phi) + \\ &\beta A\cos(\omega t + \phi) + \gamma A\cos(\omega t + \phi)[a^2 - cA^2\cos^2(\omega t + \phi)]^q = 0. \end{aligned} \tag{2.16}$$

Setting $c = 1$, and $\beta = \omega^2$, reduces Equation (2.16) to

$$-\omega A^2\alpha\cos(\omega t + \phi)\sin(\omega t + \phi) + \gamma A\cos(\omega t + \phi)[a^2 - A^2\cos^2(\omega t + \phi)]^q = 0, \tag{2.17}$$

which easily holds when $q = \frac{1}{2}$, $a = \pm A$ and $\gamma = \alpha\omega$. With these parameters, the desired Equation (1.7) takes the form

$$\ddot{x} + \alpha x\dot{x} + \omega^2 x + \epsilon\alpha\omega x\sqrt{A^2 - x^2} = 0, \tag{2.18}$$

where $\varepsilon = \pm 1$, which is completely integrable with the exact harmonic and isochronous solution to Equation (2.13) when $\varepsilon = +1$ and with the following solution:

$$x(t) = A\sin(\omega t + \phi_1), \tag{2.19}$$

when $\varepsilon = -1$. The constant ϕ and ϕ_1 can be obtained using the initial conditions. Solutions (2.13) and (2.19) show that Equation (2.18) has an isochronous center for any arbitrary value of $\alpha \neq 0$, or $\gamma \neq 0$, in contrast to the predictions of the qualitative theory of differential equations. Then, the following theorem is proved.

Theorem 2.2. *Consider Equation (2.18). If*

(i) $\varepsilon = 1$, *then Equation (2.18) has the exact harmonic solution (2.13).*

(ii) $\varepsilon = -1$, *then Equation (2.18) has the exact harmonic solution (2.19).*

Now, we can compare these results with the solutions obtained by numerical integration using the fourth-order Runge-Kutta (RK4) algorithm.

3. Comparison of analytical results with numerical solutions

3.1. Case of solution (2.13). The following initial conditions:

$$x(0) = x_o, \quad \dot{x}(0) = \vartheta_o, \tag{3.1}$$

result in the following:

$$x_o = A\cos\phi, \quad \vartheta_o = -\omega A\sin\phi. \tag{3.2}$$

From this,

$$\phi = -\arctan\left(\frac{\vartheta_o}{\omega x_o}\right). \tag{3.3}$$

In this situation, the general solution (2.13) becomes

$$x(t) = A\cos\left[\omega t - \arctan\left(\frac{\vartheta_o}{\omega x_o}\right)\right]. \tag{3.4}$$

Hence, Figure 3 shows the graphical comparison of the numerical solution in the solid line of Equation (2.18) to the analytical result (3.4) in the circle line under the conditions that $x_o = 0.5$, $\vartheta_o = 0.025$, $\alpha = 0.01$, $\omega = 0.5$, $A = 0.5$, and $\varepsilon = 1$.

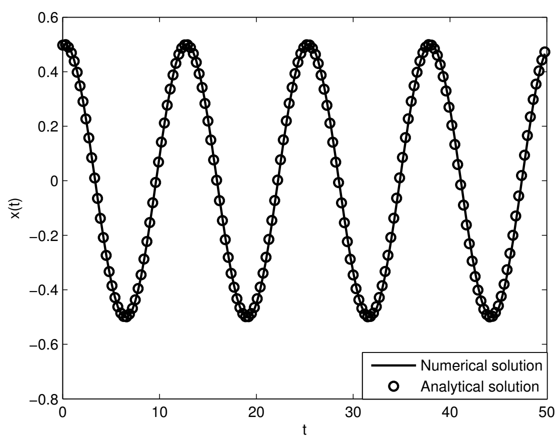

Figure 3. Comparison of Solution (3.4) with numerical solution of Equation (2.18). The values are: $A = 0.5$, $\omega = 0.5$, $\alpha = 0.01$, $x_o = 0.5$, $\vartheta_o = 0.025$, and $\varepsilon = 1$.

