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Bipolar Fuzzy Sublattices and Ideals

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Abstract. In this article, we introduce and study the theory of bipolar fuzzy sublattices (BFLs) and bipolar fuzzy ideals (BFIs) of a lattice, and some interesting properties of these BFLs and BFIs are established. Moreover, we study the properties of BFIs under lattice homomorphisms and also an application of BFLs.

1. Introduction

Zadeh [4] introduced the concept of fuzzy sets (FSs) in 1965, and it has become a thriving area of research in a variety of fields. Following that, several researchers applied this concept to various algebraic structures. The fuzzy set theory has various expansions, such as vague sets (VSs), interval-valued fuzzy sets (IVFSs), intuitionistic fuzzy sets (IFS) and so on. The IFS was introduced by Atanassov [2] in 1986 as a generalization of the FS. In both the FS and IFS, the membership value range is in [0,1]. Later, Ajmal and Thomas [5] specifically applied the concept of FSs in lattice theory and developed the theory of fuzzy sublattices (FSLs). Thereafter, Thomas and Nair [3] introduced the concept of intuitionistic fuzzy sublattices (IFSLs) in 2011. Characterization of intuitionistic fuzzy ideals and filters based on lattice operations were studied by Milles [11] in 2017. Later, rough vague lattices were studied by Rao [13] in 2019. Vague lattices were introduced by Rao [12] in 2020. In

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2020, Milles [8, 9] researched on the principal intuitionistic fuzzy ideals and filters on a lattice and the lattice of intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations. Zhang [10] studied intuitionistic fuzzy filters on residuated lattices. Nowadays, bipolarity is playing a vital role in many areas. This has become a thriving area of research in many fields like artificial intelligence (AI), machine learning (ML) etc. Lee [1] introduced the concept of bipolar fuzzy sets (BFSs) in 2000, with membership values ranging from [-1,1]. Eswarlal and Kalyani [6, 7] investigated bipolar vague cosets, homomorphism, and anti homomorphism in bipolar vague normal groups (BVNGs), and used bipolar vague sets (BVSs) to solve MCDM problems.

In this paper, we introduce the concepts of BFLs and BFIs of a lattice. Some interesting characterizations and properties of these BFLs and BFIs are established. In addition, we study the properties of BFIs under lattice homomorphisms and also an application of BFLs.

2. Preliminaries

Throughout this paper, unless otherwise stated, L always represents a lattice $(L, \vee, \wedge)$ and L^1 represents a lattice $(L^1, \vee, \wedge)$.

Here, we will review a few standard definitions that are relevant to this work.

Definition 2.1. *[4] A mapping $\delta : Z \to [0, 1]$ is represented as a fuzzy set (FS) in a non-empty set Z.*

Definition 2.2. *[3] Let δ be a FS in L. Then δ is called a fuzzy sublattice (FL) of L if for all $\mathcal{T}, k \in L$, (i) $\delta(\mathcal{T} \vee k) \geq \min\{\delta(\mathcal{T}), \delta(k)\}$, (ii) $\delta(\mathcal{T} \wedge k) \geq \min\{\delta(\mathcal{T}), \delta(k)\}$.*

Definition 2.3. *[1] Suppose X is a universal set. A bipolar fuzzy set (BFS) B_δ in X is an object having the form $B_\delta = \{< \mathcal{T}, B_\delta^P(\mathcal{T}), B_\delta^N(\mathcal{T}) >| \ \mathcal{T} \in X\}$ determined by a positive and a negative membership function, respectively, where $B_\delta^P : X \to [0, 1]$ and $B_\delta^N : X \to [-1, 0]$. For convenience, the BFS B_δ is denoted by $B_\delta = (B_\delta^P, B_\delta^N)$.*

Definition 2.4. *[1] Let B_δ and B_ω be BFSs in a non-empty set X.*
(i) B_δ is a subset of B_ω, denoted by $B_\delta \subseteq B_\omega$, if for each $\mathcal{T} \in X$, $B_\delta^P(\mathcal{T}) \leq B_\omega^P(\mathcal{T})$ and $B_\delta^N(\mathcal{T}) \geq B_\omega^N(\mathcal{T})$.
(ii) The complement of B_δ, denoted by $B_\delta^c = ((B_\delta^c)^P, (B_\delta^c)^N)$, is a BFS in X defined as: for each $\mathcal{T} \in X$, $B_\delta^c(\mathcal{T}) = (1 - B_\delta^P(\mathcal{T}), -1 - B_\delta^N(\mathcal{T}))$, i.e., $(B_\delta^c)^P(\mathcal{T}) = 1 - B_\delta^P(\mathcal{T})$ and $(B_\delta^c)^N(\mathcal{T}) = -1 - B_\delta^N(\mathcal{T})$.
(iii) The intersection of B_δ and B_ω, denoted by $B_\delta \cap B_\omega$, is a BFS in X defined as: for each $\mathcal{T} \in X, (B_\delta \cap B_\omega)(\mathcal{T}) = (B_\delta^P(\mathcal{T}) \wedge B_\omega^P(\mathcal{T}), B_\delta^N(\mathcal{T}) \vee B_\omega^N(\mathcal{T}))$.
(iv) The union of B_δ and B_ω, denoted by $B_\delta \cup B_\omega$, is a BFS in X defined as: for each $\mathcal{T} \in X, (B_\delta \cup B_\omega)(\mathcal{T}) = (B_\delta^P(\mathcal{T}) \vee B_\omega^P(\mathcal{T}), B_\delta^N(\mathcal{T}) \wedge B_\omega^N(\mathcal{T}))$.

3. Bipolar fuzzy sublattices and ideals

In this section, we introduce and study BFLs and BFIs and their characterizations.

