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Evolutes of Fronts in de Sitter and Hyperbolic Spheres

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Abstract. The evolute of a regular curve is a classical object from the viewpoint of differential geometry. We study some types of curves such as framed curves, framed immersion curves, frontal curves and front curves in 2-dimensional de Sitter and hyperbolic spaces. Also, we investigate the evolutes and some of their properties of fronts at singular points under some conditions. Finally, some computational examples in support of our main results are given and plotted.

1. Introduction

In 1915, Einstein formulated general relativity as a theory of space, time and gravitation in semi-Euclidean space. However, this subject has remained dormant for much of its history because its understanding requires advanced mathematics knowledge. Since the end of the twentieth century, semi-Euclidean geometry has been an active area of mathematical research, and it has been applied to a variety of subjects related to differential geometry and general relativity.

It is well known that many important results in the theory of curves in $\mathbb{R}^3$ were initiated by G. Monge and G. Darboux pionnered the moving frame idea. Thereafter, Frenet defined his moving frame and special equations which are playing an important role in mechanics and kinematics as well as in differential geometry [1].

At the beginning of the twentieth century, A. Einstein's theory opened a door to use of new geometries. One of them, Minkowski space-time, which is simultaneously the geometry of special relativity. It is worth mentioning that the importance of the theory of singularity as a developing area which is

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related to nonlinear sciences, it has been extensively applied in studying the classifications of singularities associated with some objects in Euclidean and semi-Euclidean spaces [2–4]. Therefore, it has been a field of research for many researchers.

In this paper, we focus on the evolutes of curves at singular points in de Sitter and hyperbolic spheres. The evolute of a plane curve is defined to be the locus of the center of its osculating circles [5]. In particular, the evolute of a regular curve is a classical object from the viewpoint of differential geometry. The evolute of a regular curve without inflection points is given by, not only, the locus of all its centers of curvature but also the envelope of its normal lines. The properties of evolutes can be discussed by Frenet-Serret formulas, distance squared functions and the theories of Lagrangian and Legendre singularities. In general, there exist singular points along the evolute of a regular curve and the singular points corresponding to the vertices of a regular curve. There are at least four vertices for a simple closed curve. One can not define the evolutes of curves at singular points. However, we can define evolutes of fronts under some conditions. In [6, 7], T. Fukunaga and M. Takahashi defined Legendre curves in Euclidean plane and studied evolutes of Legendre curves. Moreover, S. Izumiya, D. He pei, T.Sano and E. Torii defined the evolute curve in hyperbolic 2-space and found its equation (see [7]).

The paper can be organized as follows: Section 3 presents a framed curve and gives its moving frame in de Sitter sphere. Moreover, we define a pair of smooth functions of this curve as a geodesic curvature for a regular curve. We define evolutes of fronts in de Sitter sphere. The evolute of a front is a generalization of the notion of an evolute of a regular curve. Therefore, we discuss some properties of evolutes without inflection points. By the representation, we give properties for an evolute of the front. For example, the evolute of a front is also a front (see Theorem 3.1). In section 4, similar to the way that considered in the study of the framed curves, fronts and the evolutes in de Sitter sphere, we do it in the hyperbolic sphere, see Theorem (4.1). We shall assume throughout the whole paper that all manifolds and maps are C^∞ unless the contrary is explicitly stated.

2. Geometric meanings and basis concepts

In this section, we present some of classical differential geometric properties of de Sitter and hyperbolic spaces of plane curves. We adopt $\mathbb{S}^2_1$ and $\mathbb{H}^2_0$ as models of de Sitter and hyperbolic spheres in Minkowski 3-space $\mathbb{E}^3_1$, respectively. Since $\mathbb{S}^2_1$ and $\mathbb{H}^2_0$ are a Riemannian manifolds, so the explicit differential geometry of the curves in these spheres is analogous to the differential geometry of the curves in the Euclidean plane (for more details see [5, 8]).

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, and $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The pseudo scalar product of x and y is defined by $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$. We call $(\mathbb{R}^3, \langle , \rangle)$ a 3-dimensional pseudo Euclidean space, or Minkowski 3-space. We write $\mathbb{E}^3_1$ instead of $(\mathbb{R}^3, \langle , \rangle)$. We say that a vector x in $\mathbb{E}^3_1$ is spacelike, lightlike or timelike if

$\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$, respectively. Now, we define two spheres in $\mathbb{E}^3_1$ as follows:

$$\mathbb{Q}^2_\epsilon = \begin{cases} \mathbb{H}^2_0 = \{x \in \mathbb{E}^3_1 \mid -x_1^2 + x_2^2 + x_3^2 = -1\}, & if\ \epsilon = - \\ \mathbb{S}^2_1 = \{x \in \mathbb{E}^3_1 \mid -x_1^2 + x_2^2 + x_3^2 = 1\}, & if\ \epsilon = +, \end{cases}$$

and we take

$$\mathbb{H}^2_0 = \begin{cases} \mathbb{H}^2_+ = \{x \in \mathbb{E}^3_1 \mid -x_1^2 + x_2^2 + x_3^2 = -1, \ \ x_1 \geq 1\} \\ \mathbb{H}^2_- = \{x \in \mathbb{E}^3_1 \mid -x_1^2 + x_2^2 + x_3^2 = -1, \ \ x_1 \leq -1\}, \end{cases}$$

where $\mathbb{H}^2_0 = \mathbb{H}^2_+ \cup \mathbb{H}^2_-$. We call $\mathbb{H}^2_0$ a hyperbolic sphere and $\mathbb{S}^2_1$ a de Sitter sphere.

Let $\gamma : I \longrightarrow \mathbb{Q}^2_\epsilon \subset \mathbb{E}^3_1$; $\quad \gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a smooth regular curve in $\mathbb{Q}^2_\epsilon$ $(i.e., \gamma'(t) \neq 0)$ for any $t \in I$, where I is an open interval. It is easy to show that $\langle \gamma'(t), \gamma'(t)\rangle > 0$. Throughout the remainder in this paper, we denote the parameter s of γ as the arc-length parameter. Let us denote $\mathbf{T}(s) = \dot{\gamma}(s)$, and we call $\mathbf{T}(s)$ a unit tangent vector of γ at s.

For any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the pseudo vector product of x and y is defined as follows:

$$x \wedge y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Here, we set a vector $\mathbf{E}(s) = \gamma(s) \wedge \mathbf{T}(s)$. By definition, we can calculate that $\langle \mathbf{E}(s), \mathbf{E}(s)\rangle = 1$ and $\langle \gamma(s), \gamma(s)\rangle = -1$. Also, we can show that $\mathbf{T}(s) \wedge \mathbf{E}(s) = -\gamma(s)$ and $\gamma(s) \wedge \mathbf{E}(s) = -\mathbf{T}(s)$. Therefore, we have a pseudo-orthonormal frame $\{\gamma(s), \mathbf{T}(s), \mathbf{E}(s)\}$ along $\gamma(s)$. The de Sitter Frenet-Serret formula of plane curve are:

$$\begin{bmatrix} \dot{\gamma}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_g \\ 0 & \kappa_g & 0 \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \mathbf{T}(s) \\ \mathbf{E}(s) \end{bmatrix}, \tag{2.1}$$

where κ_g is the geodesic curvature of γ in $\mathbb{S}^2_1$, which is given by

$$\kappa_g(s) = det(\gamma(s)\ \mathbf{T}(s)\ \dot{\mathbf{T}}(s)). \tag{2.2}$$

For the general parameter t, we get $\mathbf{T}(t) = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$ and $\mathbf{E}(t) = \gamma(t) \wedge \mathbf{T}(t)$. Then, de Sitter Frenet-Serret formula of $\gamma(t)$ is expressed as:

$$\begin{bmatrix} \gamma'(t) \\ \mathbf{T}'(t) \\ \mathbf{E}'(t) \end{bmatrix} = \begin{bmatrix} 0 & \|\gamma'(t)\| & 0 \\ -\|\gamma'(t)\| & 0 & \|\gamma'(t)\|\kappa_g \\ 0 & \|\gamma'(t)\|\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \gamma(t) \\ \mathbf{T}(t) \\ \mathbf{E}(t) \end{bmatrix}, \tag{2.3}$$

where

$$\kappa_g(t) = \frac{det\,(\gamma(t)\ \gamma'(t)\ \gamma''(t))}{\|\gamma'(t)\|^3}. \tag{2.4}$$

