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New Generalizations of sup-Hesitant Fuzzy Ideals of Semigroups

Uraiwan Jittburus[1], Pongpun Julatha[1,*], Attaphol Pumila[1], Napaporn Chunsee[2], Aiyared Iampan[3], Rukchart Prasertpong[4]

[1]*Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand*

[2]*Faculty of Science and Technology, Uttaradit Rajabhat University, Uttaradit 53000, Thailand*

[3]*Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand*

[4]*Division of Mathematics and Statistics, Faculty of Science and Technology, Nakhon Sawan Rajabhat University, Nakhon Sawan 60000, Thailand*

[*]*Corresponding author:* pongpun.j@psru.ac.th

Abstract. As general concepts of sup-hesitant fuzzy right (resp., left, interior, two-sided) ideals of semigroups, the concepts of $\sup_{\alpha}^{+}$-hesitant fuzzy right (resp., left, interior, two-sided) ideals and $\sup_{\beta}^{-}$-hesitant fuzzy right (resp., left, interior, two-sided) ideals are introduced and their properties are investigated. Then, the concepts are established by fuzzy sets, Łukasiewicz fuzzy sets, Łukasiewicz anti-fuzzy sets, Pythagorean fuzzy sets, hesitant fuzzy sets, hybrid sets, interval-valued fuzzy sets and cubic sets. Finally, we characterize which is intra-regular, completely regular, simple semigroups or another type of semigroups in terms of $\sup_{\alpha}^{+}$-type and $\sup_{\beta}^{-}$-type of hesitant fuzzy sets.

1. Introduction

The fuzzy set theory presented by Zadeh [46] has been successfully and widely applied in many areas such as robotics, expert, computer science, finite state machine, control engineering, logic theory, automata theory, group theory, graph theory and semigroup theory. Furthermore, in the literature, a number of concepts of fuzzy sets and their generalizations and extensions have been

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introduced and studied, for instance, Łukasiewicz fuzzy sets [23], Łukasiewicz anti-fuzzy sets [22], anti-type of fuzzy sets [25, 37], negative fuzzy sets [18], bipolar fuzzy sets [29, 48], interval-valued fuzzy sets [47], intuitionistic fuzzy sets [4], Pythagorean fuzzy sets [44,45], rough sets [2,34], hesitant fuzzy sets [41, 42], cubic sets [19] and hybrid sets [1, 24].

On semigroups, Kuroki [27, 28] applied fuzzy sets to semigroups. Mordeson et al. [30] explained semigroup theory to fuzzy semigroup theory and showed their applications in coding theory, languages and fuzzy finite state machines. Shabir and Nawaz [37], Khan and Asif [25], Julatha and Siripitukdet [17] studied anti-type of fuzzy sets based on ideal theory in semigroups. Chinnadurai and Arulselvam [7] introduced Pythagorean fuzzy sets based on ideal theory in semigroups and investigated their properties. Narayanan and Manikantan [33], and Thillaigovindan and Chinnadurai [40] studied interval-valued fuzzy sets in semigroups. Jun and Khan [20], Umar et al. [43], and Muhiuddin [31] studied cubic sets in semigroups. Anis et al. [1], Elavarasan et al. [9] studied hybrid sets in semigroups. Jun et al. [21] and Talee et al. [39] studied hesitant fuzzy sets in semigroup. Studying hesitant fuzzy sets, in the meaning of the supremum of their images, on semigroups, Jittburus and Julatha [12] introduced sup-hesitant fuzzy ideals of semigroups and investigated properties via sets, fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets. Phummee et al. [35] introduced sup-hesitant fuzzy interior ideals of semigroups and studied its properties by sets, fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets. Julatha et al. [13] introduced sup-hesitant fuzzy right (left) ideals of semigroups and studied their characterizations in terms of sets, fuzzy sets, Pythagorean fuzzy sets, interval-valued fuzzy sets, hesitant fuzzy sets, cubic sets and hybrid sets. Many researchers have taken intense and eager interest in the novel area of hesitant fuzzy sets on algebraic structures in the meaning of the supremum of their images (see [1, 10, 12, 14–16, 32, 35, 36, 38]).

As previously stated, it motivated us to study hesitant fuzzy sets on semigroups in the meaning of the supremum of their images. We will introduce concepts of $\sup_{\alpha}^{+}$-hesitant fuzzy right (resp., left, interior, two-sided) ideals and $\sup_{\beta}^{-}$-hesitant fuzzy right (resp., left, interior, two-sided) ideals and investigate their properties. Also, we will show that every sup-hesitant fuzzy right (resp., left, interior, two-sided) ideal of a semigroup is both a $\sup_{\alpha}^{+}$-hesitant fuzzy right (resp., left, interior, two-sided) ideal and a $\sup_{\beta}^{-}$-hesitant fuzzy right (resp., left, interior, two-sided) ideal, but the converse is not true. Later, the concepts will be established by fuzzy sets, Łukasiewicz fuzzy sets, Łukasiewicz anti-fuzzy sets, Pythagorean fuzzy sets, hesitant fuzzy sets, hybrid sets, interval-valued fuzzy sets and cubic sets. Finally, we will characterize which is intra-regular, left (right) regular, completely regular, left (right) simple and simple semigroups in terms of $\sup_{\alpha}^{+}$-type and $\sup_{\beta}^{-}$-type of hesitant fuzzy sets.

2. Preliminaries

In this section we first give some basic definitions and results which will be used in this paper.

In what follows, unless otherwise specified, let $\mathcal{A}$ be a semigroup, $\mathcal{B}$ be a nonempty set, $\wp(\mathcal{B})$ be the power set of $\mathcal{B}$ and $\blacktriangledown, \blacktriangle \in \wp([0,1])$. A nonempty subset $\mathcal{B}$ of $\mathcal{A}$ is called a right ideal (resp., a left ideal, an interior ideal) of $\mathcal{A}$ if

$$\mathcal{B}\mathcal{A} \subseteq \mathcal{B} \text{ (resp., } \mathcal{A}\mathcal{B} \subseteq \mathcal{B},\ \mathcal{A}\mathcal{B}\mathcal{A} \subseteq \mathcal{B})$$

and an ideal of $\mathcal{A}$ if $\mathcal{B}$ is both a right ideal and left ideal of $\mathcal{A}$.

A fuzzy subset (FS) [46] of $\mathcal{B}$ is defined to be a function $\xi : \mathcal{B} \to [0,1]$ where $[0,1]$ is the unit interval. For FSs ξ and η of $\mathcal{B}$, define $\xi \le \eta$ if $\xi(p) \le \eta(p)$ for all $p \in \mathcal{B}$. A FS ξ of $\mathcal{A}$ is called

(1) a fuzzy right ideal (FRI) [30] of $\mathcal{A}$ if $\xi(p) \le \xi(pq)$ for all $p, q \in \mathcal{A}$,
(2) a fuzzy left ideal (FLI) [30] of $\mathcal{A}$ if $\xi(q) \le \xi(pq)$ for all $p, q \in \mathcal{A}$,
(3) a fuzzy ideal (FI) [30] of $\mathcal{A}$ if it is both a FRL and a FLI of $\mathcal{A}$, that is, $\max\{\xi(p), \xi(q)\} \le \xi(pq)$ for all $p, q \in \mathcal{A}$,
(4) a fuzzy interior ideal (FII) [30] of $\mathcal{A}$ if $\xi(w) \le \xi(pwq)$ for all $p, q, w \in \mathcal{A}$,
(5) an anti-fuzzy right ideal (AFRI) [37] of $\mathcal{A}$ if $\xi(pq) \le \xi(p)$ for all $p, q \in \mathcal{A}$,
(6) an anti-fuzzy left ideal (AFLI) [37] of $\mathcal{A}$ if $\xi(pq) \le \xi(q)$ for all $p, q \in \mathcal{A}$,
(7) an anti-fuzzy ideal (AFI) [37] of $\mathcal{A}$ if it is both an AFRI and an AFLI of $\mathcal{A}$, that is, $\xi(pq) \le \min\{\xi(p), \xi(q)\}$ for all $p, q \in \mathcal{A}$, and
(8) an anti-fuzzy interior ideal (AFII) [25] of $\mathcal{A}$ if $\xi(pwq) \le \xi(w)$ for all $p, q, w \in \mathcal{A}$.

