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Tri-Endomorphisms on BCH-Algebras

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Abstract. In this paper, we use the concept of endomorphisms and bi-endomorphisms as a model to create tri-endomorphisms on of BCH-algebras. We introduce the concepts of left tri-endomorphisms, central tri-endomorphisms, right tri-endomorphisms, and complete tri-endomorphisms of BCH-algebras and provide some properties. In addition, we obtain the properties between those tri-endomorphisms and some subsets of BCH-algebras.

1. Introduction

The algebraic structures of BCK-algebras and BCI-algebras were studied by Iséki and his colleague [4, 5]. In 1983, Hu and Li [3] generalized a new class of algebras from BCI-algebras, namely, a BCH-algebra. Next, Bandru and Rafi [1] introduced a new algebra, called a G-algebra. BCH-algebras are also being studied extensively later, [2, 3].

In this paper, we use the concept of endomorphisms and bi-endomorphisms as a model to create tri-endomorphisms. We introduce the concepts of left tri-endomorphisms, central tri-endomorphisms, right tri-endomorphisms, and complete tri-endomorphisms of BCH-algebras and provide some properties.

Before studying, we will review the definitions and well-known results.

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Definition 1.1. [3] *A BCH-algebra is a non-empty set X with an element 0 and a binary operation $*$ satisfying the following conditions:*

(BCH1) $(\forall x \in X)(x * x = 0)$,

(BCH2) $(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y)$,

(BCH3) $(\forall x, y, z \in X)((x * y) * z = (x * z) * y)$.

In a BCH-algebra $X = (X, *, 0)$, the binary relation $\leq$ on X is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0).$$

Example 1.1. *Let $X = \{0, a, b, c\}$ with the following Cayley table as follows:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	c	c
b	b	0	0	b
c	c	0	0	0

Then $X = (X, *, 0)$ is a BCH-algebra.

Proposition 1.1. *[2,3] Let $X = (X, *, 0)$ be a BCH-algebra. Then the following hold: for all $x, y \in X$,*

(BCH4) $(\forall x, y \in X)(x * (x * y) \leq y)$,

(BCH5) $(\forall x \in X)(x * 0 = 0 \Rightarrow x = 0)$,

(BCH6) $(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y))$,

(BCH7) $(\forall x \in X)(x * 0 = x)$,

(BCH8) $(\forall x, y \in X)((x * y) * x = 0 * y)$,

(BCH9) $(\forall x, y \in X)(x \leq y \Rightarrow 0 * x = 0 * y)$.

For a BCH-algebra $X = (X, *, 0)$, some interesting subsets of X play a significant rule in the investigation of its properties described below.

Definition 1.2. *A non-empty subset Y of a BCH-algebra $X = (X, *, 0)$ is called a subalgebra of X if $x * y \in Y$ for all $x, y \in Y$. A non-empty subset I of a BCH-algebra $X = (X, *, 0)$ is called an ideal of X if*

(1) $0 \in I$,

(2) $(\forall x, y \in X)(x * y \in I, x \in I \Rightarrow y \in I)$.

2. Main results

In this section, we introduce the concepts of left tri-endomorphisms, central tri-endomorphisms, right tri-endomorphisms, and complete tri-endomorphisms of BCH-algebras as follows.

Definition 2.1. *Let $X = (X, *, 0)$ be a BCH-algebra. A mapping $f : X^3 \to X$ is called*

(1) a left tri-endomorphism on X *if* $(\forall w, x, y, z \in X)(f(x * w, y, z) = f(x, y, z) * f(w, y, z))$,

(2) a central tri-endomorphism on X *if* $(\forall w, x, y, z \in X)(f(x, y * w, z) = f(x, y, z) * f(x, w, z))$,

(3) a right tri-endomorphism on X *if* $(\forall w, x, y, z \in X)(f(x, y, z * w) = f(x, y, z) * f(x, y, w))$,

(4) a complete tri-endomorphism on X *if* $(\forall a, b, c, x, y, z \in X)(f(x * a, y * b, z * c) = f(x, y, z) * f(a, b, c))$.

Example 2.1. *In Example 1.1, we define* $f_l : X^3 \to X$ *by*

$$f_l(x, y, z) = \begin{cases} x & \text{if } y = z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then f_l *is a left tri-endomorphism on* X.

Proposition 2.1. *Let* $X = (X, *, 0)$ *be a BCH-algebra and* f_l *be a left tri-endomorphism on* X. *Then*

(1) $(\forall y, z \in X)(f_l(0, y, z) = 0)$,

(2) $(\forall w, x, y, z \in X)(x \leq w \Rightarrow f_l(x, y, z) \leq f_l(w, y, z))$.

Proof. (1) Let $y, z \in X$. Then, by BCH1, we have $f_l(0, y, z) = f_l(0*0, y, z) = f_l(0, y, z)*f_l(0, y, z) = 0$.