3.2. Case of solution (2.19). Using the following initial conditions:

$$x_o = A\sin\phi_1, \quad \vartheta_o = \omega A\cos\phi_1, \tag{3.5}$$

such that

$$\phi_1 = \ arccotan\left(\frac{\vartheta_o}{\omega x_o}\right), \tag{3.6}$$

solution (2.19) is rewritten in the form

$$x(t) = A\sin\left[\omega t + \ arccotan\left(\frac{\vartheta_o}{\omega x_o}\right)\right]. \tag{3.7}$$

From this perspective, the graphical comparison of Formula (3.7) to the numerical solution of Equation (2.18) when $\varepsilon = -1$, is shown in Figure 4. The numerical solution is plotted as a solid line, while the analytical result (3.7) is represented as a circle line with the typical values $x_o = 0.5$, $\vartheta_o = 0.025$, $\alpha = 0.01$, $\omega = 0.5$, $A = 0.5$, and $\varepsilon = -1$.

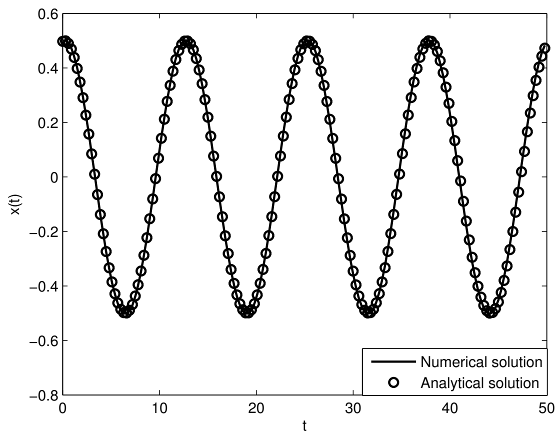

Figure 4. Comparison of solution (3.7) to the numerical solution of Equation (2.18). Typical values are: $x_o = 0.5$, $\vartheta_o = 0.025$, $\alpha = 0.01$, $\omega = 0.5$, $A = 0.5$, and $\varepsilon = -1$.

As shown in Figure 3 and Figure 4, the numerical solutions match the analytical results. Now, using Equation (2.18), we formulate the nonpolynomial differential equations of the modified Emden-type with exact algebraic limit cycles.

4. Nonpolynomial differential equations of the modified Emden-type

Although the existence of limit cycles of differential equations has been widely investigated, there are a few works devoted to exhibiting their exact and explicit expressions in the literature. In addition, the existence of limit cycles of nonpolynomial differential systems or equations [14, 15] is investigated much less often, contrary to the vast literature that can be found on the study of polynomial differential systems in connection with the second part of the Hilbert $16th$ problem [16]. In this context, consider the equation

$$\ddot{x} + (\dot{x}^2 + x^2 + \alpha x - 1)\dot{x} + x + \epsilon\alpha x\sqrt{1 - x^2} = 0. \tag{4.1}$$

When $\alpha = 0$, Equation (4.1) reduces to the well-known hybrid Rayleigh-Van der Pol equation, which is as follows:

$$\ddot{x} + (\dot{x}^2 + x^2 - 1)\dot{x} + x = 0, \tag{4.2}$$

with following the exact harmonic solution:

$$x(t) = \cos(t), \tag{4.3}$$

which corresponds to the algebraic limit cycle of degree 2 given by

$$x^2 + y^2 - 1 = 0, \tag{4.4}$$

where $y = \dot{x}$. Using Equation (4.4), Equation (4.1) becomes Equation (2.18), where $\omega = A = 1$. Therefore, it is straightforward to show that Equation (4.1) has the exact harmonic solution (4.3) for $\varepsilon = +1$. Indeed, inserting Equations (4.3), (2.14) and (2.15) with $A = \omega = 1$, and $\varphi = 0$, into Equation (4.1) yields

$$\begin{aligned} -cost + (\sin^2 t + \cos^2 t + \alpha\ cost - 1)(-\ sint) + \\ cost + \alpha\ cost\sqrt{1 - \cos^2 t} = -\alpha\ cost\ sint + \alpha\ cost\ sint = 0, \end{aligned} \tag{4.5}$$

proving that Equation (4.3) is the exact harmonic solution of Equation (4.1), where the trigonometric equation

$$\cos^2 t + \sin^2 t = 1, \tag{4.6}$$

has been used.Thus, we have proven the following result.