Theorem 3.1. *Let $B_\delta = (B_\delta^P, B_\delta^N)$ be a BFS in L. Then for all $\mathcal{T}, k \in L$, the following conditions are equivalent:*
(i) $\mathcal{T} \leq k \Rightarrow (B_\delta^P(\mathcal{T}) \geq B_\delta^P(k), B_\delta^N(\mathcal{T}) \leq B_\delta^N(k))$,
(ii) $B_\delta^P(\mathcal{T} \wedge k) \geq \max\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}, B_\delta^N(\mathcal{T} \wedge k) \leq \min\{B_\delta^N(\mathcal{T}), B_\delta^N(k)\}$,
(iii) $B_\delta^P(\mathcal{T} \vee k) \leq \min\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}, B_\delta^N(\mathcal{T} \vee k) \geq \max\{B_\delta^N(\mathcal{T}), B_\delta^N(k)\}$.

Proof. For any $\mathcal{T}, k \in L$, we have $\mathcal{T} \wedge k \leq \mathcal{T}$ and $\mathcal{T} \wedge k \leq k$.
Then from (i), we have $B_\delta^P(\mathcal{T} \wedge k) \geq B_\delta^P(\mathcal{T}), B_\delta^N(\mathcal{T} \wedge k) \leq B_\delta^N(\mathcal{T})$,
$B_\delta^P(\mathcal{T} \wedge k) \geq B_\delta^P(k)$, and $B_\delta^N(\mathcal{T} \wedge k) \leq B_\delta^N(k)$.
Thus $B_\delta^P(\mathcal{T} \wedge k) \geq \max\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}$ and $B_\delta^N(\mathcal{T} \wedge k) \leq \min\{B_\delta^N(\mathcal{T}), B_\delta^N(k)\}$.
Now for any $\mathcal{T}, k \in L$, we have $\mathcal{T} \leq \mathcal{T} \vee k$ and $k \leq \mathcal{T} \vee k$,
using (i) we have $B_\delta^P(\mathcal{T}) \geq B_\delta^P(\mathcal{T} \vee k), B_\delta^P(k) \geq B_\delta^P(\mathcal{T} \vee k), B_\delta^N(\mathcal{T}) \leq B_\delta^N(\mathcal{T} \vee k)$, and $B_\delta^N(k) \leq B_\delta^N(\mathcal{T} \vee k)$.
Thus $B_\delta^P(\mathcal{T} \vee k) \leq \min\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}$ and $B_\delta^N(\mathcal{T} \vee k) \geq \max\{B_\delta^N(\mathcal{T}), B_\delta^N(k)\}$. Hence, (ii) and (iii) are valid.

Suppose that (ii) is true. Let $\mathcal{T}, k \in L$ be such that $\mathcal{T} \leq k$.
Then $\mathcal{T} \wedge k = \mathcal{T} \Rightarrow B_\delta^P(\mathcal{T}) = B_\delta^P(\mathcal{T} \wedge k) \geq \max\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}$ and $B_\delta^N(\mathcal{T}) = B_\delta^N(\mathcal{T} \wedge k) \leq \min\{B_\delta^N(\mathcal{T}), B_\delta^N(k)\}$.
Thus $B_\delta^P(\mathcal{T}) \geq B_\delta^P(k)$ and $B_\delta^N(\mathcal{T}) \leq B_\delta^N(k)$.

Finally, suppose (iii) holds. Let $\mathcal{T}, k \in L$ be such that $\mathcal{T} \leq k$.
Then $\mathcal{T} \vee k = k \Rightarrow B_\delta^P(k) = B_\delta^P(\mathcal{T} \wedge k) \leq \min\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}$ and $B_\delta^N(\mathcal{T}) = B_\delta^N(\mathcal{T} \vee k) \geq \max\{B_\delta^N(\mathcal{T}), B_\delta^N(k)\}$.
Thus $B_\delta^P(\mathcal{T}) \geq B_\delta^P(k)$ and $B_\delta^N(\mathcal{T}) \leq B_\delta^N(k)$.

Hence, the proof is completed. □

Similar to Theorem 3.1, we get the following theorem.

Theorem 3.2. *Let $B_\delta = (B_\delta^P, B_\delta^N)$ be a BFS in L. Then for all $\mathcal{T}, k \in L$, the following conditions are equivalent:*
(i) $\mathcal{T} \leq k \Rightarrow B_\delta^P(\mathcal{T}) \leq B_\delta^P(k), B_\delta^N(\mathcal{T}) \geq B_\delta^N(k)$,
(ii) $B_\delta^P(\mathcal{T} \wedge k) \leq \min\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}, B_\delta^N(\mathcal{T} \wedge k) \geq \max\{B_\delta^N(\mathcal{T}), B_\delta^N(k)\}$,
(iii) $B_\delta^P(\mathcal{T} \vee k) \geq \max\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}, B_\delta^N(\mathcal{T} \vee k) \leq \min\{B_\delta^N(\mathcal{T}), B_\delta^N(k)\}$.

Definition 3.1. *Let $B_\delta = (B_\delta^P, B_\delta^N)$ be a BFS in L. Then B_δ is called a bipolar fuzzy sublattice (BFL) of L if the following conditions are satisfied for all $\mathcal{T}, k \in L$,*
(i) $B_\delta^P(\mathcal{T} \vee k) \geq \min\{B_\delta^P(\mathcal{T}), B_\delta^P(k)\}$,

(ii) $B^P_\delta(\mathcal{T} \wedge k) \geq \min\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$,
(iii) $B^N_\delta(\mathcal{T} \vee k) \leq \max\{B^N_\delta(\mathcal{T}), B^N_\delta(k)\}$,
(iv) $B^N_\delta(\mathcal{T} \wedge k) \leq \max\{B^N_\delta(\mathcal{T}), B^N_\delta(k)\}$.

Example 3.1. *Consider the lattice L of "divisors of 10". Then $L = \{1, 2, 5, 10\}$. Let $B_\delta = (B^P_\delta, B^N_\delta)$ be given by*

$$< 1, 0.6, -0.4 >, < 2, 0.1, -0.5 >, < 5, 0.3, -0.4 >, < 10, 0.2, -0.7 > .$$

We can routinely prove that B_δ is a BFL of L.