A Pythagorean fuzzy set (PFS) P [44, 45] in $\mathcal{B}$ is an object having the form $P = \{(p, \xi(p), \eta(p)) \mid p \in \mathcal{B}\}$ where the functions $\xi : \mathcal{B} \to [0,1]$ and $\eta : \mathcal{B} \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and $0 \le (\xi(p))^2 + (\eta(p))^2 \le 1$ for all $p \in \mathcal{B}$. For the sake of simplicity, we shall use the symbol (ξ, η) of the PFS $\{(p, \xi(p), \eta(p)) \mid p \in \mathcal{B}\}$. A PFS (ξ, η) in $\mathcal{A}$ is called

(1) a Pythagorean fuzzy right ideal (PFRI) [7] of $\mathcal{A}$ if ξ is a FRI and η is an AFRI of $\mathcal{A}$,
(2) a Pythagorean fuzzy left ideal (PFLI) [7] of $\mathcal{A}$ if ξ is a FLI and η is an AFLI of $\mathcal{A}$,
(3) a Pythagorean fuzzy ideal (PFI) [7] of $\mathcal{A}$ if it is both a PFRI and a PFLI of $\mathcal{A}$, and
(4) a Pythagorean fuzzy interior ideal (PFII) [7] of $\mathcal{A}$ if ξ is a FII and η is an AFII of $\mathcal{A}$.

By an interval number $\breve{a}$ we mean an interval $[a^-, a^+]$, where $a^-, a^+ \in [0,1]$ and $a^- \le a^+$. We denote $\mathcal{D}([0,1])$ for the set of all interval numbers. Then, we obtain $\mathcal{D}([0,1]) \subseteq \wp([0,1])$. For $\breve{a} = [a^-, a^+], \breve{b} = [b^-, b^+] \in \mathcal{D}([0,1])$, the operations $\precsim$, $=$ and $\prec$ in case of two elements in $\mathcal{D}([0,1])$ are defined by:

(1) $\breve{a} \precsim \breve{b} \Leftrightarrow a^- \le b^-$ and $a^+ \le b^+$,
(2) $\breve{a} = \breve{b} \Leftrightarrow a^- = b^-$ and $a^+ = b^+$, and
(3) $\breve{a} \prec \breve{b} \Leftrightarrow \breve{a} \precsim \breve{b}$ and $\breve{a} \ne \breve{b}$.

An interval-valued fuzzy set (IvFS) [47] on $\mathcal{B}$ is defined to be a function $\breve{\lambda} : \mathcal{B} \to \mathcal{D}([0,1])$, $\breve{\lambda}(p) \mapsto [\breve{\lambda}^L(p), \breve{\lambda}^U(p)]$ where $\breve{\lambda}^L$ and $\breve{\lambda}^U$ are FSs of $\mathcal{B}$ such that $\breve{\lambda}^L \le \breve{\lambda}^U$. For FSs ξ and η of $\mathcal{B}$ with $\xi \le \eta$, we define the IvFS $[\xi, \eta]$ on $\mathcal{B}$ by $[\xi, \eta](p) = [\xi(p), \eta(p)]$ for all $p \in \mathcal{B}$. An IvFS $\check{\lambda} = [\check{\lambda}^L, \check{\lambda}^U]$ on $\mathcal{A}$ is called

(1) an interval-valued fuzzy right ideal (IvFRI) [33, 40] of $\mathcal{A}$ if $\check{\lambda}(p) \precsim \check{\lambda}(pq)$ for all $p, q \in \mathcal{A}$, that is, $\check{\lambda}^L$ and $\check{\lambda}^U$ are FRIs of $\mathcal{A}$,

(2) an interval-valued fuzzy left ideal (IvFLI) [33, 40] of $\mathcal{A}$ if $\check{\lambda}(q) \precsim \check{\lambda}(pq)$ for all $p, q \in \mathcal{A}$, that is, $\check{\lambda}^L$ and $\check{\lambda}^U$ are FLIs of $\mathcal{A}$,

(3) an interval-valued fuzzy ideal (IvFI) [33, 40] of $\mathcal{A}$ if it is both an IvFRI and an IvFLI of $\mathcal{A}$, that is, $\check{\lambda}^L$ and $\check{\lambda}^U$ are FIs of $\mathcal{A}$, and

(4) an interval-valued fuzzy interior ideal (IvFII) [40] of $\mathcal{A}$ if $\check{\lambda}(w) \precsim \check{\lambda}(pwq)$ for all $p, q, w \in \mathcal{A}$, that is, $\check{\lambda}^L$ and $\check{\lambda}^U$ are FIIs of $\mathcal{A}$.

A cubic set [19] in $\mathcal{B}$ is defined to be a function $\langle \check{\lambda}, \eta \rangle : \mathcal{B} \to \mathcal{D}([0, 1]) \times [0, 1], p \mapsto (\check{\lambda}(p), \eta(p))$ where $\check{\lambda} : \mathcal{B} \to \mathcal{D}([0, 1])$ and $\eta : \mathcal{B} \to [0, 1]$. A cubic set $\langle \check{\lambda}, \eta \rangle$ in $\mathcal{A}$ is called

(1) a cubic right ideal (CuRI) [20] of $\mathcal{A}$ if $\check{\lambda}$ is an IvFRI and η is an AFRI of $\mathcal{A}$,

(2) a cubic left ideal (CuLI) [20] of $\mathcal{A}$ if $\check{\lambda}$ is an IvFLI and η is an AFLI of $\mathcal{A}$,

(3) a cubic ideal (CuI) [20] of $\mathcal{A}$ if it is both a CuRI and a CuLI of $\mathcal{A}$, and

(4) a cubic interior ideal (CuII) [31] of $\mathcal{A}$ if $\check{\lambda}$ is an IvFII and η is an AFII of $\mathcal{A}$.

A hesitant fuzzy set (HFS) [41, 42] on $\mathcal{B}$ is defined to be a function $\widehat{\varepsilon} : \mathcal{B} \to \wp([0, 1])$. Note that every IvFS on $\mathcal{B}$ is a HFS on $\mathcal{B}$. A HFS $\widehat{\varepsilon}$ on $\mathcal{A}$ is called

(1) a hesitant fuzzy right ideal (HFRI) [21] of $\mathcal{A}$ if $\widehat{\varepsilon}(p) \subseteq \widehat{\varepsilon}(pq)$ for all $p, q \in \mathcal{A}$,

(2) a hesitant fuzzy left ideal (HFLI) [21] of $\mathcal{A}$ if $\widehat{\varepsilon}(q) \subseteq \widehat{\varepsilon}(pq)$ for all $p, q \in \mathcal{A}$,

(3) a hesitant fuzzy ideal (HFI) [12, 21] of $\mathcal{A}$ if it is both a HFRL and a HFLI of $\mathcal{A}$, that is, $\widehat{\varepsilon}(p) \cup \widehat{\varepsilon}(q) \subseteq \widehat{\varepsilon}(pq)$ for all $p, q \in \mathcal{A}$,

(4) a hesitant fuzzy interior ideal (HFII) [35, 39] of $\mathcal{A}$ if $\widehat{\varepsilon}(w) \subseteq \widehat{\varepsilon}(pwq)$ for all $p, q, w \in \mathcal{A}$.

A hybrid set in $\mathcal{A}$ over a set $\mathcal{B}$ is defined to be a function $(\widehat{\varepsilon}, \eta) : \mathcal{A} \to \wp(\mathcal{B}) \times [0, 1], p \mapsto (\widehat{\varepsilon}(p), \eta(p))$ where $\widehat{\varepsilon} : \mathcal{A} \to \wp(\mathcal{B})$ and $\eta : \mathcal{A} \to [0, 1]$. Note that every cubic set in $\mathcal{A}$ is a hybrid set in $\mathcal{A}$ over $[0, 1]$. A hybrid set $(\widehat{\varepsilon}, \eta)$ in $\mathcal{A}$ over $[0, 1]$ is called

(1) a hybrid right ideal (HyRI) [1] of $\mathcal{A}$ over $[0, 1]$ if $\widehat{\varepsilon}$ is a HFRI and η is an AFRI of $\mathcal{A}$,

(2) a hybrid left ideal (HyLI) [1] of $\mathcal{A}$ over $[0, 1]$ if $\widehat{\varepsilon}$ is a HFLI and η is an AFLI of $\mathcal{A}$,

(3) a hybrid ideal (HyI) [1] of $\mathcal{A}$ over $[0, 1]$ if it is both a HyRI and a HyLI of $\mathcal{A}$ over $[0, 1]$, and

(4) a hybrid interior ideal (HyII) [13] of $\mathcal{A}$ over $[0, 1]$ if $\widehat{\varepsilon}$ is a HFII and η is an AFII of $\mathcal{A}$.