(2) Let $w, x, y, z \in X$ be such that $x \leq w$. Then, by (1), we have $0 = f_l(0, y, z) = f_l(x * w, y, z) = f_l(x, y, z) * f_l(w, y, z)$. Hence, $f_l(x, y, z) \leq f_l(w, y, z)$. □

Similarly, the properties of central and right tri-endomorphisms are easily obtained.

Proposition 2.2. *Let* $X = (X, *, 0)$ *be a BCH-algebra and* f_c *be a central tri-endomorphism on* X. *Then*

(1) $(\forall x, z \in X)(f_c(x, 0, z) = 0)$,

(2) $(\forall w, x, y, z \in X)(y \leq w \Rightarrow f_c(x, y, z) \leq f_c(x, w, z))$.

Proposition 2.3. *Let* $X = (X, *, 0)$ *be a BCH-algebra and* f_r *be a right tri-endomorphism on* X. *Then*

(1) $(\forall x, y \in X)(f_r(x, y, 0) = 0)$,

(2) $(\forall w, x, y, z \in X)(z \leq w \Rightarrow f_r(x, y, z) \leq f_r(x, y, w))$.

Theorem 2.1. *Let* $X = (X, *, 0)$ *be a BCH-algebra and* f *be a complete tri-endomorphism on* X. *Then*

(1) $f(0, 0, 0) = 0$,

(2) if S *is a subalgebra of* X, *then* $f(S^3)$ *is also a subalgebra of* X,

(3) if S *is an ideal of* X *and* f *is bijective, then* $f(S^3)$ *is also an ideal of* X,

(4) if f *is a left tri-endomorphism on* X, *then* $f(x, y, z) * f(x, 0, 0) = 0$ *for any* $x, y, z \in X$,

(5) if f *is a central tri-endomorphism on* X, *then* $f(x, y, z) * f(0, y, 0) = 0$ *for any* $x, y, z \in X$,

(6) if f *is a right tri-endomorphism on* X, *then* $f(x, y, z) * f(0, 0, z) = 0$ *for any* $x, y, z \in X$,

(7) if f is a left and right (central and right, left and central) tri-endomorphism on X, then $f(x, y, z) = 0$ for any $x, y, z \in X$, i.e., f is the zero map.

Proof. (1) By BCH1, we have $f(0, 0, 0) = f(0 * 0, 0 * 0, 0 * 0) = f(0, 0, 0) * f(0, 0, 0) = 0$.

(2) Suppose that S is a subalgebra of X. Let $a, b \in f(S^3)$. Then there exist $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S^3$ such that $a = f(x_1, y_1, z_1)$ and $b = f(x_2, y_2, z_2)$. Thus $a * b = f(x_1, y_1, z_1) * f(x_2, y_2, z_2) = f(x_1 * x_2, y_1 * y_2, z_1 * z_2) \in f(S^3)$. Hence, $f(S^3)$ is a subalgebra of X.

(3) Suppose that S is an ideal of X and f is bijective. Since $0 \in S$ and by (1), we have $0 = f(0, 0, 0) \in f(S^3)$. Assume that $x*y \in f(S^3)$ and $x \in f(S^3)$. There exist $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S^3$ such that $x * y = f(x_1, y_1, z_1)$ and $x = f(x_2, y_2, z_2)$. Since f is surjective, there exists $(a, b, c) \in X^3$ such that $y = f(a, b, c)$. Thus $f(S^3) \ni f(x_1, y_1, z_1) = x * y = f(x_2, y_2, z_2) * f(a, b, c) = f(x_2 * a, y_2 * b, z_2 * c)$. Since f is injective, we have $x_2 * a, y_2 * b, z_2 * c \in S$. Since S is an ideal of X, we get $a, b, c \in S$. Thus $y = f(a, b, c) \in f(S^3)$. Hence, $f(S^3)$ is an ideal of X.

(4)-(6) It is obvious from Propositions 2.1-2.3.

(7) Suppose that f is a left and right tri-endomorphism on X. Let $x, y, z \in X$. Then, by Propositions 2.1 and 2.3, BCH1, BCH7 $0 = f(0, y, z) = f(x*x, y*0, z*0) = f(x, y, z)*f(x, 0, 0) = f(x, y, z)*0 = f(x, y, z)$. Hence, f is the zero map on X. □