Also, the hyperbolic Frenet-Serret formula is given by:

$$\begin{bmatrix} \dot{\gamma}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \mathbf{T}(s) \\ \mathbf{E}(s) \end{bmatrix}, \tag{2.5}$$

where κ_g is the geodesic curvature of the curve γ in $\mathbb{H}_0^2$, which is defined as

$$\kappa_g(s) = det(\gamma(s)\ \mathbf{T}(s)\ \dot{\mathbf{T}}(s)). \tag{2.6}$$

We have the following hyperbolic Frenet-Serret formula of $\gamma(t)$

$$\begin{bmatrix} \gamma'(t) \\ \mathbf{T}'(t) \\ \mathbf{E}'(t) \end{bmatrix} = \begin{bmatrix} 0 & \|\gamma'(t)\| & 0 \\ \|\gamma'(t)\| & 0 & \|\gamma'(t)\|\kappa_g \\ 0 & -\|\gamma'(t)\|\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \gamma(t) \\ \mathbf{T}(t) \\ \mathbf{E}(t) \end{bmatrix}, \tag{2.7}$$

Definition 2.1. *Under the assumption $\kappa_g^2 \neq \pm 1$, the evolute of a regular curve γ is defined as $\mathcal{E}_\gamma : I \longrightarrow \mathbb{Q}_\epsilon^2$;*

$$\mathcal{E}_\gamma(t) = \frac{1}{\sqrt{|\kappa_g^2(t) - 1|}} \left(\kappa_g(t)\gamma(t) + \epsilon \mathbf{E}(t)\right); \tag{2.8}$$

$\mathcal{E}_\gamma$ is called hyperbolic evolute or de Sitter evolute of γ when $\epsilon = 1$ or $\epsilon = -1$, respectively (see [5, 8]).

Remark 2.1. *$\mathcal{E}_\gamma(t)$ is located in $\mathbb{H}_0^2$ with $\kappa_g^2(t) > 1$, and it is in $\mathbb{S}_1^2$ with $\kappa_g^2(t) < 1$.*

3. Evolutes of fronts in de Sitter sphere $\mathbb{S}_1^2$

If γ has a singular point, we can not construct a moving frame of γ in a traditional way. However, we could define a moving frame of a front curve. For the case of Euclidean plane, there are some creative works (see [6, 7, 9]). Now, we give the following definitions.

Definition 3.1. *We say that $(\gamma, \nu) : I \longrightarrow \mathbb{S}_1^2 \times \mathbb{S}_1^2$ is a framed curve, if $\langle \gamma(t), \nu(t) \rangle = 0$ and $\langle \gamma'(t), \nu(t) \rangle = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, namely, $(\gamma'(t), \nu'(t)) \neq (0, 0)$, we call (γ, ν) a framed immersion curve.*

Definition 3.2. *We say that $\gamma : I \longrightarrow \mathbb{S}_1^2$ is a frontal curve if there exists a smooth mapping $\nu : I \longrightarrow \mathbb{S}_1^2$ such that (γ, ν) is a framed curve. We also say that $\gamma : I \longrightarrow \mathbb{S}_1^2$ is a front curve if there exists a smooth mapping $\nu : I \longrightarrow \mathbb{S}_1^2$ such that (γ, ν) is a framed immersion curve.*

Throughout this paper, we assume that the pair (γ, ν) is co-oriented and the singular points of γ are finite.

Let $(\gamma, \nu) : I \longrightarrow \mathbb{S}_1^2 \times \mathbb{S}_1^2$ be a framed curve. If γ is singular at t_0, then we can't define a frame in a traditional way. However, ν always exists even if t is a singular point of γ. We take $\mu = \nu \wedge \gamma$ and call the pair $\{\gamma, \nu, \mu\}$ a moving frame of γ and then, de Sitter Frenet-Serret formula is given by:

$$\begin{bmatrix} \gamma'(t) \\ \nu'(t) \\ \mu'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & m(t) \\ 0 & 0 & n(t) \\ -m(t) & n(t) & 0 \end{bmatrix} \begin{bmatrix} \gamma(t) \\ \nu(t) \\ \mu(t) \end{bmatrix},$$

where $n(t) = \langle \nu'(t), \mu(t) \rangle$, $\nu(t)$ and $\mu(t)$ are both unit spacelike vectors. We declare that $(\gamma, -\nu)$ is also a framed curve. In this case, $m(t)$ dose not change, but $n(t)$ changes to $-n(t)$. If (γ, ν) is a framed immersion, we have $(m(t), n(t)) \neq (0, 0)$ for each $t \in I$. The pair (m, n) is an important pair of functions of the framed curves as the geodesic curvature of a regular curve. We call the pair (m, n) a geodesic curvature of the framed curve. Also, we have $\nu \wedge \mu = \gamma$ and $\gamma \wedge \mu = \nu$ (for more details see [8, 10]).

In what follows, some properties of the meant curves are introduced.

Proposition 3.1. *If γ be a regular curve and $(\gamma, \nu) : I \longrightarrow \mathbb{S}^2_1 \times \mathbb{S}^2_1$ be its framed curve, then the relationship between their geodesic curvatures is expressed as:*

$$\kappa_g(t) = -\frac{n(t)}{|m(t)|}. \tag{3.1}$$

Proof. By direct calculations, we have

$$\begin{aligned} \gamma'(t) &= m(t)\mu(t), \\ \gamma''(t) &= m'(t)\mu(t) + m(t)\mu'(t) \\ &= m'(t)\mu(t) + m(t)(-m(t)\gamma(t) + n(t)\nu(t)), \end{aligned}$$

where

$$\begin{aligned} \kappa_g(t) &= \frac{\det(\gamma(t), \gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3} \\ &= \frac{\det\left(\gamma(t), m\mu(t), -m^2\gamma(t) + mn\nu(t) + m'\mu(t)\right)}{|m^3|}, \end{aligned}$$

so, we get

$$\kappa_g(t) = -\frac{n(t)}{|m(t)|}.$$

For a framed immersion (γ, ν), we say that t_0 is an inflection point of the front γ if $n(t_0) = 0$. And then the condition $m(t_0) \neq 0$ and $n(t_0) = 0$, is equivalent to the condition $\kappa_g(t_0) = 0$. $\square$

If γ is not a regular curve, then we can not define the evolute as in equation (2.8), since the geodesic curvature may be divergence at a singular point. However, we can give the definition of the evolute of a front and then focus on its properties. Hence, the notion of a parallel curve of γ can be presented as follows:

Let $(\gamma,\nu): I \longrightarrow \mathbb{S}^2_1 \times \mathbb{S}^2_1$ be a framed curve which has the geodesic curvature (m,n). So, we can define a parallel curve $\gamma_\upsilon : I \longrightarrow \mathbb{S}^2_1$ of γ as follows:

$$\gamma_\upsilon(t) = \frac{1}{\sqrt{|\upsilon^2-1|}}\left(\gamma(t)+\upsilon\nu(t)\right), \tag{3.2}$$

where $\upsilon \in \mathbb{R}$ and $\upsilon \neq \pm 1$.