For a HFS $\widehat{\varepsilon}$ on $\mathcal{B}$, a nonempty subset $\mathcal{Z}$ of $\mathcal{B}$, $k \in [0, 1]$ and $\blacktriangledown \in \wp([0, 1])$, we define

(1) the element SUP$\blacktriangledown$ [12, 35] of $[0, 1]$ by

$$\text{SUP}\blacktriangledown = \begin{cases} \sup \blacktriangledown & \text{if } \blacktriangledown \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

(2) the subset $\mathcal{S}[\widehat{\varepsilon}; \blacktriangledown]$ [12, 35] of $\mathcal{B}$ by $\mathcal{S}[\widehat{\varepsilon}; \blacktriangledown] = \{p \in \mathcal{B} \mid \text{SUP}\widehat{\varepsilon}(p) \geq \text{SUP}\blacktriangledown\}$,

(3) the HFS $\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ [14, 16] on $\mathcal{B}$ by $\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(p) = \{k \in \blacktriangledown \mid \text{SUP}\widehat{\varepsilon}(p) \geq k\}$ for all $p \in \mathcal{B}$,

(4) the characteristic hesitant fuzzy set (CHFS) $\widehat{\chi}_{\mathcal{Z}}$ of $\mathcal{Z}$ on B by

$$\widehat{\chi}_{\mathcal{Z}} \colon B \to \wp([0,1]), p \mapsto \begin{cases} [0,1] & \text{if } p \in \mathcal{Z}, \\ \emptyset & \text{otherwise}, \end{cases}$$

(5) the FS $\text{f}^{\widehat{\varepsilon}}$ of $\mathcal{B}$ by $\text{f}^{\widehat{\varepsilon}}(p) = \text{SUP}\widehat{\varepsilon}(p)$ for all $p \in \mathcal{B}$,

(6) a supremum complement $\widehat{\omega}$ [36] of $\widehat{\varepsilon}$ on $\mathcal{B}$ if $\widehat{\omega}$ is a HFS on $\mathcal{B}$ such that $\text{SUP}\widehat{\omega}(p) = 1 - \text{SUP}\widehat{\varepsilon}(p)$ for all $p \in \mathcal{B}$,

(7) the HFS $\widehat{\varepsilon}^*$ by $\widehat{\varepsilon}^*(p) = \{1 - \text{SUP}\widehat{\varepsilon}(p)\}$ for all $p \in \mathcal{B}$.

Let $\mathcal{SC}(\widehat{\varepsilon})$ be the set of all supremum complements of $\widehat{\varepsilon}$. Then, we obtain that

(1) $\widehat{\varepsilon}^* \in \mathcal{SC}(\widehat{\varepsilon})$,

(2) $\text{f}^{\widehat{\omega}}(p) = 1 - \text{SUP}\widehat{\varepsilon}(p)$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $p \in \mathcal{B}$,

(3) $\text{SUP}(\widehat{\varepsilon}^*)^*(p) = \text{SUP}\widehat{\varepsilon}(p) = \text{f}^{\widehat{\varepsilon}}(p)$ for all $p \in \mathcal{B}$,

(2) $\text{SUP}\widehat{\varepsilon}(p) = 1 - (1 - \text{SUP}\widehat{\varepsilon}(p)) = 1 - \text{SUP}\widehat{\omega}(p)$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $p \in \mathcal{B}$,

(5) $\text{SUP}\breve{\lambda}(p) = \sup \breve{\lambda}(p) = \breve{\lambda}^U(p)$ for every IvFS $\breve{\lambda}$ on $\mathcal{B}$ and for all $p \in \mathcal{B}$,

(6) $\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1])}$ is both a HFS and an IvFS on $\mathcal{B}$.

Jittburus and Julatha [12] introduced a sup-hesitant fuzzy ideal, which is a generalization of the concepts of an IvFI and a HFI, of a semigroup and studied its properties via sets, FSs, HFSs and IvFSs in the following.

Definition 2.1. *[12] A HFS $\widehat{\varepsilon}$ on $\mathcal{A}$ is called a* sup-*hesitant fuzzy ideal of $\mathcal{A}$ related to $\blacktriangledown$ (briefly, $\blacktriangledown$-*sup-*hesitant fuzzy ideal) of $\mathcal{A}$ if the set $\mathcal{S}[\widehat{\varepsilon};\blacktriangledown]$ is an ideal of $\mathcal{A}$. We say that $\widehat{\varepsilon}$ is a* sup-*hesitant fuzzy ideal (*sup-*HFI) of $\mathcal{A}$ if $\widehat{\varepsilon}$ is a $\blacktriangledown$-*sup-*hesitant fuzzy ideal of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0,1])$ when $\mathcal{S}[\widehat{\varepsilon};\blacktriangledown] \neq \emptyset$.*

Theorem 2.1. *[12] Every HFI of $\mathcal{A}$ is a* sup-*HFI of $\mathcal{A}$.*

Theorem 2.2. *[12] Every IvFI of $\mathcal{A}$ is a* sup-*HFI of $\mathcal{A}$.*

Theorem 2.3. *[12] Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$. The followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a* sup-*HFI of $\mathcal{A}$,*

(2) *$\text{f}^{\widehat{\varepsilon}}$ is a FI of $\mathcal{A}$,*

(3) *$\text{SUP}\widehat{\varepsilon}(pq) \geq \max\{\text{SUP}\widehat{\varepsilon}(p), \text{SUP}\widehat{\varepsilon}(q)\}$ for all $p, q \in \mathcal{A}$,*

(4) *$\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ is a HFI of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0,1])$.*

Theorem 2.4. *[12] Let $\mathcal{B}$ be a nonempty subset of $\mathcal{A}$. Then $\mathcal{B}$ is an ideal of $\mathcal{A}$ if and only if the CHFS $\widehat{\chi}_{\mathcal{B}}$ is a* sup-*HFI of $\mathcal{A}$.*

Phummee et al. [35] introduced a sup-hesitant fuzzy interior ideal, shown a generalization of the concepts of a sup-HFI, an IvFII and a HFII, of a semigroup and studied its properties via sets, FSs, HFSs and IvFSs.

Definition 2.2. *[35] A HFS $\widehat{\varepsilon}$ on $\mathcal{A}$ is called a* sup-*hesitant fuzzy interior ideal of $\mathcal{A}$ related to $\blacktriangledown$ (briefly, $\blacktriangledown$-*sup-*hesitant fuzzy interior ideal) of $\mathcal{A}$ if the set $\mathcal{S}[\widehat{\varepsilon};\blacktriangledown]$ is an interior ideal of $\mathcal{A}$. We say that $\widehat{\varepsilon}$ is a* sup-*hesitant fuzzy interior ideal (*sup-*HFII) of $\mathcal{A}$ if $\widehat{\varepsilon}$ is a $\blacktriangledown$-*sup-*hesitant fuzzy interior ideal of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0,1])$ when $\mathcal{S}[\widehat{\varepsilon};\blacktriangledown] \neq \emptyset$.*

Theorem 2.5. *[35] Every* sup-*HFI of $\mathcal{A}$ is a* sup-*HFII of $\mathcal{A}$.*

Theorem 2.6. *[35] Every HFII of $\mathcal{A}$ is a* sup-*HFII of $\mathcal{A}$.*

Theorem 2.7. *[35] Every IvFII of $\mathcal{A}$ is a* sup-*HFII of $\mathcal{A}$.*

Theorem 2.8. *[35] Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$. The followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* sup-*HFII of $\mathcal{A}$,*
(2) $f^{\widehat{\varepsilon}}$ *is a FII of $\mathcal{A}$,*
(3) $\mathrm{SUP}\widehat{\varepsilon}(pwq) \geq \mathrm{SUP}\widehat{\varepsilon}(w)$ *for all $p, q, w \in \mathcal{A}$,*
(4) $\mathcal{H}_{\mathrm{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ *is a HFII of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0,1])$.*

Theorem 2.9. *[35] If $\mathcal{B}$ is a nonempty subset of $\mathcal{A}$, then $\mathcal{B}$ is an interior ideal of $\mathcal{A}$ if and only if $\widehat{\chi}_{\mathcal{B}}$ is a* sup-*HFII of $\mathcal{A}$.*

Julatha et al. [13] introduced a sup-hesitant fuzzy right (left) ideal, shown a generalization of the concept of a HFRI (HFLI) and an IvFRI (IvFLI), of a semigroup and studied its properties via sets, FSs, PFSs, HFSs, IvFSs, cubic sets and hybrid sets.