Theorem 4.1. *Consider Equation (4.1). Let $\varepsilon = 1$. Then, Equation (4.1) has the exact harmonic solution (4.3).*

Remark **4.1**

If $\varepsilon = -1$, one can easily verify that Equation (4.1) has the following exact harmonic solution:

$$x(t) = \ sint. \tag{4.7}$$

Remark **4.2**

It is worth noting that Equation (4.1) does not satisfy the Lienard-Levinson-Smith theorem for the existence of at least one limit cycle, but can exhibit an algebraic limit cycle of degree 2 given by $x^2 + y^2 - 1 = 0$ for $\varepsilon = 1$ and $\alpha \succ 0$. Indeed, Equation (4.1) has the form of the generalized Lienard equation, as follows:

$$\ddot{x} + h(x, \dot{x})\dot{x} + g(x) = 0, \tag{4.8}$$

where

$$h(x, \dot{x}) = \dot{x}^2 + x^2 + \alpha x - 1, \tag{4.9}$$

and

$$g(x) = \alpha x\sqrt{1 - x^2} + x, \tag{4.10}$$

where $\alpha \succ 0$ and $\varepsilon = 1$. This theorem [4, 17–19] primarily requires that

(i) $g(0) = 0$, $xg(x) \succ 0$ for $|x| \succ 0$,

(ii) $\int_0^{\infty} g(x)dx = \int_0^{-\infty} g(x)dx = \infty$,

(iii) $h(0, 0) \prec 0$,

(iv) there exists $x_0 \succ 0$ such that $h(x, \dot{x}) \succeq 0$ for $|x| \succeq x_0$. Thus, let

$$G(x) = \int_0^x g(s)ds = \int_0^x (s + \alpha s\sqrt{1 - s^2})ds \tag{4.11}$$

or

$$G(x) = \frac{1}{2}x^2 - \frac{\alpha}{3}(1 - x^2)^{\frac{3}{2}}. \tag{4.12}$$

The results is that $G(x) \longrightarrow i\infty$ as $x \longrightarrow \infty$ where i is a purely imaginary number. Thus, condition (ii) is not satisfied, so according to the Lienard-Levinson-Smith theorem, Equation (4.1) does not have a limit cycle. Additionally, it is possible to show that condition (iv) is not satisfied. From Equation (4.9), $h(x, \dot{x}) \succeq 0$ involves $\dot{x}^2 + x^2 + \alpha x - 1 \succeq 0$ for $x \succeq 0$, under Theorem 4.1 from which $\dot{x}^2 + x^2 - 1 = 0$, and $\alpha \succ 0$. Thus, we have $x_0 = 0$, which is incompatible with condition (iv). The phase portraits and vector field of Equation (4.1) are exhibited in Figures 5, 6 and 7 for $\alpha = 0.1$, 1 and 10, respectively, which show the existence of an algebraic limit cycle of degree 2 given by Equation (4.4), contrary to the prediction of the Lienard-Levinson-Smith theorem.