Proposition 3.2. *If γ_υ is a regular curve, then*

$$\kappa_{g\upsilon}(t) = \frac{-n(t)-\upsilon m(t)}{|m(t)+\upsilon n(t)|}. \tag{3.3}$$

Proof. From Eq.(3.2), we have

$$\gamma'_\upsilon(t) = \frac{m+\upsilon n}{\sqrt{|\upsilon^2-1|}}\mu(t),$$
$$\gamma''_\upsilon(t) = \frac{-m^2-\upsilon mn}{\sqrt{|\upsilon^2-1|}}\gamma(t) + \frac{mn+\upsilon n^2}{\sqrt{|\upsilon^2-1|}}\nu(t) + \frac{m'+\upsilon n'}{\sqrt{|\upsilon^2-1|}}\mu(t),$$

then, we find

$$\gamma_\upsilon \wedge \gamma'_\upsilon = \frac{1}{(\upsilon^2-1)}\begin{vmatrix} -\gamma(t) & \nu(t) & \mu(t) \\ 1 & \upsilon & 0 \\ 0 & 0 & m+\upsilon n \end{vmatrix}$$

$$= \frac{-\upsilon(m+\upsilon n)\gamma(t)-(m+\upsilon n)\nu(t)}{(\upsilon^2-1)}.$$

Since

$$\kappa_{g\upsilon}(t) = \frac{\det\left(\gamma_\upsilon(t),\gamma'_\upsilon(t),\gamma''_\upsilon(t)\right)}{\|\gamma'_\upsilon(t)\|^3}.$$

We obtain

$$|m(t)+\upsilon n(t)|\kappa_{g\upsilon}(t) = (-n(t)-\upsilon m(t)).$$

Thus, this completes the proof. □

Proposition 3.3. *For a framed immersion curve $(\gamma,\nu): I \longrightarrow \mathbb{S}^2_1\times\mathbb{S}^2_1$, the parallel curve $\gamma_\upsilon : I \longrightarrow \mathbb{S}^2_1$ is a front for each $\upsilon \neq \pm 1$.*

Proof. We take $\nu_\upsilon : I \longrightarrow \mathbb{S}^2_1$ by

$$\nu_\upsilon(t) = \frac{1}{\sqrt{|\upsilon^2-1|}}\left(\upsilon\gamma(t)+\nu(t)\right),$$

since

$$\begin{cases} \gamma_\upsilon(t) = \dfrac{1}{\sqrt{|\upsilon^2-1|}}\left(\gamma(t)+\upsilon\nu(t)\right), \\ \gamma'_\upsilon(t) = \dfrac{1}{\sqrt{|\upsilon^2-1|}}\left(\gamma'(t)+\upsilon\nu'(t)\right). \end{cases}$$

If $\gamma_\upsilon'(t_0) = 0$ at a point $t_0 \in I$, then we have

$$\gamma'(t_0) + \upsilon\nu'(t_0) = 0.$$

Also, if $\nu'(t_0) = 0$, then $\gamma'(t_0) = 0$. It is contradicted with the fact that (γ, ν) is a framed immersion and hence $(\gamma_\upsilon, \nu_\upsilon)$ is a framed immersion. By $\|\nu(t)\| = 1$, we have $\langle \nu(t), \nu'(t)\rangle = 0$. Then

$$\langle \gamma_\upsilon'(t), \nu_\upsilon(t)\rangle = \frac{1}{(\upsilon^2 - 1)}\langle \gamma' + \upsilon\nu', \upsilon\gamma + \nu\rangle = 0,$$

therefore, it leads to the curve γ_υ is a front. □

Proposition 3.4. *Let (γ, ν) be a framed curve. If γ is a regular curve and $\upsilon \neq 1/\kappa_g$, then a parallel curve γ_υ is also a regular curve and its evolute is given by*

$$\mathcal{E}_{\gamma\upsilon}(t) = -\mathcal{E}_\gamma(t). \tag{3.4}$$

Proof. Since

$$\begin{aligned}
\gamma_\upsilon(t) &= \frac{1}{\sqrt{|\upsilon^2 - 1|}}\left(\gamma(t) + \upsilon\mathbf{E}(t)\right),\\
\gamma_\upsilon'(t) &= \frac{\|\gamma'\|}{\sqrt{|\upsilon^2 - 1|}}\left(1 + \upsilon\kappa_g\right)\mathbf{T}(t),\\
\gamma_\upsilon''(t) &= \frac{\|\gamma'\|}{\sqrt{|\upsilon^2 - 1|}}\left(-\|\gamma'\|(1 + \upsilon\kappa_g)\gamma(t) + \upsilon\kappa_g'\mathbf{T}(t) + \|\gamma'\|\kappa_g(1 + \upsilon\kappa_g)\mathbf{E}(t)\right).
\end{aligned}$$

By the assumption $\upsilon \neq 1/\kappa_g$, γ_υ is a regular curve. Therefore, we get

$$\gamma_\upsilon \wedge \gamma_\upsilon' = \frac{1}{(\upsilon^2 - 1)}\begin{vmatrix} -\gamma(t) & \mathbf{T}(t) & \mathbf{E}(t)\\ 1 & 0 & \upsilon\\ 0 & \|\gamma'\|(1 + \upsilon\kappa_g) & 0 \end{vmatrix}$$

$$= \frac{\|\gamma'\|\upsilon(1 + \upsilon\kappa_g)\gamma(t) + \|\gamma'\|(1 + \upsilon\kappa_g)\mathbf{E}(t)}{(\upsilon^2 - 1)},$$

and hence

$$\langle \gamma_\upsilon \wedge \gamma_\upsilon', \gamma_\upsilon''\rangle = \frac{\|\gamma'\|^3\upsilon(1 + \upsilon\kappa_g)^2 + \|\gamma'\|^3\kappa_g(1 + \upsilon\kappa_g)^2}{|\upsilon^2 - 1|^{\frac{3}{2}}},$$

we get

$$\kappa_{g\upsilon}(t) = \frac{\kappa_g + \upsilon}{|1 + \upsilon\kappa_g|},$$

and

$$\mathbf{T}_\upsilon(t) = \frac{\gamma_\upsilon'}{\|\gamma_\upsilon'\|} = \frac{1 + \upsilon\kappa_g}{|1 + \upsilon\kappa_g|}\mathbf{T}(t).$$

Since $\mathbf{E}_\upsilon = \gamma_\upsilon \wedge \mathbf{T}_\upsilon$, we obtain

$$\mathbf{E}_\upsilon(t) = \frac{1 + \upsilon\kappa_g}{|1 + \upsilon\kappa_g|}\frac{1}{\sqrt{|\upsilon^2 - 1|}}\left(\mathbf{E}(t) + \upsilon\gamma(t)\right).$$

Thus, from (2.8) we find

$$
\begin{aligned}
\mathcal{E}_{\gamma_\upsilon}(t) &= \frac{1}{\sqrt{\left|\kappa_{g\upsilon}^2(t)-1\right|}}\left(\kappa_{g\upsilon}(t)\gamma_\upsilon(t)-\mathbf{E}_\upsilon(t)\right) \\
&= \frac{1}{\sqrt{\left|\left(\frac{\upsilon+\kappa_g}{|1+\upsilon\kappa_g|}\right)^2-1\right|}}\left(\frac{\upsilon+\kappa_g}{|1+\upsilon\kappa_g|}\frac{(\gamma(t)+\upsilon\mathbf{E}(t))}{\sqrt{|\upsilon^2-1|}}-\frac{1+\upsilon\kappa_g}{|1+\upsilon\kappa_g|}\frac{(\mathbf{E}(t)+\upsilon\gamma(t))}{\sqrt{|\upsilon^2-1|}}\right) \\
&= \frac{1}{\sqrt{\left|\left(\frac{\upsilon+\kappa_g}{|1+\upsilon\kappa_g|}\right)^2-1\right|}}\frac{1}{|1+\upsilon\kappa_g|\sqrt{|\upsilon^2-1|}}\left((1-\upsilon^2)\kappa_g\gamma(t)-(1-\upsilon^2)\mathbf{E}(t)\right) \\
&= \frac{1}{\sqrt{\left|(\upsilon^2-1)-\kappa_g^2(\upsilon^2-1)\right|}}\frac{(1-\upsilon^2)}{\sqrt{|\upsilon^2-1|}}\left(\kappa_g\gamma(t)-\mathbf{E}(t)\right) \\
&= -\frac{1}{\sqrt{\left|\kappa_g^2(t)-1\right|}}\left(\kappa_g(t)\gamma(t)-\mathbf{E}(t)\right) \\
&= -\mathcal{E}_\gamma(t).
\end{aligned}
$$

Thus, the proof is completed. □

Definition 3.3. *Let $(\gamma,\nu): I \longrightarrow \mathbb{S}_1^2\times\mathbb{S}_1^2$ be a framed immersion curve. We define an evolute $\mathcal{E}_\gamma : I \longrightarrow \mathbb{S}_1^2$ of γ as follows:*

If t is a regular point, then

$$
\mathcal{E}_\gamma(t) = \frac{1}{\sqrt{\left|\kappa_g^2(t)-1\right|}}\left(\kappa_g(t)\gamma(t)-\mathbf{E}(t)\right). \tag{3.5}
$$

If t_0 is a singular point, for any $t\in(t_0-\delta, t_0+\delta)$, we get

$$
\mathcal{E}_{\gamma_\upsilon}(t) = \frac{-1}{\sqrt{\left|\kappa_{g\upsilon}^2(t)-1\right|}}\left(\kappa_{g\upsilon}(t)\gamma_\upsilon(t)-\mathbf{E}_\upsilon(t)\right), \tag{3.6}
$$

where δ is a sufficiently small positive real number and $\upsilon\in\mathbb{R}$ satisfies the condition $\upsilon\neq 1/\kappa_g(t)$.