Definition 2.3. *[13] Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$.*

(1) $\widehat{\varepsilon}$ *is called a* sup-*hesitant fuzzy left ideal (*sup-*HFLI) of $\mathcal{A}$ if $(\forall p, q \in \mathcal{A})(\mathrm{SUP}\widehat{\varepsilon}(q) \leq \mathrm{SUP}\widehat{\varepsilon}(pq))$.*
(2) $\widehat{\varepsilon}$ *is called a* sup-*hesitant fuzzy right ideal (*sup-*HFRI) of $\mathcal{A}$ if $(\forall p, q \in \mathcal{A})(\mathrm{SUP}\widehat{\varepsilon}(p) \leq \mathrm{SUP}\widehat{\varepsilon}(pq))$.*

Theorem 2.10. *[13] Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$. The followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* sup-*HFRI (*sup-*HFLI) of $\mathcal{A}$,*
(2) $f^{\widehat{\varepsilon}}$ *is a FRI (FLI) of $\mathcal{A}$,*
(3) $\mathcal{H}_{\mathrm{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ *is a HFRI (HFLI) of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0,1])$.*

Theorem 2.11. *[13] If $\mathcal{B}$ is a nonempty subset of $\mathcal{A}$, then $\mathcal{B}$ is a right ideal (resp., left ideal) of $\mathcal{A}$ if and only if $\widehat{\chi}_{\mathcal{B}}$ is a* sup-*HFRI (resp.,* sup-*HFLI) of $\mathcal{A}$.*

3. Generalized sup-hesitant fuzzy ideals

In what follows, let α and β be elements of $[0,1]$, unless otherwise specified. We introduce the concepts of $\sup_{\alpha}^{+}$-hesitant fuzzy left (resp., right, interior, two-sided) ideals and $\sup_{\beta}^{-}$-hesitant fuzzy left (resp., right, interior, two-sided) ideals of semigroups and investigate their properties. The concepts are established by FSs, PFSs, HFSs, IvFSs, Łukasiewicz fuzzy sets, Łukasiewicz anti-fuzzy sets, hybrid sets and cubic sets.

3.1. $\sup_\alpha^+$**-hesitant fuzzy ideals.** In this part, we introduce a $\sup_\alpha^+$-hesitant fuzzy right ideal, a $\sup_\alpha^+$-hesitant fuzzy left ideal, a $\sup_\alpha^+$-hesitant fuzzy interior ideal and a $\sup_\alpha^+$-hesitant fuzzy two-sided ideal of a semigroup, and investigate some of their properties. Also, it is shown that a $\sup_\alpha^+$-hesitant fuzzy left (resp., right, interior, two-sided) ideal of a semigroup is a generalization of the concept of a sup-hesitant fuzzy left (resp., right, interior, two-sided) ideal.

For a HFS $\widehat{\varepsilon}$ on $\mathcal{A}$, and $\blacktriangledown, \blacktriangle \in \wp([0,1])$, we define

1. $\mathrm{SUP}_\alpha^+\blacktriangledown = \min\{\mathrm{SUP}\blacktriangledown + \alpha, 1\}$,
2. $\blacktriangledown \sqsubseteq_\alpha^+ \blacktriangle$ if and only if $\mathrm{SUP}_\alpha^+\blacktriangledown \leq \mathrm{SUP}_\alpha^+\blacktriangle$,
3. $\blacktriangledown \sqsubset_\alpha^+ \blacktriangle$ if and only if $\mathrm{SUP}_\alpha^+\blacktriangledown < \mathrm{SUP}_\alpha^+\blacktriangle$,
4. $\blacktriangledown \cong_\alpha^+ \blacktriangle$ if and only if $\mathrm{SUP}_\alpha^+\blacktriangledown = \mathrm{SUP}_\alpha^+\blacktriangle$.

We denote $\blacktriangledown \sqsubseteq \blacktriangle$ (resp., $\blacktriangledown \sqsubset \blacktriangle$, $\blacktriangledown \cong \blacktriangle$) for $\blacktriangledown \sqsubseteq_0^+ \blacktriangle$ (resp., $\blacktriangledown \sqsubset_0^+ \blacktriangle$, $\blacktriangledown \cong_0^+ \blacktriangle$). Then we have

1. $\mathrm{SUP}_0^+\blacktriangledown = \mathrm{SUP}\blacktriangledown$,
2. $\blacktriangledown \sqsubseteq \blacktriangle$ if and only if $\mathrm{SUP}\blacktriangledown \leq \mathrm{SUP}\blacktriangle$,
3. $\blacktriangledown \sqsubset \blacktriangle$ if and only if $\mathrm{SUP}\blacktriangledown < \mathrm{SUP}\blacktriangle$,
4. $\blacktriangledown \cong \blacktriangle$ if and only if $\mathrm{SUP}\blacktriangledown = \mathrm{SUP}\blacktriangle$,
5. $\blacktriangledown \cong_\alpha^+ \blacktriangle$ if and only if $\blacktriangledown \sqsubseteq_\alpha^+ \blacktriangle$ and $\blacktriangle \sqsubseteq_\alpha^+ \blacktriangledown$.

For elements $\breve{a} = [a^-, a^+]$ and $\breve{b} = [b^-, b^+]$ in $\mathcal{D}([0,1])$, then the following are true:

1. if $\breve{a} \precsim \breve{b}$, then $\breve{a} \sqsubseteq \breve{b}$, and
2. if $\breve{a} = \breve{b}$, then $\breve{a} \cong \breve{b}$.

Definition 3.1. *A HFS $\widehat{\varepsilon}$ on $\mathcal{A}$ is called*

1. *a $\sup_\alpha^+$-hesitant fuzzy right ideal ($\sup_\alpha^+$-HFRI) of $\mathcal{A}$ if $(\forall p, q \in \mathcal{A})(\widehat{\varepsilon}(p) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(pq))$,*
2. *a $\sup_\alpha^+$-hesitant fuzzy left ideal ($\sup_\alpha^+$-HFLI) of $\mathcal{A}$ if $(\forall p, q \in \mathcal{A})(\widehat{\varepsilon}(q) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(pq))$,*
3. *a $\sup_\alpha^+$-hesitant fuzzy two-sided ideal (or a $\sup_\alpha^+$-hesitant fuzzy ideal ($\sup_\alpha^+$-HFI)) of $\mathcal{A}$ if it is both a $\sup_\alpha^+$-HFRI and a $\sup_\alpha^+$-HFLI of $\mathcal{A}$,*
4. *a $\sup_\alpha^+$-hesitant fuzzy interior ideal ($\sup_\alpha^+$-HFII) of $\mathcal{A}$ if $(\forall p, q, w \in \mathcal{A})(\widehat{\varepsilon}(w) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(pwq))$.*

Example 3.1. *Let $\mathcal{A} = \{(1,1), (0,1), (0,0), (1,0)\}$. Then $\mathcal{A}$ is a semigroup with respect to multiplication defined as follows: $(p_1, p_2)(p_3, p_4) = (p_1, p_4)$ for all $p_1, p_2, p_3, p_4 \in \{0,1\}$.*

1. *A HFS $\widehat{\varepsilon}_1$ of $\mathcal{A}$ is defined by*

$$\widehat{\varepsilon}_1((0,0)) = (0, 0.8),\ \widehat{\varepsilon}_1((0,1)) = \{0.2, 0.4, 0.9\},\ \widehat{\varepsilon}_1((1,0)) = \emptyset \text{ and } \widehat{\varepsilon}_1((1,1)) = \{0\}.$$