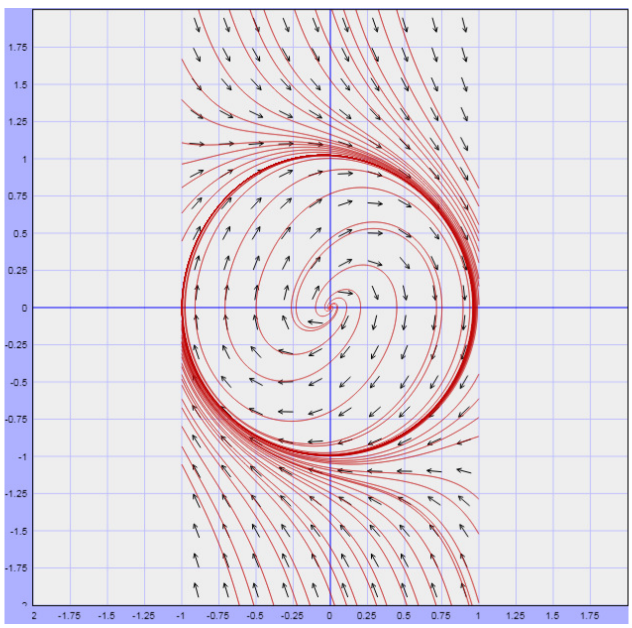

Figure 5. Phase portrait and vector field of Equation (4.1) for $\alpha = 0.1$ and $\varepsilon = 1$.

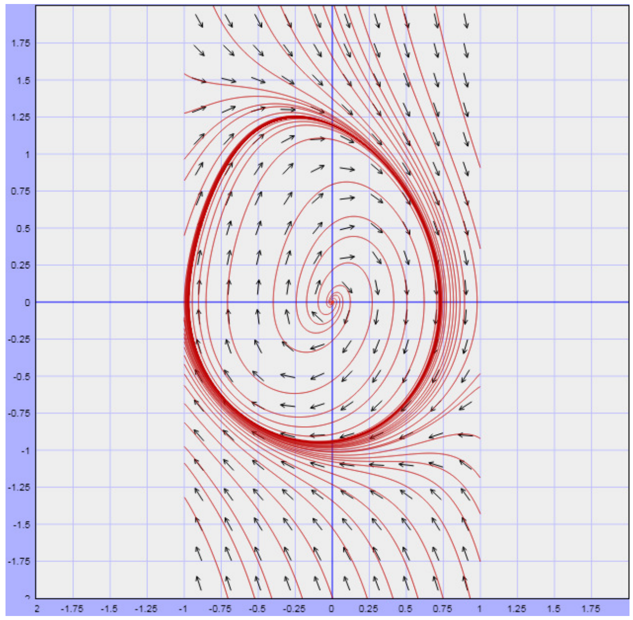

Figure 6. Phase portrait and vector field of Equation (4.1) for $\alpha = 1$ and $\varepsilon = 1$.

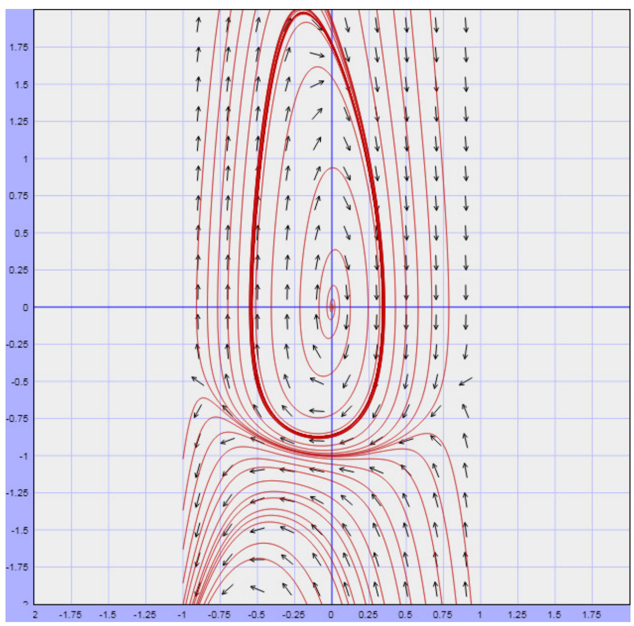

Figure 7. Phase portrait and vector field of Equation (4.1) for $\alpha = 10$ and $\varepsilon = 1$.