Definition 3.2. *Let $B_\delta = (B^P_\delta, B^N_\delta)$ be a BFS in L. Then B_δ is called a bipolar fuzzy ideal (BFI) of L if the following conditions are satisfied for all $\mathcal{T}, k \in L$,*
(i) $B^P_\delta(\mathcal{T} \vee k) \geq \min\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$, *(ii)* $B^P_\delta(\mathcal{T} \wedge k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$, *(iii)* $B^N_\delta(\mathcal{T} \vee k) \leq \max\{B^N_\delta(\mathcal{T}), B^N_\delta(k)\}$, *(iv)* $B^N_\delta(\mathcal{T} \wedge k) \leq \min\{B^N_\delta(\mathcal{T}), B^N_\delta(k)\}$.

Example 3.2. *Consider the lattice L in Example 3.1. Let $B_\delta = (B^P_\delta, B^N_\delta)$ be given by*

$$< 1, 0.7, -0.5 >, < 2, 0.5, -0.3 >, < 5, 0.6, -0.5 >, < 10, 0.4, -0.2 > .$$

We can routinely prove that B_δ is a BFI of L.

Theorem 3.3. *If J and M are two BFLs (BFIs) of a lattice L, then $J \cap M$ is a BFL (BFI) of L.*

Proof. Let $J = (\delta^P_J, \delta^N_J)$ and $M = (\delta^P_M, \delta^N_M)$ be two BFLs of L. Now,

$$\begin{aligned}
\delta^P_{J\cap M}(\mathcal{T} \vee k) &= \min\{\delta^P_J(\mathcal{T} \vee k), \delta^P_M(\mathcal{T} \vee k)\} \\
&\geq \min\{\min\{\delta^P_J(\mathcal{T}), \delta^P_M(k)\}, \min\{\delta^P_J(\mathcal{T}), \delta^P_M(k)\} \\
&= \min\{\min\{\delta^P_J(\mathcal{T}), \delta^P_M(\mathcal{T})\}, \min\{\delta^P_J(k), \delta^P_M(k)\} \\
&= \min\{\delta^P_{J\cap M}(\mathcal{T}), \delta^P_{J\cap M}(k)\}.
\end{aligned}$$

Thus

$$\delta^P_{J\cap M}(\mathcal{T} \vee k) \geq \min\{\delta^P_{J\cap M}(\mathcal{T}), \delta^P_{J\cap M}(k)\} \text{ for all } \mathcal{T}, k \in L.$$

Similarly, we get

$$\delta^P_{J\cap M}(\mathcal{T} \wedge k) \geq \min\{\delta^P_{J\cap M}(\mathcal{T}), \delta^P_{J\cap M}(k)\} \text{ for all } \mathcal{T}, k \in L.$$

Now,

$$\begin{aligned}
\delta^N_{J\cap M}(\mathcal{T} \vee k) &= \max\{\delta^N_J(\mathcal{T} \vee k), \delta^N_M(\mathcal{T} \vee k)\} \\
&\leq \max\{\max\{\delta^N_J(\mathcal{T}), \delta^N_M(k)\}, \max\{\delta^N_J(\mathcal{T}), \delta^N_M(k)\} \\
&= \max\{\max\{\delta^N_J(\mathcal{T}), \delta^N_M(\mathcal{T})\}, \max\{\delta^N_J(k), \delta^N_M(k)\} \\
&= \max\{\delta^N_{J\cap M}(\mathcal{T}), \delta^N_{J\cap M}(k)\}.
\end{aligned}$$

Thus

$$\delta^N_{J\cap M}(\mathcal{T}\vee k)\leq \max\{\delta^N_{J\cap M}(\mathcal{T}),\delta^N_{J\cap M}(k)\} \text{ for all } \mathcal{T},k\in L.$$

Similarly, we get

$$\delta^N_{J\cap M}(\mathcal{T}\wedge k)\leq \max\{\delta^N_{J\cap M}(\mathcal{T}),\delta^N_{J\cap M}(k)\} \text{ for all } \mathcal{T},k\in L.$$

Hence, $J\cap M$ is a BFL of L.

Similarly, we can prove that $J\cap M$ is a BFI of L if J and M are BFIs of L. □

Example 3.3. *Consider the lattice L given in Example 3.1. If*

$$J=\{<1,0.7,-0.3>,<2,0.4,-0.4>,<5,0.1,-0.3>,<10,0.2,-0.5>\}$$

and

$$M=\{<1,0.6,-0.4>,<2,0.1,-0.5>,<5,0.3,-0.4>,<10,0.2,-0.7>\}$$

are two BFLs of L, then

$$J\cap M=\{<1,0.6,-0.3>,<2,0.1,-0.4>,<5,0.1,-0.3>,<10,0.2,-0.5>\}$$

is a BFL of L.

Example 3.4. *Consider the lattice L given in Example 3.1. If*

$$J=\{<1,0.7,-0.5>,<2,0.5,-0.6>,<5,0.6,-0.5>,<10,0.5,-0.6>\}$$

and

$$M=\{<1,0.7,-0.3>,<2,0.4,-0.2>,<5,0.2,-0.1>,<10,0.2,-0.1>\}$$

are two BFIs of L, then

$$J\cap M=\{<1,0.7,-0.3>,<2,0.4,-0.2>,<5,0.2,-0.1>,<10,0.2,-0.1>\}$$

is a BFI of L.

Remark 3.1. *The union of two BFLs of a lattice L need not be a BFL. Consider the lattice L given in Example 3.1. If*

$$J=\{<1,0.7,-0.3>,<2,0.4,-0.4>,<5,0.1,-0.3>,<10,0.2,-0.5>\}$$

and

$$M=\{<1,0.6,-0.4>,<2,0.1,-0.5>,<5,0.3,-0.4>,<10,0.2,-0.7>\}$$

are two BFLs of L, then

$$J\cup M=\{<1,0.7,-0.4>,<2,0.4,-0.5>,<5,0.3,-0.4>,<10,0.2,-0.7>\}.$$

Here, $\delta^P_{J\cup M}(2\vee 5)=\delta^P_{J\cup M}(10)=0.2\ngeq 0.3=\min\{0.4,0.3\}=\min\{\delta^P_{J\cup M}(2),\delta^P_{J\cup M}(5)\}$. Hence, $J\cup M$ is not a BFL of L.