Now, we give another representation of the evolute by using the moving frame $\{\gamma(t),\nu(t),\mu(t)\}$ and its geodesic curvature $\{m(t), n(t)\}$.

Theorem 3.1. *Under the condition $|m(t)|\neq|n(t)|$, the evolute of a front curve $\mathcal{E}_\gamma(t): I\longrightarrow\mathbb{S}_1^2$ is represented by*

$$
\mathcal{E}_\gamma(t) = \frac{1}{\sqrt{|n^2(t)-m^2(t)|}}\left(-n(t)\gamma(t)+m(t)\nu(t)\right), \tag{3.7}
$$

and $\mathcal{E}_\gamma(t)$ is a front curve.

Proof. (i) Suppose that γ is a regular curve. Since $\gamma'(t) = m(t)\mu(t)$, we have $|m(t)| \neq 0$ and

$$\mathbf{T}(t) = \frac{m(t)}{|m(t)|}\mu(t), \quad \mathbf{E}(t) = -\frac{m(t)}{|m(t)|}\nu(t).$$

From Eqs.(3.1) and (3.5), we get

$$\begin{aligned}
\mathcal{E}_\gamma(t) &= \frac{1}{\sqrt{|\kappa_g^2(t) - 1|}}\left(\kappa_g(t)\gamma(t) - \mathbf{E}(t)\right) \\
&= \frac{1}{\sqrt{\left|\left(-\frac{n(t)}{|m(t)|}\right)^2 - 1\right|}}\left(-\frac{n(t)}{|m(t)|}\gamma(t) + \frac{m(t)}{|m(t)|}\nu(t)\right) \\
&= \frac{1}{\sqrt{|n^2(t) - m^2(t)|}}\left(-n(t)\gamma(t) + m(t)\nu(t)\right).
\end{aligned}$$

(ii) Suppose that t_0 is a singular point of γ and consider γ_υ in de Sitter sphere, also we know that γ_υ is a regular curve around the neighbourhood of t_0 with $\upsilon \neq 1/\kappa_g(t)$.

From Eq.(3.2), we get

$$\gamma_\upsilon'(t) = \frac{m + \upsilon n}{\sqrt{|\upsilon^2 - 1|}}\mu(t),$$

then, $|m + \upsilon n| \neq 0$ and

$$\mathbf{T}_\upsilon(t) = \frac{m + \upsilon n}{|m + \upsilon n|}\mu(t),$$

where $\mathbf{E}_\upsilon(t) = \gamma_\upsilon(t) \wedge \mathbf{T}_\upsilon(t)$, we have

$$\mathbf{E}_\upsilon(t) = \frac{m + \upsilon n}{|m + \upsilon n|}\frac{1}{\sqrt{|\upsilon^2 - 1|}}\left(-\upsilon\gamma(t) - \nu(t)\right).$$

Therefore, from Eq.(3.3), we find

$$\kappa_{g\upsilon} = -\frac{n + \upsilon m}{|m + \upsilon n|},$$

and from Eqs.(3.4) and (3.6), we get

$$\begin{aligned}
\mathcal{E}_\gamma(t) &= -\mathcal{E}_{\gamma\upsilon}(t) \\
&= \frac{1}{\sqrt{|\kappa_{g\upsilon}^2(t) - 1|}}\left(\kappa_{g\upsilon}(t)\gamma_\upsilon(t) - \mathbf{E}_\upsilon(t)\right) \\
&= \frac{1}{\sqrt{\left|\left(-\frac{(n + \upsilon m)}{|m + \upsilon n|}\right)^2 - 1\right|}}\left(\left(-\frac{(n + \upsilon m)}{|m + \upsilon n|}\right)\frac{(\gamma(t) + \upsilon\nu(t))}{\sqrt{|\upsilon^2 - 1|}} + \frac{m + \upsilon n}{|m + \upsilon n|}\frac{(\upsilon\gamma(t) + \nu(t))}{\sqrt{|\upsilon^2 - 1|}}\right) \\
&= \frac{1}{\sqrt{|(n + \upsilon m)^2 - (m + \upsilon n)^2|}}\frac{\left(n\gamma(t) + \upsilon^2 m\nu(t) - \upsilon^2 n\gamma(t) - m\nu(t)\right)}{\sqrt{|\upsilon^2 - 1|}} \\
&= \frac{1}{\sqrt{|(\upsilon^2 - 1)(n^2 - m^2)|}}\frac{(\upsilon^2 - 1)(m\nu(t) - n\gamma(t))}{\sqrt{|\upsilon^2 - 1|}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{|n^2 - m^2|}} \left(-n\gamma(t) + m\nu(t)\right) \\
&= \mathcal{E}_\gamma(t).
\end{aligned}
$$

If we take $\tilde{\nu}(t) = \mu(t)$, then $(\mathcal{E}_\gamma(t), \tilde{\nu}(t))$ is a framed immersion. In fact, we have

$$
\begin{aligned}
\mathcal{E}'_\gamma(t) &= \frac{mm' - nn'}{|n^2 - m^2|^{\frac{3}{2}}} \left(-n\gamma(t) + m\nu(t)\right) + \frac{1}{\sqrt{|n^2 - m^2|}} \left(-n'\gamma(t) + m'\nu(t)\right) \\
&= \frac{m'n - mn'}{|n^2 - m^2|^{\frac{3}{2}}} \left(-m\gamma(t) + n\nu(t)\right) \\
&= \frac{d}{dt}\left(\frac{m}{n}\right) \frac{n^2}{|n^2 - m^2|^{\frac{3}{2}}} \left(-m\gamma(t) + n\nu(t)\right),
\end{aligned}
$$

we have $\langle \mathcal{E}_\gamma(t), \tilde{\nu}(t) \rangle = \langle \mathcal{E}'_\gamma(t), \tilde{\nu}(t) \rangle = 0$. And (γ, ν) is a framed immersion satisfying $(m(t), n(t)) \neq (0, 0)$. Since, $\tilde{\nu}(t) = \mu(t)$, we get $\tilde{\nu}'(t) = -m\gamma(t) + n\nu(t) \neq 0$. It follows that $\mathcal{E}_\gamma(t)$ is a front. Hence, this completes the proof. □

4. Evolutes of fronts in hyperbolic sphere $\mathbb{H}^2_0$

In the hyperbolic sphere $\mathbb{H}^2_0$, if β has a singular point, we can not construct a moving frame of β in a traditional way. However, we could define a moving frame of a front curve. So, we give the following definitions.

Definition 4.1. *We say that $(\beta, \nu) : I \longrightarrow \mathbb{H}^2_0 \times \mathbb{H}^2_0$ is a framed curve, if $\langle \beta(t), \nu(t) \rangle = 0$ and $\langle \beta'(t), \nu(t) \rangle = 0$ for all $t \in I$. Moreover, if (β, ν) is an immersion, namely, $(\beta'(t), \nu'(t)) \neq (0, 0)$, we call (β, ν) a framed immersion curve.*

Definition 4.2. *We say that $\beta : I \longrightarrow \mathbb{H}^2_0$ is a frontal curve if there exists a smooth mapping $\nu : I \longrightarrow \mathbb{H}^2_0$ such that (β, ν) is a framed curve. We also say that $\beta : I \longrightarrow \mathbb{H}^2_0$ is a front curve if there exists a smooth mapping $\nu : I \longrightarrow \mathbb{H}^2_0$ such that (β, ν) is a framed immersion curve.*

Let $(\beta, \nu) : I \longrightarrow \mathbb{H}^2_0 \times \mathbb{H}^2_0$ be a framed curve. If β is singular at t_0, we can't define a frame in a traditional way. However, ν always exists even if t is a singular point of β. We take $\mu = \nu \wedge \beta$. We call the pair $\{\beta, \nu, \mu\}$ is a moving frame of β and the hyperbolic Frenet-Serret matrix is given by

$$
\begin{bmatrix} \beta'(t) \\ \nu'(t) \\ \mu'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & m(t) \\ 0 & 0 & n(t) \\ m(t) & -n(t) & 0 \end{bmatrix} \begin{bmatrix} \beta(t) \\ \nu(t) \\ \mu(t) \end{bmatrix},
$$

where $n(t) = \langle \nu'(t), \mu(t) \rangle$, $\nu(t)$ and $\mu(t)$ are both unit spacelike vectors. We declare that $(\beta, -\nu)$ is also a framed curve. In this case, $n(t)$ dose not change, but $m(t)$ changes to $-m(t)$. If (β, ν) is a framed immersion, we have $(m(t), n(t)) \neq (0, 0)$ for each $t \in I$ and call the pair (m, n) geodesic curvature of the framed curve (for more details see [8, 10]).