As shown by these figures, the limit cycle shape is controlled by the parameter α. When $\varepsilon = -1$, the phase paths are not closed trajectories, as shown in Figures 8 and 9 for $\alpha = 0.1$, and $\alpha = 10$, respectively, while Equation (4.1) admits the exact harmonic solution $\sin(t)$. However, for small enough α, such as $0 \prec \alpha \prec 0.1$, the phase portraits show the existence of limit cycles. In this way, Figure 10 exhibits the phase portrait and vector field of Equation (4.1) for $\varepsilon = -1$, and $\alpha = 0.001$, showing the existence of an algebraic limit cycle of degree 2 given by Equation (4.4).

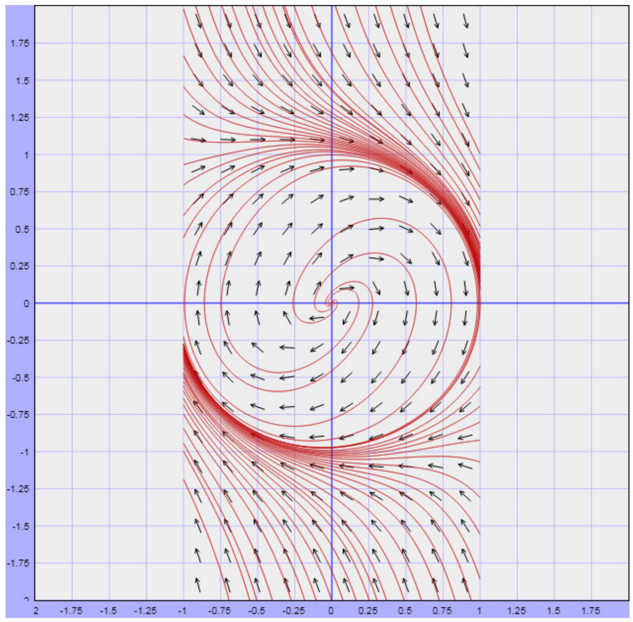

Figure 8. Phase portrait and vector field of Equation (4.1) for $\alpha = 0.1$ and $\varepsilon = -1$.

Now, the objective is to investigate a more general nonpolynomial differential equation. Therefore, consider the following theorem.

Theorem 4.2. *Let*

$$\ddot{x} + [\dot{x}^2 + x^2 + \alpha x + \dot{x}\sum_{\ell=0}^{n} x^{2\ell+1} - 1]\dot{x} + x^{2n+3} + \alpha x\sqrt{1 - x^2} = 0, \tag{4.13}$$

where $n \succeq 0$, is an integer. Then, Equation (4.13) has the following exact harmonic solution:

$$x(t) = \ cost. \tag{4.14}$$

Proof of Theorem 4.2

From Equation (4.14),

$$\dot{x} = -\sin t. \tag{4.15}$$

and

$$\ddot{x} = -\cos t. \tag{4.16}$$

Substituting Equations (4.6), (4.14), (4.15) and (4.16) into Equation (4.13), results in the following:

$$\begin{aligned}
&-\cos t-\left[\sin^2+\cos^2+\alpha-\sin t\sum_{\ell=0}^{n}\cos^{2\ell+1}t-1\right]\sin t+\cos^{2n+3}t+\alpha\cos t\sqrt{1-\cos^2 t}\\
=&-\cos t-\alpha\cos t\sin t+\sin^2 t\sum_{\ell=0}^{n}\cos^{2\ell+1}t+\cos^{2n+3}t+\alpha\cos t\sqrt{1-\cos^2 t}\\
=&-\cos t-\alpha\cos t\sin t+\left(1-\cos^2 t\right)\sum_{\ell=0}^{n}\cos^{2\ell+1}t+\cos^{2n+3}t+\alpha\cos t\sqrt{1-\cos^2 t}\\
=&-\cos t-\alpha\cos t\sin t+\sum_{\ell=0}^{n}\cos^{2\ell+1}t-\sum_{\ell=0}^{n}\cos^{2\ell+3}t+\cos^{2n+3}t+\alpha\cos t\sin t\\
=&-\cos t+\cos t+\sum_{\ell=1}^{n}\cos^{2\ell+1}t-\cos^{2n+3}t-\sum_{\ell=0}^{n-1}\cos^{2\ell+3}t+\cos^{2n+3}t\\
=&\sum_{\ell=1}^{n}\cos^{2\ell+1}t-\sum_{\ell=0}^{n-1}\cos^{2\ell+3}t\\
=&\sum_{\ell=1}^{n}\cos^{2\ell+1}t-\left(\cos^3 t+\cos^5 t+\ldots.+\cos^{2n+1}t\right)\\
=&\sum_{\ell=1}^{n}\cos^{2\ell+1}t-\sum_{\ell=1}^{n}\cos^{2\ell+1}t\\
=&0.
\end{aligned} \tag{4.17}$$

Thus, Theorem 4.2 is proved.