Remark 3.2. *Every BFI of L is a BFL, but the converse need not be true. Consider the lattice L given in Example 3.1. Then the BFI given by*

$$J = \{<1, 0.7, -0.5>, <2, 0.5, -0.6>, <5, 0.6, -0.5>, <10, 0.5, -0.6>\}$$

is a BFL of L. But the BFL given by

$$M = \{<1, 0.5, -0.3>, <2, 0.4, -0.4>, <5, 0.4, -0.3>, <10, 0.7, -0.5>\}$$

is not a BFI of L as $\delta^P_M(2 \wedge 10) = 0.4 \ngeq 0.7 = \max\{\delta^P_M(2), \delta^P_B(10)\}$.

Remark 3.3. *The union of two BFIs of a lattice L need not be a BFI. Consider the lattice L given in Example 3.1. If*

$$J = \{<1, 0.7, -0.5>, <2, 0.5, -0.6>, <5, 0.6, -0.5>, <10, 0.5, -0.6>\}$$

and

$$M = \{<1, 0.7, -0.3>, <2, 0.4, -0.2>, <5, 0.2, -0.1>, <10, 0.2, -0.1>\}$$

are two BFIs of L, then

$$J \cup M = \{<1, 0.7, -0.5>, <2, 0.5, -0.6>, <5, 0.6, -0.5>, <10, 0.5, -0.6>\}.$$

Here, $\delta^N_{J\cup M}(2 \wedge 5) = \delta^N_{J\cup M}(1) = -0.5 \nleq -0.6 = \{-0.6, -0.5\} = \min\{\delta^N_{J\cup M}(2), \delta^N_{J\cup M}(5)\}$*. Hence,* $J \cup M$ *is not a BFI of L.*

Theorem 3.4. *Let* $B_\delta = (B^P_\delta, B^N_\delta)$ *be a BFL of L. Then for all* $\mathcal{T}, k \in L$*, the following four statements hold:*
(i) $B^P_\delta(\mathcal{T} \wedge k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\} \Leftrightarrow (\mathcal{T} \leq k \Rightarrow B^P_\delta(\mathcal{T}) \geq B^P_\delta(k))$,
(ii) $B^P_\delta(\mathcal{T} \vee k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\} \Leftrightarrow (\mathcal{T} \leq k \Rightarrow B^P_\delta(\mathcal{T}) \leq B^P_\delta(k))$,
(iii) $B^N_\delta(\mathcal{T} \wedge k) \leq \min\{B^N_\delta(\mathcal{T}), B^N_\delta(k)\} \Leftrightarrow (\mathcal{T} \leq k \Rightarrow B^N_\delta(\mathcal{T}) \leq B^N_\delta(k))$,
(iv) $B^N_\delta(\mathcal{T} \vee k) \leq \min\{B^N_\delta(\mathcal{T}), B^N_\delta(k)\} \Leftrightarrow (\mathcal{T} \leq k \Rightarrow B^N_\delta(\mathcal{T}) \geq B^N_\delta(k))$.

Proof. Let $\mathcal{T}, k \in L$.

(i) Suppose $B^P_\delta(\mathcal{T} \wedge k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$. If $\mathcal{T} \leq k$, then $\mathcal{T} \wedge k = \mathcal{T}$. Since $B^P_\delta(\mathcal{T} \wedge k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$, we have $B^P_\delta(\mathcal{T}) = B^P_\delta(\mathcal{T} \wedge k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$. Hence, $B^P_\delta(\mathcal{T}) \geq B^P_\delta(k)$.

Conversely, suppose $(\mathcal{T} \leq k \Rightarrow B^P_\delta(\mathcal{T}) \geq B^P_\delta(k))$. Then $B^P_\delta(\mathcal{T} \wedge k) \geq B^P_\delta(\mathcal{T})$ and $B^P_\delta(\mathcal{T} \wedge k) \geq B^P_\delta(k)$. Hence, $B^P_\delta(\mathcal{T} \wedge k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$.

(ii) Suppose $B^P_\delta(\mathcal{T} \vee k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$. If $\mathcal{T} \leq k$, then $\mathcal{T} \vee k = k$. Since $B^P_\delta(\mathcal{T} \vee k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$, we have $B^P_\delta(k) = B^P_\delta(\mathcal{T} \vee k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$. Hence, $B^P_\delta(\mathcal{T}) \leq B^P_\delta(k)$.

Conversely, suppose $(\mathcal{T} \leq k \Rightarrow B^P_\delta(\mathcal{T}) \leq B^P_\delta(k))$. Then $B^P_\delta(\mathcal{T}) \leq B^P_\delta(\mathcal{T} \vee k)$ and $B^P_\delta(k) \leq B^P_\delta(\mathcal{T} \vee k)$. Hence, $B^P_\delta(\mathcal{T} \vee k) \geq \max\{B^P_\delta(\mathcal{T}), B^P_\delta(k)\}$.

(iii) Suppose $B_\delta^N(\mathcal{T}\wedge k)\le \min\{B_\delta^N(\mathcal{T}),B_\delta^N(k)\}$. If $\mathcal{T}\le k$, then $\mathcal{T}\wedge k=\mathcal{T}$. Since $B_\delta^N(\mathcal{T}\wedge k)\le \min\{B_\delta^N(\mathcal{T}),B_\delta^N(k)\}$, we have $B_\delta^N(\mathcal{T})=B_\delta^N(\mathcal{T}\wedge k)\le \min\{B_\delta^N(\mathcal{T}),B_\delta^N(k)\}$. Hence, $B_\delta^N(\mathcal{T})\le B_\delta^N(k)$.

Conversely, suppose $(\mathcal{T}\le k\Rightarrow B_\delta^N(\mathcal{T})\le B_\delta^N(k))$. Then $B_\delta^N(\mathcal{T}\wedge k)\le B_\delta^N(\mathcal{T})$ and $(B_\delta^N(\mathcal{T}\wedge k)\le B_\delta^N(k)$. Hence, $B_\delta^N(\mathcal{T}\wedge k)\le \min\{B_\delta^N(\mathcal{T}),B_\delta^N(k)\}$.