Let $T_l(X)$ (resp., $T_c(X)$, $T_r(X)$ and $T(X)$) be the set of all left tri-endomorphisms (resp., right, central and complete tri-endomorphisms) on a BCH-algebra $X = (X, *, 0)$. We define an operation $\star$ on $T_l(X)$ by $(\forall x, y, z \in X)((f \star g)(x, y, z) = f(x, y, z) * g(x, y, z))$. Let $f \in T_l(X)$ and $x, y, z \in X$. Then $(f \star f)(x, y, z) = f(x, y, z) * f(x, y, z) = 0$. This means that $f \star f = 0_X$, where $0_X : X^3 \to X$ is a function that maps all members to 0. Let $f, g \in T_l(X)$ be such that $f \star g = 0_X$ and $g \star f = 0_X$. Then for all $x, y, z \in X$, $0 = (f \star g)(x, y, z) = f(x, y, z) * g(x, y, z)$ and $0 = (g \star f)(x, y, z) = g(x, y, z) * f(x, y, z)$. Since $g(x, y, z), f(x, y, z) \in X$, we have $f(x, y, z) = g(x, y, z)$ for all $x, y, z \in X$. Hence, $f = g$. Let $f, g, h \in T_l(X)$ and $x, y, z \in X$. Then $((f \star g) \star h)(x, y, z) = (f \star g)(x, y, z) * h(x, y, z) = \big(f(x, y, z)*g(x, y, z)\big)*h(x, y, z) = \big(f(x, y, z)*h(x, y, z)\big)*g(x, y, z) = (f \star h)(x, y, z)*g(x, y, z) = ((f \star h) \star g)(x, y, z)$. Hence, $(f \star g) \star h = (f \star h) \star g$.

Theorem 2.2. *$(T_l(X), \star, 0_X), (T_c(X), \star, 0_X), (T_r(X), \star, 0_X)$, and $(T(X), \star, 0_X)$ are BCH-algebras.*

Let $X = (X, *, 0)$ be a BCH-algebra. We define the binary operation $\diamond$ on X^3 as follows: $(\forall (a, b, c), (x, y, z) \in X^3)((a, b, c) \diamond (x, y, z) = (a * x, b * y, c * z))$. Then $X^3 = (X, \diamond, (0, 0, 0))$ is a BCH-algebra.

Theorem 2.3. *Let $X = (X, *, 0)$ be a BCH-algebra and S_1, S_2, S_3 be subsets of X. Then*

(1) $S_1 \times S_2 \times S_3$ is a subalgebra of X^3 if and only if S_1, S_2 and S_3 are subsets of X,

(2) $S_1 \times S_2 \times S_3$ is an ideal of X^3 if and only if S_1, S_2 and S_3 are ideals of X.

Proof. (1) Suppose that $S_1 \times S_2 \times S_3$ is a subalgebra of X^3. Firstly, we will show that S_1 is a subalgebra of X. Let $a, b \in S_1$. Let $x \in S_2$ and $u \in S_3$. Then $(a, x, u), (b, x, u) \in S_1 \times S_2 \times S_3$. Thus $(a * b, 0, 0) = (a * b, x * x, u * u) = (a, x, u) \diamond (b, x, u) \in S_1 \times S_2 \times S_3$, that is, $a * b \in S_1$. Hence, S_1 is a subalgebra of X. On the other hand, we can show that S_2 and S_3 are subalgebras of X.

Conversely, let $(x, y, z), (a, b, c) \in S_1 \times S_2 \times S_3$. Then $x * a \in S_1, y * b \in S_2$, and $z * c \in S_3$, so $(x, y, z) \diamond (a, b, c) = (x * a, y * b, z * c) \in S_1 \times S_2 \times S_3$. Hence, $S_1 \times S_2 \times S_3$ is a subalgebra of X^3.

(2) Suppose that $S_1 \times S_2 \times S_3$ is an ideal of X^3. Since $(0, 0, 0) \in S_1 \times S_2 \times S_3$, we have $0 \in S_i$ for all $i = 1, 2, 3$. Assume that $a * x \in S_1$ and $a \in S_1$. Let $b \in S_2$ and $c \in S_3$. Then $(a, b, c) \in S_1 \times S_2 \times S_3$ and $(x, b, c) \in X^3$. Thus $(a, b, c) \diamond (x, b, c) = (a * x, b * b, c * c) = (a * x, 0, 0) \in S_1 \times S_2 \times S_3$. Since $S_1 \times S_2 \times S_3$ is an ideal of X^3, we have $(x, b, c) \in S_1 \times S_2 \times S_3$, that is, $x \in S_1$. Hence, S_1 is an ideal of X. Similarly, we can show that S_2 and S_3 are ideals of X.

Conversely, suppose that S_1, S_2 and S_3 are ideals of X. Since $0 \in S_i$ for all $i = 1, 2, 3$, we have $(0, 0, 0) \in S_1 \times S_2 \times S_3$. Assume that $(a, b, c) * (x, y, z) \in S_1 \times S_2 \times S_3$ and $(a, b, c) \in S_1 \times S_2 \times S_3$. We get $(a * x, b * y, c * z) \in S_1 \times S_2 \times S_3$. Since $a * x, a \in S_1$, we have $x \in S_1$. Moreover, we can obtain that $y \in S_2$ and $z \in S_3$. This implies that $(x, y, z) \in S_1 \times S_2 \times S_3$. Hence, $S_1 \times S_2 \times S_3$ is an ideal of X^3. □

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