In what follows, some important properties of the meant curves are introduced.

Proposition 4.1. *Let β be a regular curve and $(\beta, \nu) : I \longrightarrow \mathbb{H}_0^2 \times \mathbb{H}_0^2$ be its framed curve, then the relationship between their geodesic curvatures is given by:*

$$\kappa_g(t) = \frac{n(t)}{|m(t)|}. \tag{4.1}$$

Proof. Direct calculations lead to

$$\begin{aligned}\beta'(t) &= m(t)\mu(t),\\ \beta''(t) &= m'(t)\mu(t) + m(t)\mu'(t)\\ &= m'(t)\mu(t) + m(t)(m(t)\beta - n(t)\nu(t)),\end{aligned}$$

where

$$\begin{aligned}\kappa_g(t) &= \frac{\det(\beta(t), \beta'(t), \beta''(t))}{\|\beta'(t)\|^3}\\ &= \frac{\det\left(\beta(t), m\mu(t), m^2\beta(t) - mn\nu(t) + m'\mu(t)\right)}{|m^3|},\end{aligned}$$

then, we get

$$\kappa_g(t) = \frac{n(t)}{|m(t)|}.$$

For the framed immersion (β, ν), we say that t_0 is an inflection point of the front β if $n(t_0) = 0$. And then the condition $m(t_0) \neq 0$ and $n(t_0) = 0$, is equivalent to the condition $\kappa_g(t_0) = 0$. $\square$

If β is not a regular curve, then we can not define the evolute as in Eq. (2.8). Since the geodesic curvature may be divergence at a singular point. Then, we can give the definition of the evolute of a front and focus on its properties.

The notion of a parallel curve of β can be presented as follows:
Let $(\beta, \nu) : I \longrightarrow \mathbb{H}_0^2 \times \mathbb{H}_0^2$ be a framed curve which has the geodesic curvature (m, n). Then, we define a parallel curve $\beta_\lambda : I \longrightarrow \mathbb{H}_0^2$ of β as

$$\beta_\lambda(t) = \frac{1}{\sqrt{|\lambda^2 - 1|}}\left(\beta(t) + \lambda\nu(t)\right), \tag{4.2}$$

where $\lambda \in \mathbb{R}$ and $\lambda \neq \pm 1$.

Proposition 4.2. *If β_λ is a regular curve, then*

$$\kappa_{g\lambda}(t) = \frac{n(t) + \lambda m(t)}{|m(t) + \lambda n(t)|}. \tag{4.3}$$

Proof. From Eq.(4.2), we have

$$\begin{aligned}\beta_\lambda'(t) &= \frac{m + \lambda n}{\sqrt{|\lambda^2 - 1|}}\mu(t),\\ \beta_\lambda''(t) &= \frac{m^2 + \lambda mn}{\sqrt{|\lambda^2 - 1|}}\beta(t) + \frac{-mn - \lambda n^2}{\sqrt{|\lambda^2 - 1|}}\nu(t) + \frac{m' + \lambda n'}{\sqrt{|\lambda^2 - 1|}}\mu(t),\end{aligned}$$

which implies that

$$\beta_\lambda \wedge \beta'_\lambda = \frac{1}{(\lambda^2-1)} \begin{vmatrix} -\beta(t) & \nu(t) & \mu(t) \\ 1 & \lambda & 0 \\ 0 & 0 & m+\lambda n \end{vmatrix}$$

$$= \frac{-\lambda(m+\lambda n)\beta(t)-(m+\lambda n)\nu(t)}{(\lambda^2-1)}.$$

Since

$$\kappa_{g\lambda}(t) = \frac{\det\left(\beta_\lambda(t), \beta'_\lambda(t), \beta''_\lambda(t)\right)}{\|\beta'_\lambda(t)\|^3},$$

then, we get

$$|m(t)+\lambda n(t)|\kappa_{g\lambda}(t) = (n(t)+\lambda m(t)),$$

which completes the proof. □

Proposition 4.3. *For a framed immersion curve $(\beta,\nu): I \longrightarrow \mathbb{H}^2_0 \times \mathbb{H}^2_0$, the parallel curve $\beta_\lambda : I \longrightarrow \mathbb{H}^2_0$ is a front for each $\lambda \neq \pm 1$.*

Proof. We take $\nu_\lambda : I \longrightarrow \mathbb{H}^2_0$ as follows:

$$\nu_\lambda(t) = \frac{1}{\sqrt{|\lambda^2-1|}}\left(\lambda\beta(t)+\nu(t)\right),$$

since

$$\begin{cases} \beta_\lambda(t) = \dfrac{1}{\sqrt{|\lambda^2-1|}}\left(\beta(t)+\lambda\nu(t)\right), \\ \beta'_\lambda(t) = \dfrac{1}{\sqrt{|\lambda^2-1|}}\left(\beta'(t)+\lambda\nu'(t)\right). \end{cases}$$

If $\beta'_\lambda(t_0)=0$ at a point $t_0 \in I$, then we have

$$\beta'(t_0)+\lambda\nu'(t_0)=0.$$

If $\nu'(t_0)=0$, then $\beta'(t_0)=0$. It is contradicted with the fact that (β,ν) is a framed immersion and hence $(\beta_\lambda,\nu_\lambda)$ is a framed immersion. By $\|\nu(t)\|=-1$, we have $\langle\nu(t),\nu'(t)\rangle=0$. Then

$$\langle\beta'_\lambda(t),\nu_\lambda(t)\rangle = \frac{1}{(\lambda^2-1)}\langle\beta'+\lambda\nu', \lambda\beta+\nu\rangle = 0,$$

and from this, the curve β_λ is a front. □

Proposition 4.4. *Let (β,ν) be a framed curve. If β is a regular curve and $\lambda \neq 1/\kappa_g$, then a parallel curve β_λ is also a regular curve and its evolute is given by*

$$\mathcal{E}_{\beta\lambda}(t) = \mathcal{E}_\beta(t). \tag{4.4}$$

Proof. Since

$$\beta_\lambda(t) = \frac{1}{\sqrt{|\lambda^2-1|}}\left(\beta(t)+\lambda\mathbf{E}(t)\right),$$

$$\beta'_\lambda(t) = \frac{\|\beta'\|}{\sqrt{|\lambda^2-1|}}\left(1-\lambda\kappa_g\right)\mathbf{T}(t),$$

$$\beta''_\lambda(t) = \frac{\|\beta'\|}{\sqrt{|\lambda^2-1|}}\left(\|\beta'\|(1-\lambda\kappa_g)\beta(t)-\lambda\kappa'_g\mathbf{T}(t)+\|\beta'\|\kappa_g(1-\lambda\kappa_g)\mathbf{E}(t)\right).$$

By the assumption $\lambda \neq 1/\kappa_g$, β_λ is a regular curve. By direct calculations, we obtain

$$\beta_\lambda \wedge \beta'_\lambda = \frac{1}{(\lambda^2-1)}\begin{vmatrix} -\beta(t) & \mathbf{T}(t) & \mathbf{E}(t) \\ 1 & 0 & \lambda \\ 0 & \|\beta'\|(1-\lambda\kappa_g) & 0 \end{vmatrix}$$

$$= \frac{\|\beta'\|\lambda(1-\lambda\kappa_g)\beta(t)+\|\beta'\|(1-\lambda\kappa_g)\mathbf{E}(t)}{(\lambda^2-1)},$$

and hence

$$\langle\beta_\lambda \wedge \beta'_\lambda, \beta''_\lambda\rangle = \frac{-\|\beta'\|^3\lambda(1-\lambda\kappa_g)^2+\|\beta'\|^3\kappa_g(1-\lambda\kappa_g)^2}{|\lambda^2-1|^{\frac{3}{2}}},$$

then, we get

$$\kappa_{g\lambda}(t) = \frac{\kappa_g-\lambda}{|1-\lambda\kappa_g|},$$

and

$$\mathbf{T}_\lambda(t) = \frac{\beta'_\lambda}{\|\beta'_\lambda\|} = \frac{1-\lambda\kappa_g}{|1-\lambda\kappa_g|}\mathbf{T}(t),$$

where $\mathbf{E}_\lambda = \beta_\lambda \wedge \mathbf{T}_\lambda$, we have

$$\mathbf{E}_\lambda(t) = \frac{1-\lambda\kappa_g}{|1-\lambda\kappa_g|}\frac{1}{\sqrt{|\lambda^2-1|}}\left(\mathbf{E}(t)+\lambda\beta(t)\right).$$