(iv) Suppose $B_\delta^N(\mathcal{T}\vee k)\le \min\{B_\delta^N(\mathcal{T}),B_\delta^N(k)\}$. If $\mathcal{T}\le k$, then $\mathcal{T}\vee k=k$. Since $(B_\delta^N(\mathcal{T}\vee k)\le \min\{B_\delta^N(\mathcal{T}),B_\delta^N(k))$, we have $B_\delta^N(k)=B_\delta^N(\mathcal{T}\vee k)\le \min\{B_\delta^N(\mathcal{T}),B_\delta^N(k)\}$. Hence, $B_\delta^N(\mathcal{T})\ge B_\delta^N(k)$.

Conversely, suppose $(\mathcal{T}\le k\Rightarrow B_\delta^N(\mathcal{T})\ge B_\delta^N(k))$. Then $B_\delta^N(\mathcal{T})\ge (B_\delta^N(\mathcal{T}\vee k)$ and $(B_\delta^N(k)\ge B_\delta^N(\mathcal{T}\vee k)\ge B_\delta^N(k)$. Hence, $B_\delta^N(\mathcal{T}\vee k)\ge \min\{B_\delta^N(\mathcal{T}),B_\delta^N(k)\}$. □

Theorem 3.5. *Let $B_\eta=(B_\eta^P,B_\eta^N)$ be a BFL of L. Then B_η is a BFI of L if and only if the following two conditions are satisfied for all $\mathcal{T},k\in L$,*
(i) $B_\eta^P(\mathcal{T}\vee k)=\min\{B_\eta^P(\mathcal{T}),B_\eta^P(k)\}$,
(ii) $B_\eta^N(\mathcal{T}\vee k)=\max\{B_\eta^N(\mathcal{T}),B_\eta^N(k)\}$.

Proof. Suppose that B_η is a BFI of L. Let $\mathcal{T},k\in L$. Then $B_\eta^P(\mathcal{T}\vee k)\ge \min\{B_\eta^P(\mathcal{T}),B_\eta^P(k)\}$ and $B_\eta^N(\mathcal{T}\vee k)\le \max\{B_\eta^N(\mathcal{T}),B_\eta^N(k)\}$. Since $\mathcal{T}\le\mathcal{T}\vee k$ and $k\le\mathcal{T}\vee k$, then by Theorem 3.4, we have $B_\eta^P(\mathcal{T})\ge B_\eta^P(\mathcal{T}\vee k)$ and $B_\eta^P(k)\ge B_\eta^P(\mathcal{T}\vee k)$. Hence, $\min\{B_\eta^P(\mathcal{T}),B_\eta^P(k)\}\ge B_\eta^P(\mathcal{T}\vee k)$. Thus $B_\eta^P(\mathcal{T}\vee k)=\min\{B_\eta^P(\mathcal{T}),B_\eta^P(k)\}$. Now, since $\mathcal{T}\le\mathcal{T}\vee k$ and $k\le\mathcal{T}\vee k$, then by Theorem 3.4, we have $B_\eta^N(\mathcal{T})\le B_\eta^N(\mathcal{T}\vee k)$ and $B_\eta^N(k)\le B_\eta^N(\mathcal{T}\vee k)$. Hence, $\max\{B_\eta^N(\mathcal{T}),B_\eta^N(k)\}\le B_\eta^N(\mathcal{T}\vee k)$. Thus $B_\eta^N(\mathcal{T}\vee k)=\max\{B_\eta^N(\mathcal{T}),B_\eta^N(k)\}$.

Conversely, suppose that $B_\eta^P(\mathcal{T}\vee k)=\min\{B_\eta^P(\mathcal{T}),B_\eta^P(k)\}$ and $B_\eta^N(\mathcal{T}\vee k)=\max\{B_\eta^N(\mathcal{T}),B_\eta^N(k)\}$ for any $\mathcal{T},k\in L$. Then it is clear that $B_\eta^P(\mathcal{T}\vee k)\ge \min\{B_\eta^P(\mathcal{T}),B_\eta^P(k)\}$ and $B_\eta^N(\mathcal{T}\vee k)\le \max\{B_\eta^N(\mathcal{T}),B_\eta^N(k)\}$ for any $\mathcal{T},k\in L$. Next, we shall show that $B_\eta^P(\mathcal{T}\wedge k)\ge \max\{B_\eta^P(\mathcal{T}),B_\eta^P(k)\}$ and $B_\eta^N(\mathcal{T}\wedge k)\le \min\{B_\eta^N(\mathcal{T}),B_\eta^N(k)\}$ for any $\mathcal{T},k\in L$.

Let $\mathcal{T},k\in L$. Since $\mathcal{T}\vee(\mathcal{T}\wedge k)=\mathcal{T}$ and $k\vee(\mathcal{T}\wedge k)=k$, we have $B_\eta^P(\mathcal{T}\vee(\mathcal{T}\wedge k))=B_\eta^P(\mathcal{T})$ and $B_\eta^P(k\vee(\mathcal{T}\wedge k))=B_\eta^P(k)$. Thus $\min\{B_\eta^P(\mathcal{T}),B_\eta^P(\mathcal{T}\wedge k)\}=B_\eta^P(\mathcal{T})$ and $\min\{B_\eta^P(k),B_\eta^P(\mathcal{T}\wedge k)\}=B_\eta^P(k)$, hence, $B_\eta^P(\mathcal{T}\wedge k)\ge B_\eta^P(\mathcal{T})$ and $B_\eta^P(\mathcal{T}\wedge k)\ge B_\eta^P(k)$. Therefore, $B_\eta^P(\mathcal{T}\wedge k)\ge \max\{B_\eta^P(\mathcal{T}),B_\eta^P(k)\}$ for any $\mathcal{T},k\in L$.