Then $\widehat{\varepsilon}_1$ is a $\sup_{0.2}^+$-HFRI of $\mathcal{A}$ but not a $\sup_{0.2}^+$-HFLI of $\mathcal{A}$ because

$$\widehat{\varepsilon}_1((1,0)(0,0)) \sqsubset_{0.2}^+ \widehat{\varepsilon}_1((0,0)).$$

(2) *A HFS* $\widehat{\varepsilon}_2$ *of* $\mathcal{A}$ *is defined by*

$$\widehat{\varepsilon}_2((0,0)) = \{0.2, 0.4, 0.5\},\ \widehat{\varepsilon}_2((0,1)) = (0.3, 0.7),\ \widehat{\varepsilon}_2((1,0)) = [0, 0.5] \text{ and } \widehat{\varepsilon}_2((1,1)) = \{0.7, 0.8, 0.9\}.$$

Then $\widehat{\varepsilon}_2$ *is a* $\sup^+_{0.3}$*-HFLI of* $\mathcal{A}$ *but not a* $\sup^+_{0.3}$*-HFRI of* $\mathcal{A}$ *because*

$$\widehat{\varepsilon}_2((1,1)(1,0)) \sqsubset^+_{0.3} \widehat{\varepsilon}_2((1,1)).$$

Example 3.2. *Let* $\mathcal{A} = \{p_1, p_2, p_3, p_4\}$ *and define the binary operation "* $\cdot$*" on* $\mathcal{A}$ *as follows:*

$\cdot$	p_1	p_2	p_3	p_4
p_1	p_1	p_1	p_1	p_1
p_2	p_1	p_1	p_1	p_1
p_3	p_1	p_1	p_2	p_1
p_4	p_1	p_1	p_2	p_2

Then $\mathcal{A}$ *is a be the semigroup under the binary operation "* $\cdot$*" [30]. Now, define HFSs* $\widehat{\varepsilon}_1$ *and* $\widehat{\varepsilon}_2$ *on* $\mathcal{A}$ *by*

$$\widehat{\varepsilon}_1(p_1) = [0.3, 0.6],\ \widehat{\varepsilon}_1(p_2) = \{0.3, 0.5\},\ \widehat{\varepsilon}_1(p_3) = \emptyset \text{ and } \widehat{\varepsilon}_1(p_4) = \{1\},$$
$$\widehat{\varepsilon}_2(p_1) = [0.3, 0.7],\ \widehat{\varepsilon}_2(p_2) = \{0.3, 0.5, 0.8\},\ \widehat{\varepsilon}_2(p_3) = \emptyset \text{ and } \widehat{\varepsilon}_2(p_4) = [0.2, 0.5].$$

Thus

(1) $\widehat{\varepsilon}_1$ *is a* $\sup^+_{0.4}$*-HFII but not a* $\sup^+_{0.4}$*-HFI of* $\mathcal{A}$ *because* $\widehat{\varepsilon}_1(p_4p_4) \sqsubset^+_{0.4} \widehat{\varepsilon}_1(p_4)$,

(2) $\widehat{\varepsilon}_2$ *is a* $\sup^+_{0.3}$*-HFI of* $\mathcal{A}$.

Proposition 3.1. *Every* sup*-HFRI (resp.,* sup*-HFLI,* sup*-HFII,* sup*-HFI) of* $\mathcal{A}$ *is a* $\sup^+_\alpha$*-HFRI (resp.,* $\sup^+_\alpha$*-HFLI,* $\sup^+_\alpha$*-HFII,* $\sup^+_\alpha$*-HFI) of* $\mathcal{A}$ *for all* $\alpha \in [0,1]$.

Proof. Assume that $\widehat{\varepsilon}$ is a sup-HFRI of $\mathcal{A}$, $\alpha \in [0,1]$ and $p, q \in \mathcal{A}$. Then $\mathrm{SUP}\widehat{\varepsilon}(pq) \geq \mathrm{SUP}\widehat{\varepsilon}(p)$ and so

$$\mathrm{SUP}^+_\alpha\widehat{\varepsilon}(pq) = \min\{\mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha, 1\} \geq \min\{\mathrm{SUP}\widehat{\varepsilon}(p) + \alpha, 1\} = \mathrm{SUP}^+_\alpha\widehat{\varepsilon}(p).$$

Hence $\widehat{\varepsilon}(p) \sqsubseteq^+_\alpha \widehat{\varepsilon}(pq)$. Therefore, we obtain that $\widehat{\varepsilon}$ is a $\sup^+_\alpha$-HFRI of $\mathcal{A}$.

Similarly, we can prove the other results. □

Example 3.3. *Let* $\mathcal{A} = \{p_1, p_2, p_3, p_4\}$ *be the semigroup defined in Example 3.2. We define a HFS* $\widehat{\varepsilon}$ *on* $\mathcal{A}$ *by*

$$\widehat{\varepsilon}(p_1) = \{0.1, 0.5, 0.8\},\ \widehat{\varepsilon}(p_2) = [0, 0.9],\ \widehat{\varepsilon}(p_3) = [0, 0.5] \text{ and } \widehat{\varepsilon}(p_4) = \emptyset.$$

Then $\widehat{\varepsilon}$ *is a* $\sup^+_\alpha$*-HFRI (resp.,* $\sup^+_\alpha$*-HFLI,* $\sup^+_\alpha$*-HFII,* $\sup^+_\alpha$*-HFI) of* $\mathcal{A}$ *for all* $\alpha \in [0.2, 1]$ *but* $\widehat{\varepsilon}$ *is not a* sup*-HFRI (resp.,* sup*-HFLI,* sup*-HFII,* sup*-HFI) of* $\mathcal{A}$*. Indeed,* $\widehat{\varepsilon}$ *is not a* sup*-HFRI and* sup*-HFI of* $\mathcal{A}$ *because* $\mathrm{SUP}\widehat{\varepsilon}(p_2p_1) < \mathrm{SUP}\widehat{\varepsilon}(p_2)$*,* $\widehat{\varepsilon}$ *is not a* sup*-HFLI of* $\mathcal{A}$ *because* $\mathrm{SUP}\widehat{\varepsilon}(p_1p_2) < \mathrm{SUP}\widehat{\varepsilon}(p_2)$*, and* $\widehat{\varepsilon}$ *is not a* sup*-HFII of* $\mathcal{A}$ *because* $\mathrm{SUP}\widehat{\varepsilon}(p_1p_2p_3) < \mathrm{SUP}\widehat{\varepsilon}(p_2)$.

From Proposition 3.1 and Example 3.3, we have that the concept of a $\sup_\alpha^+$-HFRI (resp., $\sup_\alpha^+$-HFLI, $\sup_\alpha^+$-HFII, $\sup_\alpha^+$-HFI) of a semigroup $\mathcal{A}$ is a generalization of the concept of a sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of $\mathcal{A}$.

Proposition 3.2. *Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$ and $k \in [0,1]$. If $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFRI (resp., $\sup_\alpha^+$-HFLI, $\sup_\alpha^+$-HFII, $\sup_\alpha^+$-HFI) of $\mathcal{A}$ for all $\alpha \in [0,k]$, then $\widehat{\varepsilon}$ is a sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of $\mathcal{A}$.*

Proof. Let $\widehat{\varepsilon}$ be a $\sup_\alpha^+$-HFRI of $\mathcal{A}$ for all $\alpha \in [0,k]$. Suppose $\widehat{\varepsilon}$ is not a sup-HFRI of $\mathcal{A}$, that is, there exist $p, q \in \mathcal{A}$ such that $\mathrm{SUP}\widehat{\varepsilon}(pq) < \mathrm{SUP}\widehat{\varepsilon}(p)$. Choose

$$\alpha = \min\{\frac{\mathrm{SUP}\widehat{\varepsilon}(p) - \mathrm{SUP}\widehat{\varepsilon}(pq)}{2}, k\}.$$

Then $\alpha \in [0,k]$ and

$$\begin{aligned}\mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha &\leq \mathrm{SUP}\widehat{\varepsilon}(pq) + (\frac{\mathrm{SUP}\widehat{\varepsilon}(p) - \mathrm{SUP}\widehat{\varepsilon}(pq)}{2})\\ &< \mathrm{SUP}\widehat{\varepsilon}(pq) + (\mathrm{SUP}\widehat{\varepsilon}(p) - \mathrm{SUP}\widehat{\varepsilon}(pq))\\ &= \mathrm{SUP}\widehat{\varepsilon}(p)\\ &\leq 1.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{SUP}_\alpha^+\widehat{\varepsilon}(p) &= \min\{\mathrm{SUP}\widehat{\varepsilon}(p) + \alpha, 1\}\\ &> \mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha\\ &= \min\{\mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha, 1\}\\ &= \mathrm{SUP}_\alpha^+\widehat{\varepsilon}(pq).\end{aligned}$$

Hence $\widehat{\varepsilon}(pq) \sqsubset_\alpha^+ \widehat{\varepsilon}(p)$. Since $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFRI of $\mathcal{A}$, we hvae

$$\widehat{\varepsilon}(pq) \sqsubset_\alpha^+ \widehat{\varepsilon}(p) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(pq).$$

This is a contradiction. Therefore, $\widehat{\varepsilon}$ is a sup-HFRI of $\mathcal{A}$.