The phase portraits and vector field of Equation (4.13) are exhibited in Figures 11, 12 and 13 for $\alpha = 0.1$ and $n = 0$, 1 and 2, respectively, which show the existence of an algebraic limit cycle of degree 2. The above allows us to present a conclusion for the work.

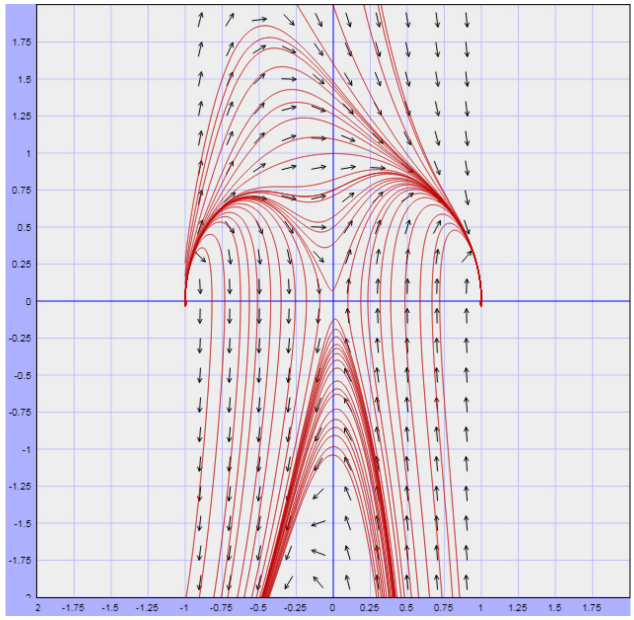

Figure 9. Phase portrait and vector field of Equation (4.1) for $\alpha = 10$ and $\varepsilon = -1$.

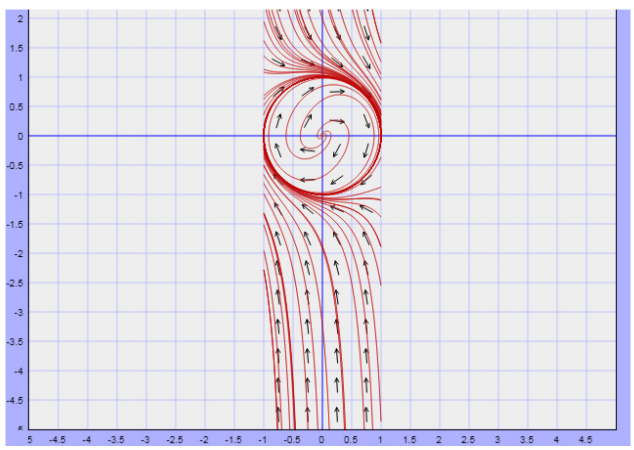

Figure 10. Phase portrait and vector field of Equation (4.1) for $\alpha = 0.001$ and $\varepsilon = -1$.