Let $\mathcal{T},k\in L$. Since $\mathcal{T}\vee(\mathcal{T}\wedge k)=\mathcal{T}$ and $k\vee(\mathcal{T}\wedge k)=k$, we have $B_\eta^N(\mathcal{T}\vee(\mathcal{T}\wedge k))=B_\eta^N(\mathcal{T})$ and $B_\eta^N(k\vee(\mathcal{T}\wedge k))=B_\eta^N(k)$. Thus $\max\{B_\eta^N(\mathcal{T}),B_\eta^N(\mathcal{T}\wedge k)\}=B_\eta^N(\mathcal{T})$ and $\max\{B_\eta^N(k),B_\eta^N(\mathcal{T}\wedge k)\}=B_\eta^N(k)$. Hence, $B_\eta^N(\mathcal{T}\wedge k)\le B_\eta^N(\mathcal{T})$ and $B_\eta^N(\mathcal{T}\wedge k)\ge B_\eta^N(k)$. Therefore, $B_\eta^N(\mathcal{T}\wedge k)\le \min\{B_\eta^N(\mathcal{T}),B_\eta^N(k)\}$ for any $\mathcal{T},k\in L$.

Hence, B_η is a BFI of L. □

4. Bipolar fuzzy ideals under lattice homomorphisms

Definition 4.1. *Let $\theta : L \to L^1$ be a mapping and $B_\delta = (B_\delta^P, B_\delta^N)$ be a BFS in L. Then the image $\theta(B_\delta)$ is defined as $\theta(B_\delta) = \{< k, \theta(B_\delta^P)(k), \theta(B_\delta^N)(k) >\mid k \in L^1\}$,*

$$\theta(B_\delta^P)(k) = \begin{cases} \sup\{B_\delta^P(\mathcal{T}) \mid \mathcal{T} \in \theta^{-1}(k)\} & \text{if } \theta^{-1}(k) \neq \emptyset, \\ 0 & \text{if otherwise} \end{cases}$$

and

$$\theta(B_\delta^N)(k) = \begin{cases} \inf\{B_\delta^N(\mathcal{T}) \mid \mathcal{T} \in \theta^{-1}(k)\} & \text{if } \theta^{-1}(k) \neq \emptyset, \\ 0 & \text{if otherwise.} \end{cases}$$

Similarly, if $B_\eta = (B_\eta^P, B_\eta^N)$ be a BFS in L^1, then $\theta^{-1}(B_\eta) = \{< \mathcal{T}, \theta^{-1}(B_\eta^P(\mathcal{T})), \theta^{-1}(B_\eta^N(\mathcal{T})) >\mid t \in L\}$, where

$$\theta^{-1}(B_\eta^P(\mathcal{T})) = B_\eta^P(\theta(\mathcal{T})) \text{ and } \theta^{-1}(B_\eta^N(\mathcal{T})) = B_\eta^N(\theta(\mathcal{T})).$$

Theorem 4.1. *Let $\theta : L \to L^1$ be an epimorphism. If B_δ is a BFI of L, then $\theta(B_\delta)$ is a BFI of L^1.*

Proof. Let $B_\delta = (B_\delta^P, B_\delta^N)$ be a BFI of L. Let $s, w \in L^1$. Then

$$\begin{aligned} \theta(B_\delta^P)(s \vee w) &= \sup\{B_\delta^P(\mathcal{T}) : \mathcal{T} \in \theta^{-1}(s \vee w)\} \\ &\geq \sup\{B_\delta^P(h \vee k) \mid h \in \theta^{-1}(s), k \in \theta^{-1}(w)\} \\ &\geq \sup\{\min\{B_\delta^P(h), B_\delta^P(k)\} \mid h \in \theta^{-1}(s), k \in \theta^{-1}(w)\} \\ &= \min\{\sup\{B_\delta^P(h) \mid h \in \theta^{-1}(s)\}, \sup\{B_\delta^P(k) \mid k \in \theta^{-1}(w)\}\} \\ &= \min\{\theta(B_\delta^P)(s), \theta(B_\delta^P)(w)\}, \end{aligned}$$

$$\begin{aligned} \theta(B_\delta^P)(s \wedge w) &= \sup\{B_\delta^P(\mathcal{T}) \mid \mathcal{T} \in \theta^{-1}(s \wedge w)\} \\ &\geq \sup\{B_\delta^P(h \wedge k) \mid h \in \theta^{-1}(s), k \in \theta^{-1}(w)\} \\ &\geq \sup\{\max\{B_\delta^P(h), B_\delta^P(k)\} \mid h \in \theta^{-1}(s), k \in \theta^{-1}(w)\} \\ &= \max\{\sup\{B_\delta^P(h) \mid h \in \theta^{-1}(s)\}, \sup\{B_\delta^P(k) \mid k \in \theta^{-1}(w)\}\} \\ &= \max\{\theta(B_\delta^P)(s), \theta(B_\delta^P)(w)\}, \end{aligned}$$

$$\begin{aligned} \theta(B_\delta^N)(s \vee w) &= \inf\{B_\delta^N(\mathcal{T}) \mid \mathcal{T} \in \theta^{-1}(s \vee w)\} \\ &\leq \inf\{B_\delta^N(h \vee k) \mid h \in \theta^{-1}(s), k \in \theta^{-1}(w)\} \\ &\leq \inf\{\max\{B_\delta^N(h), B_\delta^N(k)\} \mid h \in \theta^{-1}(s), k \in \theta^{-1}(w)\} \\ &= \max\{\inf\{B_\delta^N(h) \mid h \in \theta^{-1}(s)\}, \inf\{B_\delta^N(k) \mid k \in \theta^{-1}(w)\}\} \\ &= \max\{\theta(B_\delta^N)(s), \theta(B_\delta^N)(w)\}, \end{aligned}$$

and

$$\begin{aligned}\theta(B_\delta^N)(s\wedge w) &= \inf\{B_\delta^N(\mathcal{T}) \mid \mathcal{T}\in\theta^{-1}(s\wedge w)\}\\ &\le \inf\{B_\delta^N(h\wedge k) \mid h\in\theta^{-1}(s), k\in\theta^{-1}(w)\}\\ &\le \inf\{\min\{B_\delta^N(h), B_\delta^N(k)\} \mid h\in\theta^{-1}(s), k\in\theta^{-1}(w)\}\\ &= \min\{\inf\{B_\delta^N(h) \mid h\in\theta^{-1}(s)\}, \inf\{B_\delta^N(k) \mid k\in\theta^{-1}(w)\}\}\\ &= \min\{\theta(B_\delta^N)(s), \theta(B_\delta^N)(w)\}.\end{aligned}$$