Similarly, we can prove the other results. □

Proposition 3.3. *If $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFRI (resp., $\sup_\alpha^+$-HFLI, $\sup_\alpha^+$-HFII, $\sup_\alpha^+$-HFI) of $\mathcal{A}$, then $\widehat{\varepsilon}$ is a $\sup_k^+$-HFRI (resp., $\sup_k^+$-HFLI, $\sup_k^+$-HFII, $\sup_k^+$-HFI) of $\mathcal{A}$ for all $k \in [\alpha, 1]$.*

Proof. Assume that $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFRI of $\mathcal{A}$, $k \in [\alpha,1]$ and $p, q \in \mathcal{A}$. Then $\widehat{\varepsilon}(p) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(pq)$, that is, $\min\{\mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha, 1\} \geq \min\{\mathrm{SUP}\widehat{\varepsilon}(p) + \alpha, 1\}$. If $\mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha \geq 1$, then

$$\mathrm{SUP}\widehat{\varepsilon}(pq) + k \geq \mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha \geq 1 \geq \mathrm{SUP}_k^+\widehat{\varepsilon}(p)$$

and so $\widehat{\varepsilon}(p) \sqsubseteq_k^+ \widehat{\varepsilon}(pq)$. On the other hand, suppose that $\text{SUP}\widehat{\varepsilon}(pq) + \alpha \geq \text{SUP}\widehat{\varepsilon}(p) + \alpha$. Then $\text{SUP}\widehat{\varepsilon}(pq) \geq \text{SUP}\widehat{\varepsilon}(p)$ and so

$$\text{SUP}\widehat{\varepsilon}(pq) + k \geq \text{SUP}\widehat{\varepsilon}(p) + k \geq \text{SUP}_k^+\widehat{\varepsilon}(p).$$

Thus $\widehat{\varepsilon}(p) \sqsubseteq_k^+ \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup_k^+$-HFRI of $\mathcal{A}$.

Similarly, we can prove the other results. □

Proposition 3.4. *Every* $\sup_\alpha^+$*-HFI of* $\mathcal{A}$ *is a* $\sup_\alpha^+$*-HFII of* $\mathcal{A}$.

Proof. Assume that $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFI of $\mathcal{A}$. Then $\widehat{\varepsilon}(w) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(wq) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(pwq)$ for all $p, q, w \in \mathcal{A}$. Therefore, $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFII of $\mathcal{A}$. □

From Proposition 3.4 and Example 3.2, we have that the concept of a $\sup_\alpha^+$-HFII of a semigroup $\mathcal{A}$ is a generalization of the concept of a $\sup_\alpha^+$-HFI of $\mathcal{A}$.

3.2. $\sup_\beta^-$**-hesitant fuzzy ideals.** In this part, we introduce a $\sup_\beta^-$-hesitant fuzzy right ideal, a $\sup_\beta^-$-hesitant fuzzy left ideal, a $\sup_\beta^-$-hesitant fuzzy interior ideal and a $\sup_\beta^-$-hesitant fuzzy two-sided ideal of a semigroup, and investigate some of their properties. Moreover, it is shown that a $\sup_\beta^-$-hesitant fuzzy left (resp., right, interior, two-sided) ideal of a semigroup is a generalization of the concept of a sup-hesitant fuzzy left (resp., right, interior, two-sided) ideal.

For a HFS $\widehat{\varepsilon}$ on $\mathcal{A}$ and $\blacktriangledown, \blacktriangle \in \wp([0,1])$, we define

(1) $\text{SUP}_\beta^-\blacktriangledown = \max\{\text{SUP}\blacktriangledown - \beta, 0\}$,
(2) $\blacktriangledown \sqsubseteq_\beta^- \blacktriangle$ if and only if $\text{SUP}_\beta^-\blacktriangledown \leq \text{SUP}_\beta^-\blacktriangle$,
(3) $\blacktriangledown \sqsubset_\beta^- \blacktriangle$ if and only if $\text{SUP}_\beta^-\blacktriangledown < \text{SUP}_\beta^-\blacktriangle$,
(4) $\blacktriangledown \cong_\beta^- \blacktriangle$ if and only if $\text{SUP}_\beta^-\blacktriangledown = \text{SUP}_\beta^-\blacktriangle$.

Then we have

(1) $\text{SUP}_0^-\blacktriangledown = \text{SUP}\blacktriangledown$,
(2) $\blacktriangledown \sqsubseteq \blacktriangle$ if and only if $\blacktriangledown \sqsubseteq_0^- \blacktriangle$,
(3) $\blacktriangledown \sqsubset \blacktriangle$ if and only if $\blacktriangledown \sqsubset_0^- \blacktriangle$,
(4) $\blacktriangledown \cong \blacktriangle$ if and only if $\blacktriangledown \cong_0^- \blacktriangle$.

Definition 3.2. *Let* $\widehat{\varepsilon}$ *be a HFS on* $\mathcal{A}$.

(1) $\widehat{\varepsilon}$ *is called a* $\sup_\beta^-$*-hesitant fuzzy right ideal* ($\sup_\beta^-$*-HFRI) of* $\mathcal{A}$ *if* $(\forall p, q \in \mathcal{A})(\widehat{\varepsilon}(p) \sqsubseteq_\beta^- \widehat{\varepsilon}(pq))$,
(2) $\widehat{\varepsilon}$ *is called a* $\sup_\beta^-$*-hesitant fuzzy left ideal* ($\sup_\beta^-$*-HFLI) of* $\mathcal{A}$ *if* $(\forall p, q \in \mathcal{A})(\widehat{\varepsilon}(q) \sqsubseteq_\beta^- \widehat{\varepsilon}(pq))$,
(3) $\widehat{\varepsilon}$ *is called a* $\sup_\beta^-$*-hesitant fuzzy two-sided ideal (or a* $\sup_\beta^-$*-hesitant fuzzy ideal* ($\sup_\beta^-$*-HFI)) of* $\mathcal{A}$ *if it is both a* $\sup_\beta^-$*-HFRI and a* $\sup_\beta^-$*-HFLI of* $\mathcal{A}$,
(4) $\widehat{\varepsilon}$ *is called a* $\sup_\beta^-$*-hesitant fuzzy interior ideal* ($\sup_\beta^-$*-HFII) of* $\mathcal{A}$ *if* $(\forall p, q, w \in \mathcal{A})(\widehat{\varepsilon}(w) \sqsubseteq_\beta^- \widehat{\varepsilon}(pwq))$.