From (2.8) we get

$$\mathcal{E}_{\beta_\lambda}(t) = \frac{1}{\sqrt{\left|\kappa^2_{g\lambda}(t)-1\right|}}\left(\kappa_{g\lambda}(t)\beta_\lambda(t)+\mathbf{E}_\lambda(t)\right)$$

$$= \frac{1}{\sqrt{\left|\left(\frac{\kappa_g-\lambda}{|1-\lambda\kappa_g|}\right)^2-1\right|}}\left(\frac{\kappa_g-\lambda}{|1-\lambda\kappa_g|}\frac{(\beta(t)+\lambda\mathbf{E}(t))}{\sqrt{\lambda^2-1}}+\frac{1-\lambda\kappa_g}{|1-\lambda\kappa_g|}\frac{(\mathbf{E}(t)+\lambda\beta(t))}{\sqrt{|\lambda^2-1|}}\right)$$

$$= \frac{1}{\sqrt{\left|\left(\frac{\kappa_g-\lambda}{|1-\lambda\kappa_g|}\right)^2-1\right|}}\frac{1}{|1-\lambda\kappa_g|\sqrt{|\lambda^2-1|}}\left((1-\lambda^2)\kappa_g\beta(t)+(1-\lambda^2)\mathbf{E}(t)\right)$$

$$= \frac{1}{\sqrt{\left|\kappa^2_g(1-\lambda^2)-(1-\lambda^2)\right|}}\frac{(1-\lambda^2)}{\sqrt{|\lambda^2-1|}}\left(\kappa_g\beta(t)+\mathbf{E}(t)\right)$$

$$= \frac{1}{\sqrt{\left|\kappa_g^2(t) - 1\right|}} \left(\kappa_g(t)\beta(t) + \mathbf{E}(t)\right)$$
$$= \mathcal{E}_\beta(t).$$

In the light of the above calculations, the proof is completed. □

Definition 4.3. *Let $(\beta, \nu) : I \longrightarrow \mathbb{H}_0^2 \times \mathbb{H}_0^2$ be a framed immersion curve. We define an evolute $\mathcal{E}_\beta : I \longrightarrow \mathbb{H}_0^2$ of β as follows:*

If t is a regular point, then we find

$$\mathcal{E}_\beta(t) = \frac{1}{\sqrt{\left|\kappa_g^2(t) - 1\right|}} \left(\kappa_g(t)\beta(t) + \mathbf{E}(t)\right). \tag{4.5}$$

If t_0 is a singular point, for any $t \in (t_0 - \delta, t_0 + \delta)$, we get

$$\mathcal{E}_{\beta_\lambda}(t) = \frac{1}{\sqrt{\left|\kappa_{g\lambda}^2(t) - 1\right|}} \left(\kappa_{g\lambda}(t)\beta_\lambda(t) + \mathbf{E}_\lambda(t)\right), \tag{4.6}$$

where δ is a sufficiently small positive real number and $\lambda \in \mathbb{R}$ satisfies the condition $\lambda \neq 1/\kappa_g(t)$.

Now, we give another representation of the evolute by using the moving frame $\{\beta(t), \nu(t), \mu(t)\}$ and its geodesic curvature $\{m(t), n(t)\}$.

Theorem 4.1. *Under the condition of $|m(t)| \neq |n(t)|$, the evolute of a front curve $\mathcal{E}_\beta(t) : I \longrightarrow \mathbb{H}_0^2$ is represented by*

$$\mathcal{E}_\beta(t) = \frac{1}{\sqrt{|n^2(t) - m^2(t)|}} \left(n(t)\beta(t) - m(t)\nu(t)\right), \tag{4.7}$$

and $\mathcal{E}_\beta(t)$ is a front curve.

Proof. (i) Suppose that β is a regular curve. Since $\beta'(t) = m(t)\mu(t)$, we have $|m(t)| \neq 0$ and

$$\mathbf{T}(t) = \frac{m(t)}{|m(t)|}\mu(t), \qquad \mathbf{E}(t) = -\frac{m(t)}{|m(t)|}\nu(t).$$

From Eqs.(4.1) and (4.5), we get

$$\mathcal{E}_\beta(t) = \frac{1}{\sqrt{\left|\kappa_g^2(t) - 1\right|}} \left(\kappa_g(t)\beta(t) + \mathbf{E}(t)\right)$$
$$= \frac{1}{\sqrt{\left|\left(\frac{n(t)}{|m(t)|}\right)^2 - 1\right|}} \left(\frac{n(t)}{|m(t)|}\beta(t) - \frac{m(t)}{|m(t)|}\nu(t)\right)$$
$$= \frac{1}{\sqrt{|n^2(t) - m^2(t)|}} \left(n(t)\beta(t) - m(t)\nu(t)\right).$$

(ii) Suppose that t_0 is a singular point of β and consider β_λ in a hyperbolic sphere, also we know that β_λ is a regular curve around the neighbourhood of t_0 with $\lambda \neq 1/\kappa_g(t)$.
From Eq.(4.2), so we get

$$\beta'_\lambda(t) = \frac{m+\lambda n}{\sqrt{|\lambda^2-1|}}\mu(t),$$

then, $|m+\lambda n| \neq 0$ and

$$\mathbf{T}_\lambda(t) = \frac{m+\lambda n}{|m+\lambda n|}\mu(t),$$

where $\mathbf{E}_\lambda(t) = \beta_\lambda(t) \wedge \mathbf{T}_\lambda(t)$, we find

$$\mathbf{E}_\lambda(t) = \frac{m+\lambda n}{|m+\lambda n|}\frac{1}{\sqrt{|\lambda^2-1|}}\left(-\lambda\beta(t)-\nu(t)\right).$$

Therefore, from Eq.(4.3), we obtain

$$\kappa_{g\lambda} = \frac{(n+\lambda m)}{|m+\lambda n|},$$

and from Eqs.(4.4) and (4.6) we have

$$\begin{aligned}
\mathcal{E}_\beta(t) &= \mathcal{E}_{\beta\lambda}(t) \\
&= \frac{1}{\sqrt{\left|\kappa^2_{g\lambda}(t)-1\right|}}\left(\kappa_{g\lambda}(t)\beta_\lambda(t)+\mathbf{E}_\lambda(t)\right) \\
&= \frac{1}{\sqrt{\left|\left(\frac{(n+\lambda m)}{|m+\lambda n|}\right)^2-1\right|}}\left(\left(\frac{(n+\lambda m)}{|m+\lambda n|}\right)\frac{(\beta(t)+\lambda\nu(t))}{\sqrt{\lambda^2-1}}+\frac{m+\lambda n}{|m+\lambda n|}\frac{(-\lambda\beta(t)-\nu(t))}{\sqrt{|\lambda^2-1|}}\right) \\
&= \frac{1}{\sqrt{|(n+\lambda m)^2-(m+\lambda n)^2|}}\frac{\left(n\beta(t)+\lambda^2 m\nu(t)-\lambda^2 n\beta(t)-m\nu(t)\right)}{\sqrt{|\lambda^2-1|}} \\
&= \frac{1}{\sqrt{|(1-\lambda^2)(n^2-m^2)|}}\frac{(1-\lambda^2)(n\beta(t)-m\nu(t))}{\sqrt{|\lambda^2-1|}} \\
&= \frac{1}{\sqrt{|n^2-m^2|}}\left(n\beta(t)-m\nu(t)\right).
\end{aligned}$$