Hence, $\theta(B_\delta)$ is a BFI of L^1. □

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Theorem 4.2. *Let $\theta : L\to L^1$ be a homomorphism. If B_η is a BFI of L^1, then $\theta^{-1}(B_\eta)$ is a BFI of L.*

Proof. Let $B_\eta = (B_\eta^P, B_\eta^N)$ be a BFI of L^1. Let $\mathcal{T}, k\in L$. Then

$$\begin{aligned}\theta^{-1}(B_\eta^P)(\mathcal{T}\vee k) &= B_\eta^P(\theta(\mathcal{T}\vee k))\\ &= B_\eta^P\{(\theta(\mathcal{T})\vee\theta(k)\}\\ &\ge \min\{B_\eta^P(\theta(\mathcal{T})), B_\eta^P(\theta(k))\}\\ &= \min\{\theta^{-1}(B_\eta^P)(\mathcal{T}), \theta^{-1}(B_\eta^P)(k)\},\end{aligned}$$

$$\begin{aligned}\theta^{-1}(B_\eta^P)(\mathcal{T}\wedge k) &= B_\eta^P(\theta(\mathcal{T}\wedge k))\\ &= B_\eta^P\{(\theta(\mathcal{T})\wedge\theta(k)\}\\ &\ge \max\{B_\eta^P(\theta(\mathcal{T})), B_\eta^P(\theta(k))\}\\ &= \max\{\theta^{-1}(B_\eta^P)(\mathcal{T}), \theta^{-1}(B_\eta^P)(k)\},\end{aligned}$$

$$\begin{aligned}\theta^{-1}(B_\eta^N)(\mathcal{T}\vee k) &= B_\eta^N(\theta(\mathcal{T}\vee k))\\ &= B_\eta^N\{(\theta(\mathcal{T})\vee\theta(k)\}\}\\ &\le \max\{B_\eta^N(\theta(\mathcal{T})), B_\eta^N(\theta(k))\\ &= \max\{\theta^{-1}(B_\eta^N)(\mathcal{T}), \theta^{-1}(B_\eta^N)(k)\},\end{aligned}$$

and

$$\begin{aligned}\theta^{-1}(B_\eta^N)(\mathcal{T}\wedge k) &= B_\eta^N(\theta(\mathcal{T}\wedge k))\\ &= B_\eta^N\{(\theta(\mathcal{T})\wedge\theta(k)\}\\ &\le \min\{B_\eta^N(\theta(\mathcal{T})), B_\eta^N(\theta(k))\}\\ &= \min\{\theta^{-1}(B_\eta^N)(\mathcal{T}), \theta^{-1}(B_\eta^N)(k)\}.\end{aligned}$$

Hence, $\theta^{-1}(B_\eta)$ is a BFI of L. □

Theorem 4.3. *Let $\theta : L \to L^1$ be a homomorphism and let μ and η be BFLs of L and L^1, respectively. Then*
(i)$\theta(\mu)$ is a BFL of L^1,
(ii) $\theta^{-1}(\eta)$ is a BFL L.

Proof. The proof is omitted since it follows the same proof of Theorems 4.1 and 4.2. □

Theorem 4.4. *If $\theta : L \to L^1$ is an surjection and B_η, B_δ are BFSs of L and L^1, respectively, then*
(i) $\theta[\theta^{-1}(B_\delta)] = B_\delta$,
(ii) $B_\eta \subseteq \theta^{-1}[\theta(B_\eta)]$.

Proof. (i) Let $\alpha \in L^1$. Then $\theta[\theta^{-1}(B_\delta^P)](\alpha) = \sup\{\theta^{-1}(B_\delta^P)(\gamma) \mid \gamma \in \theta^{-1}(\alpha)\} = \sup\{B_\delta^P(\theta(\gamma)) \mid \gamma \in L, \theta(\gamma) = \alpha\} = B_\delta^P(\alpha)$ because θ is onto, for every $\alpha \in L^1$, there exists γ in L such that $\theta(\gamma) = \alpha$. Similarly, $\theta[\theta^{-1}(B_\delta^N)](\alpha) = B_\delta^N(\alpha)$. Hence, $\theta[\theta^{-1}(B_\delta)] = B_\delta$.

(ii) Let $\gamma \in L$. Then $\theta^{-1}[\theta(B_\eta^P)](\gamma) = \theta(B_\eta^P)(\theta(\gamma)) = \sup\{B_\eta^P(\gamma) \mid \gamma \in \theta^{-1}[\theta(\gamma)]\} \geq B_\eta^P(\gamma)$ and $\theta^{-1}[\theta(B_\eta^N)](\gamma) = \theta(B_\eta^N)(\theta(\gamma)) = \inf\{B_\eta^N(\gamma) \mid \gamma \in \theta^{-1}[\theta(\gamma)]\} \leq B_\eta^N(\gamma)$. Hence, $B_\eta \subseteq \theta^{-1}[\theta(B_\eta)]$. □

Definition 4.2. *Let $f : L \to L^1$ be a surjection and $B_\delta = (B_\delta^P, B_\delta^N)$ be a BFS in L. Then B_δ is said to be f-invariant if for any $w, s \in L^1$ such that $f(w) = f(s)$ implies $B_\delta^P(w) = B_\delta^P(s)$ and $B_\delta^N(w) = B_\delta^N(s)$.*

From Theorem 4.4 and Definition 4.2, we have the following theorem.