Example 3.4. *Let* $\mathcal{A} = \{(1,1), (0,1), (0,0), (1,0)\}$ *be a semigroup defined in Example 3.1.*

(1) *A HFS* $\widehat{\varepsilon}_1$ *of* $\mathcal{A}$ *is defined by*

$$\widehat{\varepsilon}_1((0,0)) = (0, 0.8],\ \widehat{\varepsilon}_1((0,1)) = \{0.4, 0.8\},\ \widehat{\varepsilon}_1((1,0)) = \emptyset \text{ and } \widehat{\varepsilon}_1((1,1)) = [0, 0.4].$$

Then $\widehat{\varepsilon}_1$ *is a* $\sup^-_{0.4}$*-HFRI of* $\mathcal{A}$ *but not a* $\sup^-_{0.4}$*-HFLI of* $\mathcal{A}$ *because*

$$\widehat{\varepsilon}_1((1,0)(0,1)) \sqsubset^-_{0.4} \widehat{\varepsilon}_1((0,1)).$$

(2) *A HFS* $\widehat{\varepsilon}_2$ *of* $\mathcal{A}$ *is defined by*

$$\widehat{\varepsilon}_2((0,0)) = \{0.2, 0.4, 0.5\},\ \widehat{\varepsilon}_2((0,1)) = (0.3, 0.6),\ \widehat{\varepsilon}_2((1,0)) = \emptyset \text{ and}$$
$$\widehat{\varepsilon}_2((1,1)) = \{0.4, 0.5, 0.6\}.$$

Then $\widehat{\varepsilon}_2$ *is a* $\sup^-_{0.5}$*-HFLI of* $\mathcal{A}$ *but not a* $\sup^-_{0.5}$*-HFRI of* $\mathcal{A}$ *because*

$$\widehat{\varepsilon}_2((0,1)(1,0)) \sqsubset^-_{0.5} \widehat{\varepsilon}_2((0,1)).$$

Example 3.5. *Let* $\mathcal{A} = \{p_1, p_2, p_3, p_4\}$ *be the semigroup defined in Example 3.2. We define a HFS* $\widehat{\varepsilon}$ *on* $\mathcal{A}$ *by*

$$\widehat{\varepsilon}(p_1) = [0.3, 0.7],\ \widehat{\varepsilon}(p_2) = \{0.3, 0.5\},\ \widehat{\varepsilon}(p_3) = \emptyset, \text{ and } \widehat{\varepsilon}(p_4) = (0, 0.7).$$

Thus

(1) $\widehat{\varepsilon}$ *is a* $\sup^-_{\beta}$*-HFII of* $\mathcal{A}$ *for all* $\beta \in [0,1]$ *but not a* $\sup^-_{k}$*-HFI of* $\mathcal{A}$ *for all* $k \in [0, 0.7)$ *because*

$$\widehat{\varepsilon}(p_4p_3) \sqsubset^-_k \widehat{\varepsilon}(p_4) \text{ for all } k \in [0, 0.7).$$

(2) $\widehat{\varepsilon}$ *is a* $\sup^-_{\beta}$*-HFI of* $\mathcal{A}$ *for all* $\beta \in [0.7, 1]$.

Proposition 3.5. *Every* sup*-HFRI (resp.,* sup*-HFLI,* sup*-HFII,* sup*-HFI) of* $\mathcal{A}$ *is a* $\sup^-_{\beta}$*-HFRI (resp.,* $\sup^-_{\beta}$*-HFLI,* $\sup^-_{\beta}$*-HFII,* $\sup^-_{\beta}$*-HFI) of* $\mathcal{A}$ *for all* $\beta \in [0,1]$.

Proof. Assume that $\widehat{\varepsilon}$ is a sup-HFRI of $\mathcal{A}$, $\beta \in [0,1]$ and $p, q \in \mathcal{A}$. Then $\mathrm{SUP}\widehat{\varepsilon}(pq) \geq \mathrm{SUP}\widehat{\varepsilon}(p)$ and thus

$$\mathrm{SUP}^-_{\beta}\widehat{\varepsilon}(pq) = \max\{\mathrm{SUP}\widehat{\varepsilon}(pq) - \beta, 0\} \geq \max\{\mathrm{SUP}\widehat{\varepsilon}(p) - \beta, 0\} = \mathrm{SUP}^-_{\beta}\widehat{\varepsilon}(p).$$

Hence $\widehat{\varepsilon}(p) \sqsubseteq^-_{\beta} \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup^-_{\beta}$-HFRI of $\mathcal{A}$.

Similarly, we can prove the other results. □

Example 3.6. *From Example 3.3, we get that the HFS* $\widehat{\varepsilon}$ *is a* $\sup^-_{\beta}$*-HFRI,* $\sup^-_{\beta}$*-HFLI,* $\sup^-_{\beta}$*-HFII and* $\sup^-_{\beta}$*-HFI of* $\mathcal{A}$ *for all* $\beta \in [0.9, 1]$*. However,* $\widehat{\varepsilon}$ *is not a* sup*-HFRI,* sup*-HFLI,* sup*-HFII and* sup*-HFI of* $\mathcal{A}$*.*

From Example 3.6 and Proposition 3.5, we obtain that the concept of a $\sup^-_{\beta}$-HFRI (resp., $\sup^-_{\beta}$-HFLI, $\sup^-_{\beta}$-HFII, $\sup^-_{\beta}$-HFI) of a semigroup $\mathcal{A}$ is a generalization of the concept of a sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of $\mathcal{A}$.

Proposition 3.6. *Let* $\widehat{\varepsilon}$ *be a HFS on* $\mathcal{A}$ *and* $k \in [0,1]$*. If* $\widehat{\varepsilon}$ *is a* $\sup^-_{\beta}$*-HFRI (resp.,* $\sup^-_{\beta}$*-HFLI,* $\sup^-_{\beta}$*-HFII,* $\sup^-_{\beta}$*-HFI) of* $\mathcal{A}$ *for all* $\beta \in [0,k]$*, then* $\widehat{\varepsilon}$ *is a* sup*-HFRI (resp.,* sup*-HFLI,* sup*-HFII,* sup*-HFI) of* $\mathcal{A}$*.*

Proof. Let $\widehat{\varepsilon}$ be a $\sup_\beta^-$-HFRI of $\mathcal{A}$ for all $\beta \in [0,k]$. Suppose that $\mathrm{SUP}\widehat{\varepsilon}(pq) < \mathrm{SUP}\widehat{\varepsilon}(p)$ for some $p, q \in \mathcal{A}$. Choose $\beta \in [0,k]$ such that $\mathrm{SUP}\widehat{\varepsilon}(p) - \beta > 0$. Then

$$\mathrm{SUP}_\beta^-\widehat{\varepsilon}(p) = \max\{\mathrm{SUP}\widehat{\varepsilon}(p) - \beta, 0\} = \mathrm{SUP}\widehat{\varepsilon}(p) - \beta.$$

Since $\widehat{\varepsilon}$ is a $\sup_\beta^-$-HFRI of $\mathcal{A}$, we have

$$\mathrm{SUP}_\beta^-\widehat{\varepsilon}(pq) \geq \mathrm{SUP}_\beta^-\widehat{\varepsilon}(p) = \mathrm{SUP}\widehat{\varepsilon}(p) - \beta > 0.$$

Thus

$$\begin{aligned}\mathrm{SUP}_\beta^-\widehat{\varepsilon}(pq) &= \mathrm{SUP}\widehat{\varepsilon}(pq) - \beta \\ &< \mathrm{SUP}\widehat{\varepsilon}(p) - \beta \\ &= \mathrm{SUP}_\beta^-\widehat{\varepsilon}(p) \\ &\leq \mathrm{SUP}_\beta^-\widehat{\varepsilon}(pq),\end{aligned}$$

which is a contradiction. Hence $\widehat{\varepsilon}$ is a sup-HFRI of $\mathcal{A}$.

Similarly, we can prove the other results. □

Proposition 3.7. *If $\widehat{\varepsilon}$ is a $\sup_\beta^-$-HFRI (resp., $\sup_\beta^-$-HFLI, $\sup_\beta^-$-HFII, $\sup_\beta^-$-HFI) of $\mathcal{A}$, then $\widehat{\varepsilon}$ is a $\sup_k^-$-HFRI (resp., $\sup_k^-$-HFLI, $\sup_k^-$-HFII, $\sup_k^-$-HFI) of $\mathcal{A}$ for all $k \in [\beta, 1]$.*

Proof. Assume that $\widehat{\varepsilon}$ is a $\sup_\beta^-$-HFRI of $\mathcal{A}$, $k \in [\beta, 1]$ and $p, q \in \mathcal{A}$. Then $\widehat{\varepsilon}(p) \sqsubseteq_\beta^- \widehat{\varepsilon}(pq)$, that is,

$$\max\{\mathrm{SUP}\widehat{\varepsilon}(pq) - \beta, 0\} \geq \max\{\mathrm{SUP}\widehat{\varepsilon}(p) - \beta, 0\} \geq \mathrm{SUP}\widehat{\varepsilon}(p) - \beta.$$