If we take $\tilde{\nu}(t) = \mu(t)$, then $(\mathcal{E}_\beta(t), \tilde{\nu}(t))$ is a framed immersion. In fact, we have

$$\begin{aligned}
\mathcal{E}'_\beta(t) &= \frac{mm'-nn'}{(n^2-m^2)^{\frac{3}{2}}}\left(n\beta(t)-m\nu(t)\right)+\frac{1}{\sqrt{n^2-m^2}}\left(n'\beta(t)-m'\nu(t)\right) \\
&= \frac{m'n-mn'}{(n^2-m^2)^{\frac{3}{2}}}\left(m\beta(t)-n\nu(t)\right) \\
&= \frac{d}{dt}\left(\frac{m}{n}\right)\frac{n^2}{(n^2-m^2)^{\frac{3}{2}}}\left(m\beta(t)-n\nu(t)\right),
\end{aligned}$$

therefore, we find $\langle \mathcal{E}_\beta(t), \tilde{\nu}(t)\rangle = \langle \mathcal{E}'_\beta(t), \tilde{\nu}(t)\rangle = 0$. And (β, ν) is a framed immersion satisfying $(m(t), n(t)) \neq (0,0)$. Since, $\tilde{\nu}(t) = \mu(t)$, we get $\tilde{\nu}'(t) = m\beta(t) - n\nu(t) \neq 0$. It follows that $\mathcal{E}_\beta(t)$ is a front and thus, the proof is completed. □

5. Computational examples

Example 5.1. *Let $\gamma : I \longrightarrow \mathbb{S}^2_1$ be a regular curve given by*

$$\gamma(t) = \left(\sinh(t^3), \cos(t^2)\cosh(t^3), \sin(t^2)\cosh(t^3)\right),$$

then, we get

$$\gamma'(t) = \left(3t^2\cosh(t^3), -2t\sin(t^2)\cosh(t^3) + 3t^2\cos(t^2)\sinh(t^3), 2t\cos(t^2)\cosh(t^3) + 3t^2\sin(t^2)\sinh(t^3)\right).$$

Since, t = 0 is a singular point on γ. If we take $\nu = (\nu_1, \nu_2, \nu_3)$, where

$$\begin{cases} \nu_1 = \frac{1}{\mathcal{P}}\left(2\cosh^2(t^3)\right) \\ \nu_2 = \frac{1}{\mathcal{P}}\left(2\sinh(t^3)\cos(t^2)\cosh(t^3) - 3t\sin(t^2)\right) \\ \nu_3 = \frac{1}{\mathcal{P}}\left(2\sin(t^2)\sinh(t^3)\cosh(t^3) + 3t\cos(t^2)\right), \end{cases} \tag{5.1}$$

and $\mathcal{P} = \sqrt{\left|9t^2 - 4\cosh^2(t^3)\right|}$, then we have $\langle\gamma(t), \nu(t)\rangle = \langle\gamma'(t), \nu(t)\rangle = 0$ and $\langle\nu(t), \nu(t)\rangle = 1$. Hence, (γ, ν) is a framed curve. Also, from the relation $\mu = \nu \wedge \gamma$, we get (see Fig.(1a)).

$$\mu(t) = \frac{1}{\mathcal{P}}\begin{vmatrix} -i & j & k \\ 2\cosh^2(t^3) & 2\sinh(t^3)\cos(t^2)\cosh(t^3) - 3t\sin(t^2) & 2\sin(t^2)\sinh(t^3)\cosh(t^3) + 3t\cos(t^2) \\ \sinh(t^3) & \cos(t^2)\cosh(t^3) & \sin(t^2)\cosh(t^3) \end{vmatrix},$$

and we get

$$\mu = \frac{1}{\mathcal{P}}\left(3t\cosh(t^3), 3t\cos(t^2)\sinh(t^3) - 2\sin(t^2)\cosh(t^3), 3t\sin(t^2)\sinh(t^3) + 2\cos(t^2)\cosh(t^3)\right). \tag{5.2}$$

From the fact that $\langle\mu(t), \mu(t)\rangle = 1$, one can obtain

$$m(t) = \|\gamma'(t)\| = t\sqrt{\left|9t^2 - 4\cosh^2(t^3)\right|}. \tag{5.3}$$

Also, from Eq.(5.1) we have

$$\begin{aligned} \nu'(t) =& \frac{-(9t - 4\cosh(t^3)\sinh(t^3))}{\left|9t^2 - 4\cosh^2(t^3)\right|^{\frac{3}{2}}}(2\cosh^2(t^3), 2\sinh(t^3)\cos(t^2)\cosh(t^3) - 3t\sin(t^2), \\ & 2\sin(t^2)\sinh(t^3)\cosh(t^3) + 3t\cos(t^2)) \\ & + \frac{1}{\sqrt{\left|9t^2 - 4\cosh^2(t^3)\right|}}(12t^2\cosh(t^3)\sinh(t^3), 6t^2\cos(t^2)(\cosh^2(t^3) + \sinh^2(t^3)) \\ & - 4t\sinh(t^3)\sin(t^2)\cosh(t^3) - 3\sin(t^2) - 6t^2\cos(t^2), 6t^2\sin(t^2)(\cosh^2(t^3) + \sinh^2(t^3)) \\ & + 4t\sinh(t^3)\cos(t^2)\cosh(t^3) - 3\cos(t^2) - 6t^2\sin(t^2)), \end{aligned}$$

which leads to

$$\begin{aligned} n(t) &= \langle \nu'(t), \mu(t) \rangle \\ &= \frac{2(-18t^3 \sinh(t^3) + 4t \sinh(t^3) \cosh(t^3) + 3 \cosh(t^3))}{9t^2 - 4\cosh^2(t^3)}, \end{aligned}$$

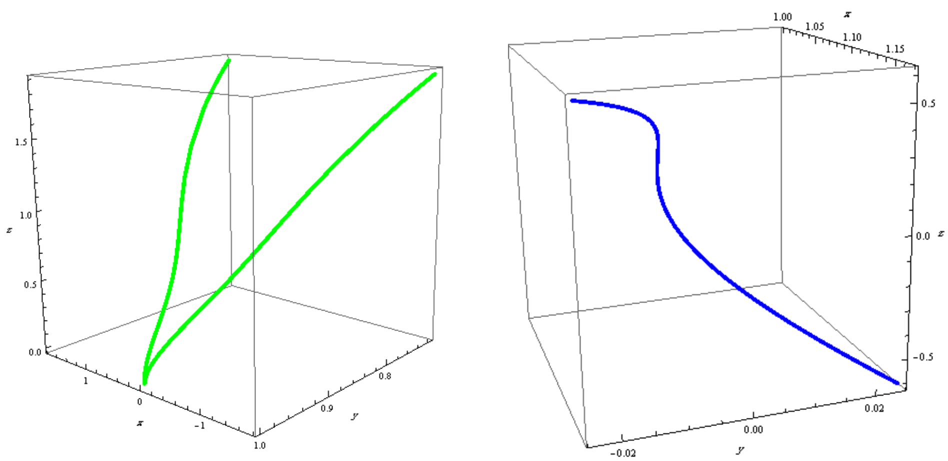
and we have $(m(0), n(0)) \neq (0,0)$, thus, γ is a frontal curve.
On the other hand, the evolute curve of γ is given as

$$\mathcal{E}_\gamma(t) = (\mathcal{E}_{\gamma 1}, \mathcal{E}_{\gamma 2}, \mathcal{E}_{\gamma 3}),$$

where

$$\begin{aligned} \mathcal{E}_{\gamma 1}(t) &= \frac{1}{\mathcal{P}\sqrt{|4\mathcal{F}^2 - t^2\mathcal{P}^4|}} \left(2\mathcal{F}\sinh(t^3) - 2t\mathcal{P}^2\cosh^2(t^3)\right), \\ \mathcal{E}_{\gamma 2}(t) &= \frac{1}{\mathcal{P}\sqrt{|4\mathcal{F}^2 - t^2\mathcal{P}^4|}} \left(2\mathcal{F}\cos(t^2)\cosh(t^3) - 2t\mathcal{P}^2 2\sinh(t^3)\cos(t^2)\cosh(t^3) - 3t^2\mathcal{P}^2\sin(t^2)\right), \\ \mathcal{E}_{\gamma 3}(t) &= \frac{1}{\mathcal{P}\sqrt{|4\mathcal{F}^2 - t^2\mathcal{P}^4|}} \left(2\mathcal{F}\sin(t^2)\cosh(t^3) - t\mathcal{P}^2\sin(t^2)\sinh(t^3)\cosh(t^3) - 3t^2\mathcal{P}^2\cos(t^2)\right), \end{aligned}$$

keep in mind that $\mathcal{F} = -18t^3\sinh(t^3) + 4t\sinh(t^3)\cosh(t^3) + 3\cosh(t^3)$.