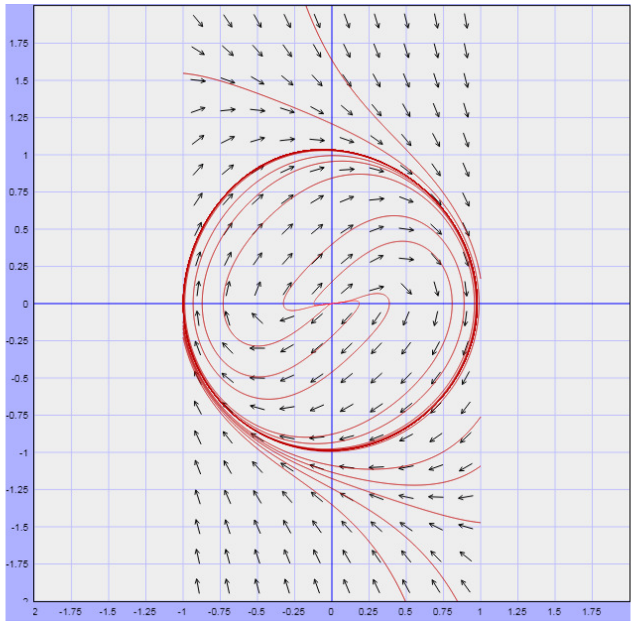

Figure 11. Phase portrait and vector field of Equation (4.13) for $\alpha = 0.1$ and $n = 0$.

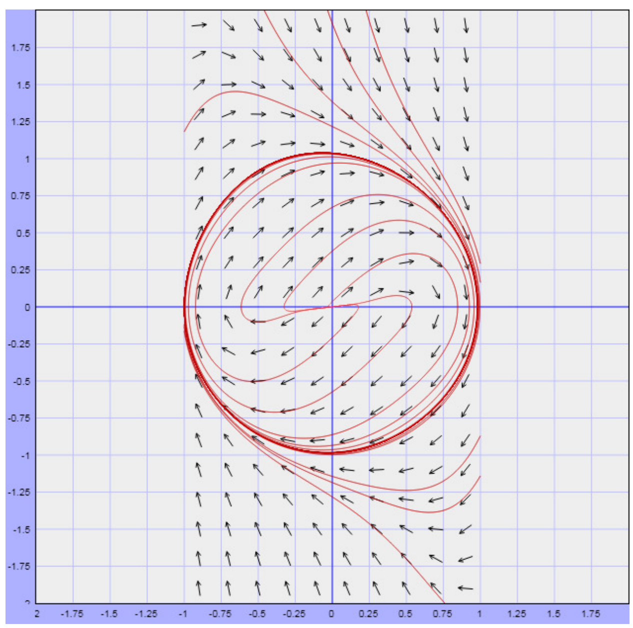

Figure 12. Phase portrait and vector field of Equation (4.13) for $\alpha = 0.1$ and $n = 1$.

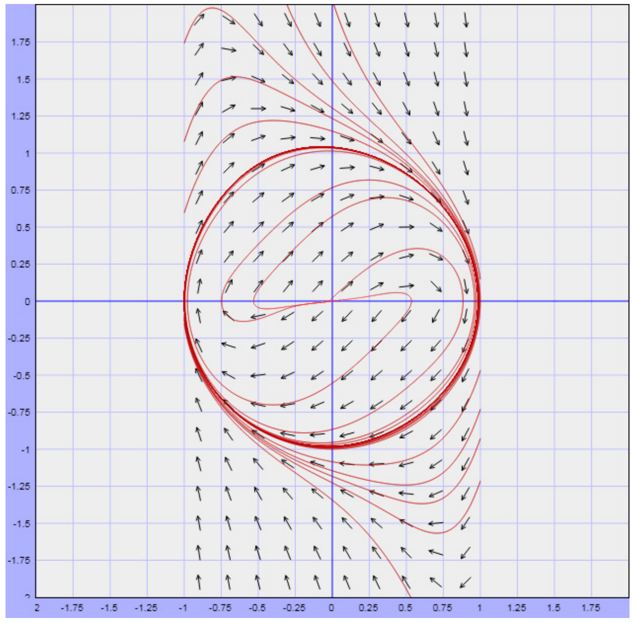

Figure 13. Phase portrait and vector field of Equation (4.13) for $\alpha = 0.1$ and $n = 2$.

5. Conclusion

This work has been devoted to investigating some modified Emden-type equations. We successfully calculated their exact and general solutions and showed the existence of algebraic limit cycles of degree 2. Numerical simulations were performed to illustrate the validity of the solution methods.

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