Theorem 4.5. *Let $f : L \to L^1$ be a surjection and $B_\delta = (B_\delta^P, B_\delta^N)$ be a BFS in L. If a BFS B_δ is f-invariant, then $f^{-1}(f(B_\delta)) = B_\delta$.*

Theorem 4.6. *Let $f : L \to L^1$ be a surjection, B_δ and B_η be BFSs of L, and B_δ^1 and B_η^1 be BFSs of L^1. Then*
(i) $B_\delta \subseteq B_\eta \Rightarrow f(B_\delta) \subseteq f(B_\eta)$,
(ii) $B_\delta^1 \subseteq B_\eta^1 \Rightarrow f^{-1}(B_\delta^1) \subseteq f^{-1}(B_\eta^1)$.

Proof. Let $B_\delta = (B_\delta^P, B_\delta^N)$ and $B_\eta = (B_\eta^P, B_\eta^N)$ be BFSs in L such that $B_\delta \subseteq B_\eta$. Then $B_\delta^P \leq B_\eta^P$ and $B_\delta^N \geq B_\eta^N$. Also, $f(B_\delta) = \{< t, f(B_\delta^P)(t), f(B_\delta^N)(t) >\mid t \in L^1\}$ and $f(B_\eta) = \{< t, f(B_\eta^P)(t), f(B_\eta^N)(t) >\mid t \in L^1\}$. Now, for any $t \in L$, we have $f(B_\delta^P)(t) = \sup\{(B_\delta^P(k) \mid k \in f^{-1}(t))\} \leq \sup\{(B_\eta^P(k) \mid k \in f^{-1}(t))\} = f(B_\eta^P)(t)$ and $f(B_\delta^N)(t) = \inf\{(B_\delta^N(k) \mid k \in f^{-1}(t))\} \leq \sup\{(B_\eta^N(k) \mid k \in f^{-1}(t))\} = f(B_\eta^N)(t)$. Hence, $f(B_\delta) \subseteq f(B_\eta)$.

Similarly, we can prove that $B_\delta^1 \subseteq B_\eta^1 \Rightarrow f^{-1}(B_\delta^1) \subseteq f^{-1}(B_\eta^1)$. □

Theorem 4.7. *If $f : L \to L^1$ is an epimorphism, then there is one to one order preserving correspondence between the BFIs of L^1 and those of L which are f-invariant.*

Proof. Let $B(L^1)$ denote the set of all BFIs of L^1 and $B(L)$ denote the set of all BFIs of L which are f-invariant. Define $\varsigma : B(L) \to B(L^1)$ and $\Psi : B(L^1) \to B(L)$ such that $\varsigma(B_\delta) = f(B_\delta)$ and $\Psi(B^1_\delta) = f^{-1}(B^1_\delta)$. By Theorems 4.1 and 4.2, we have ς and Ψ are well-defined. Also by Theorems 4.4 and 4.5, we have ς and Ψ are the inverse to each other which gives that the one-to-one correspondence. Also by Theorem 4.6, we get $B_\delta \subseteq B_\eta \Rightarrow f(B_\delta) \subseteq f(B_\eta)$. Hence, the correspondence is order preserving. □

5. An application of bipolar fuzzy sublattices

The single pattern: the one-minute microwave [15].

The one-minute microwave is a simple system with the following requirements:

1. There is a single button available for the user.

2. If the door is closed and the button is pushed, the oven will be energized for one minute.

3. If the button is pushed while the oven is energized, the cooking time is increased by one minute.

4. If the door is open, pushing the button has no effect.

5. The oven has a light that is turned on when the door is open, and also when the oven is cooking. Otherwise, the light is off.

6. Opening the door stops the cooking and clears the timer (i.e., the remaining cooking time is set to zero).

7. When the cooking is complete (oven times out) a beeper sounds and the light is turned off.

Here in this application, we consider the one-minute microwave as a lattice $L = \{$button, timer, oven-door$\}$ with the operations ON and OFF. The final output will be cooking the food or not cooking the food. When we consider the button there may be two cases that is the button may be pressed or unpressed.

When we consider the timer the cases will be the timer may be initiated or uninitiated.

When we consider the oven-door then there may be a chance that the door is closed or open.

To check whether it forms a bipolar fuzzy lattice first let us know what the possible cases arise.

The cases will be like, oven-door closed and button pressed, oven-door closed and button unpressed, button pressed and timer initiated, timer initiated and oven-door opened, timer initiated and oven-door closed, oven-door closed and button unpressed etc.

We shall represent B for the button, T for timer, D for oven-door, and join operator as 'ON' and meet operator as 'OFF'.

If we take the operator 'ON' between B and T then it is considered as a button pressed and timer initiated. Then cooking will be done. So, the important thing here in this case is pushing the button. So $B \vee T = B$. If we take the operator 'OFF' between B and T then it is considered the button is unpressed. In this case, there is no question about whether the timer is initiated or not. So $B \wedge T = T$. Similarly, if we take the operator 'ON' between B and D then it is considered the button is pressed and the door is closed. Then cooking will be done. So, $B \vee D = B$. If we take the operator 'OFF' between B and D then it is considered the button is unpressed. In this case, there is no question about whether the timer is initiated or not. So, $B \wedge D = D$.

Similarly, if we take the operator 'ON' between T and D then it is considered the timer is initiated and oven-door is closed. Then cooking will be done. So, $T \vee D = T$.

If we take the operator 'OFF' between T and D then it is considered the timer is uninitiated and door is open. So, $T \wedge D = B$ (here the minimum considered to be B, because the possibility that the timer is uninitiated is that the button is unpressed).

Let us consider a BFS in L, as $B_\delta = \{(B, 0.5, -0.5), (T, 0.4, -0.6)(D, 0.3, -0.5)\}$. In this set the positive value shows the button pressed, timer initiated, door closed, and the negative values show the button unpressed, timer uninitiated, and door open. Now, we check B_δ forms a BF-lattice or not. We can routinely prove that B_δ is a BFL of L.

6. Conclusion and future work

In this article, we have introduced the concepts of BFLs and BFIs of a lattice. Interesting properties of these BFLs and BFIs are developed. Moreover, we investigated the properties of BFIs under lattice homomorphism and an application of BFLs is given.

Our future work is to develop the bipolar fuzzy prime ideals, bipolar fuzzy principal ideals, quotient ideals, bipolar fuzzy filters, and bipolar fuzzy prime filters of a lattice.

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