(a) (b)

Figure 1. (A) The curve $\gamma(t)$ with singular point at $t = 0$. (B) The curve $\nu(t) = (\nu_1, \nu_2, \nu_3)$.

Example 5.2. *Consider the de Sitter asteroid curve $\beta : I \longrightarrow \mathbb{S}^2_1$, $\beta(t) = (\beta_1, \beta_2, \beta_3)$ expressed as [8]*

$$\begin{cases} \beta_1 = \sqrt{|\cos^6(t) + \sin^6(t) - 1|} \\ \beta_2 = \cos^3(t) \\ \beta_3 = \sin^3(t), \end{cases} \tag{5.4}$$

then, we get

$$\beta'(t) = 3\sin(t)\cos(t)\left(\frac{\sin^4(t)-\cos^4(t)}{\sqrt{|\cos^6(t)+\sin^6(t)-1|}}, -\cos(t), \sin(t)\right).$$

It is obvious that β is singular at $t = 0$, $\pi/2$, π and $3\pi/2$.
If we take $\nu = (\nu_1, \nu_2, \nu_3)$, where

$$\begin{cases} \nu_1 = \frac{1}{\mathcal{Q}}\left(\sin(t)\cos(t)\sqrt{|\cos^6(t)+\sin^6(t)-1|}\right) \\ \nu_2 = \frac{1}{\mathcal{Q}}\left(\sin(t)\left(\cos^4(t)-1\right)\right) \\ \nu_3 = \frac{1}{\mathcal{Q}}\left(\cos(t)\left(\sin^4(t)-1\right)\right), \end{cases} \tag{5.5}$$

with the knowledge that $\mathcal{Q} = \sqrt{\left|1-\sin^2(t)\cos^2(t)\right|}$,
then by a straightforward calculations, we have $\langle\beta(t), \nu(t)\rangle = \langle\beta'(t), \nu(t)\rangle = 0$ and $\langle\nu(t), \nu(t)\rangle = 1$. Hence, (β, ν) is a framed curve. From the equation $\mu = \nu \wedge \beta$, we have (see Fig.(2a) and Fig.(2b)).

$$\mu = \frac{1}{\mathcal{Q}}\begin{vmatrix} -i & j & k \\ \frac{\sin(t)\cos(t)}{(|\cos^6(t)+\sin^6(t)-1|)^{-\frac{1}{2}}} & \sin(t)\left(\cos^4(t)-1\right) & \cos(t)\left(\sin^4(t)-1\right) \\ \sqrt{|\cos^6(t)+\sin^6(t)-1|} & \cos^3(t) & \sin^3(t) \end{vmatrix},$$

which gives

$$\mu(t) = \frac{\sqrt{|\cos^6(t)+\sin^6(t)-1|}}{\sqrt{\left|1-\sin^2(t)\cos^2(t)\right|}}\left(\frac{\sin^4(t)-\cos^4(t)}{\sqrt{|\cos^6(t)+\sin^6(t)-1|}}, -\cos(t), \sin(t)\right). \tag{5.6}$$

From the previous equation, we have $\langle\mu(t), \mu(t)\rangle = 1$. Then, we obtain

$$m(t) = \|\beta'(t)\| = 3\sin(t)\cos(t)\sqrt{\frac{\cos^6(t)+\sin^6(t)}{|\cos^6(t)+\sin^6(t)-1|}}. \tag{5.7}$$

Also, from Eq.(5.1) we have

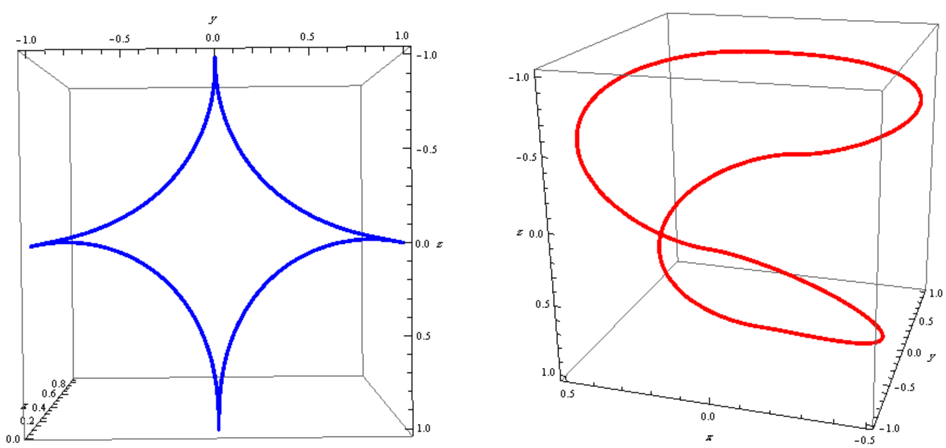
$$\begin{aligned} \nu'(t) = &\frac{\sin(t)\cos(t)(\cos^2(t)-\sin^2(t))}{\left|1-\sin^2(t)\cos^2(t)\right|^{\frac{3}{2}}}(\sin(t)\cos(t)\sqrt{|\cos^6(t)+\sin^6(t)-1|}, \\ &\sin(t)(\cos^4(t)-1), \cos(t)(\sin^4(t)-1)) \\ &+ \frac{1}{\sqrt{\left|1-\sin^2(t)\cos^2(t)\right|}}(\sqrt{|\cos^6(t)+\sin^6(t)-1|}(\cos^2(t)-\sin^2(t) \\ &+ \frac{3\sin^2(t)\cos^2(t)(\sin^4(t)-\cos^4(t))}{\cos^6(t)+\sin^6(t)-1}), \cos^3(t)(\cos^2(t)-4\sin^2(t)) \\ &-\cos(t), -\sin^3(t)(\sin^2(t)-4\cos^2(t))+\sin(t)). \end{aligned}$$

Hence, we get

$$\begin{aligned} n(t) &= \langle \nu'(t), \mu(t) \rangle \\ &= \frac{\sqrt{|\cos^6(t)+\sin^6(t)-1|}}{1-\sin^2(t)\cos^2(t)} \left(3\sin^2(t)\cos^2(t)\left(1-\frac{(\cos^2(t)-\sin^2(t))^2}{\cos^6(t)+\sin^6(t)-1}\right)+1\right). \end{aligned} \tag{5.8}$$

In the light of the above calculations, we have $(m(0), n(0)) \neq (0,0)$, thus, β is a frontal curve. Also, from Eqs.(3.7), (5.4), (5.5), (5.7), and (5.8), we get the evolute curve of β as $\mathcal{E}_\beta(t) = (\mathcal{E}_{\beta 1}, \mathcal{E}_{\beta 2}, \mathcal{E}_{\beta 3})$, where

$$\begin{aligned} \mathcal{E}_{\beta 1}(t) &= \frac{-\mathcal{Q}n+m}{\sqrt{|n^2-m^2|}}\left(\sqrt{|\cos^6(t)+\sin^6(t)-1|}\,(1+\sin(t)\cos(t))\right), \\ \mathcal{E}_{\beta 2}(t) &= \frac{-\mathcal{Q}n+m}{\sqrt{|n^2-m^2|}}\left(\cos^3(t)\,(1+\sin(t)\cos(t))-\sin(t)\right), \\ \mathcal{E}_{\beta 3}(t) &= \frac{-\mathcal{Q}n+m}{\sqrt{|n^2-m^2|}}\left(\sin^3(t)\,(1+\sin(t)\cos(t))-\cos(t)\right). \end{aligned}$$



(a) (b)

Figure 2. (A) The *de Sitter* astroid curve β. (B) The curve $\nu(t) = (\nu_1, \nu_2, \nu_3)$.

6. Conclusion

In 2-dimensional de Sitter and hyperbolic spaces, some types of curves such as framed curves, framed immersion curves, frontal curves and front curves are studied. Also, the evolutes and some of their properties of fronts at singular points under some conditions are investigated. Finally, two computational examples in support of our main results are given and plotted.

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