If $0 \geq \mathrm{SUP}\widehat{\varepsilon}(p) - \beta$, then

$$\mathrm{SUP}_k^-\widehat{\varepsilon}(pq) \geq 0 \geq \mathrm{SUP}\widehat{\varepsilon}(p) - \beta \geq \mathrm{SUP}\widehat{\varepsilon}(p) - k$$

and so $\mathrm{SUP}_k^-\widehat{\varepsilon}(pq) \geq \mathrm{SUP}_k^-\widehat{\varepsilon}(p)$, which implies that $\widehat{\varepsilon}(p) \sqsubseteq_k^- \widehat{\varepsilon}(pq)$. On the other hand, suppose that $\mathrm{SUP}\widehat{\varepsilon}(pq) - \beta \geq \mathrm{SUP}\widehat{\varepsilon}(p) - \beta$. Then $\mathrm{SUP}\widehat{\varepsilon}(pq) \geq \mathrm{SUP}\widehat{\varepsilon}(p)$. Thus $\mathrm{SUP}\widehat{\varepsilon}(pq) - k \geq \mathrm{SUP}\widehat{\varepsilon}(p) - k$. Hence $\mathrm{SUP}_k^-\widehat{\varepsilon}(pq) \geq \mathrm{SUP}_k^-\widehat{\varepsilon}(p)$, which implies that $\widehat{\varepsilon}(p) \sqsubseteq_k^- \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup_k^-$-HFRI of $\mathcal{A}$.

Similarly, we can prove the other results. □

Proposition 3.8. *Every $\sup_\beta^-$-HFI of $\mathcal{A}$ is a $\sup_\beta^-$-HFII of $\mathcal{A}$.*

Proof. Assume that $\widehat{\varepsilon}$ is a $\sup_\beta^-$-HFI of $\mathcal{A}$. Then $\widehat{\varepsilon}(w) \sqsubseteq_\beta^- \widehat{\varepsilon}(wq) \sqsubseteq_\beta^- \widehat{\varepsilon}(pwq)$ for all $p, q, w \in \mathcal{A}$. Therefore, $\widehat{\varepsilon}$ is a $\sup_\beta^-$-HFII of $\mathcal{A}$. □

From Proposition 3.8 and Example 3.5, we have that the concept of a $\sup_\beta^-$-HFII of a semigroup $\mathcal{A}$ is a generalization of the concept of a $\sup_\beta^-$-HFI of $\mathcal{A}$.

Proposition 3.9. *Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$. Then the followings are true:*

(1) $\widehat{\varepsilon}$ *is a* sup-*HFRI of* $\mathcal{A}$ *if and only if* $\widehat{\varepsilon}(p) \sqsubseteq \widehat{\varepsilon}(pq)$ *for all* $p, q \in \mathcal{A}$,

(2) $\widehat{\varepsilon}$ *is a* sup-*HFLI of* $\mathcal{A}$ *if and only if* $\widehat{\varepsilon}(q) \sqsubseteq \widehat{\varepsilon}(pq)$ *for all* $p, q \in \mathcal{A}$,

(3) $\widehat{\varepsilon}$ *is a* sup-*HFII of* $\mathcal{A}$ *if and only if* $\widehat{\varepsilon}(w) \sqsubseteq \widehat{\varepsilon}(pwq)$ *for all* $p, q, w \in \mathcal{A}$.

Proof. It follows from Proposition 3.6. □

Proposition 3.10. *Let* $\widehat{\varepsilon}$ *be a HFS on* $\mathcal{A}$. *Then the followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* sup-*HFRI (resp.,* sup-*HFLI,* sup-*HFII,* sup-*HFI) of* $\mathcal{A}$,

(2) $\widehat{\varepsilon}$ *is a* sup^-_β-*HFRI (resp.,* sup^-_β-*HFLI,* sup^-_β-*HFII,* sup^-_β-*HFI) of* $\mathcal{A}$ *for all* $\beta \in [0, 1]$,

(3) $\widehat{\varepsilon}$ *is a* sup^+_α-*HFRI (resp.,* sup^+_α-*HFLI,* sup^+_α-*HFII,* sup^+_α-*HFI) of* $\mathcal{A}$ *for all* $\alpha \in [0, 1]$.

Proof. It follows from Propositions 3.1, 3.2, 3.5 and 3.6. □

3.3. **Fuzzy sets, Łukasiewicz fuzzy sets, Łukasiewicz anti-fuzzy sets and Pythagorean fuzzy sets.** In this part, we characterize sup^+_α-HFRIs, sup^+_α-HFLIs, sup^+_α-HFIIs, sup^+_α-HFIs, sup^-_β-HFRIs, sup^-_β-HFLIs, sup^-_β-HFIIs, and sup^-_β-HFIs of semigroups in terms of FSs, PFSs, Łukasiewicz fuzzy sets and Łukasiewicz anti-fuzzy sets.

For a FS ξ of $\mathcal{A}$, consider the FS

$$\xi^+_\alpha : \mathcal{A} \to [0, 1], p \mapsto \min\{\xi(p) + \alpha, 1\},$$

which is called an α-Łukasiewicz anti-fuzzy set [22] of ξ in $\mathcal{A}$. In case that $0 \leq \alpha \leq 1 - \sup\{\xi(p) \mid p \in \mathcal{A}\}$, the Łukasiewicz anti-fuzzy set ξ^+_α is called a fuzzy α-translation [8] of ξ of type I.

For a FS ξ of $\mathcal{A}$, consider the FS

$$\xi^-_\beta : \mathcal{A} \to [0, 1], p \mapsto \max\{\xi(p) - \beta, 0\}.$$

Then $\xi^-_\beta(p) = \max\{\xi(p) + (1 - \beta) - 1, 0\}$ for all $p \in \mathcal{A}$ and so ξ^-_β is called an $1 - \beta$-Łukasiewicz fuzzy set [23] of ξ in $\mathcal{A}$. In case that $0 \leq \beta \leq \inf\{\xi(p) \mid p \in \mathcal{A}\}$, the Łukasiewicz fuzzy set ξ^-_β is called a fuzzy β-translation [8] of ξ of type II. Then we have the following results:

(1) $\xi^-_0 = \xi = \xi^+_0$,

(2) $\xi^-_\beta \leq \xi \leq \xi^+_\alpha$,

(3) $\xi^-_k \leq \xi^-_\beta$ for all $k \in [\beta, 1]$,

(4) $\xi^+_k \leq \xi^+_\alpha$ for all $k \in [0, \alpha]$,

(5) the FS $(f^{\widehat{\varepsilon}})^+_\alpha$ is an α-Łukasiewicz anti-fuzzy set of $f^{\widehat{\varepsilon}}$ in $\mathcal{A}$ and $(f^{\widehat{\varepsilon}})^+_\alpha(p) = \text{SUP}^+_\alpha\widehat{\varepsilon}(p)$ for each HFS $\widehat{\varepsilon}$ on $\mathcal{A}$ and $p \in \mathcal{A}$,

(6) the FS $(f^{\widehat{\varepsilon}})^-_\beta$ is an $1 - \beta$-Łukasiewicz fuzzy set of $f^{\widehat{\varepsilon}}$ in $\mathcal{A}$ and $(f^{\widehat{\varepsilon}})^-_\beta(p) = \text{SUP}^-_\beta\widehat{\varepsilon}(p)$ for each HFS $\widehat{\varepsilon}$ on $\mathcal{A}$ and $p \in \mathcal{A}$.

Theorem 3.1. *For a HFS* $\widehat{\varepsilon}$ *on* $\mathcal{A}$, *the followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* sup^+_α-*HFRI (resp.,* sup^+_α-*HFLI,* sup^+_α-*HFII,* sup^+_α-*HFI) of* $\mathcal{A}$,

(2) $(f^{\widehat{\varepsilon}})^+_\alpha$ *is a FRI (resp., FLI, FII, FI) of* $\mathcal{A}$, *and*

(3) $(f^{\widehat{\varepsilon}})^+_k$ *is a FRI (resp., FLI, FII, FI) of* $\mathcal{A}$ *for all* $k \in [\alpha, 1]$.

Proof. (1) ⇒ (3). Assume that $\widehat{\varepsilon}$ is a $\sup^+_\alpha$-HFRI of $\mathcal{A}$, $k \in [\alpha, 1]$ and $p, q \in \mathcal{A}$. Then $\widehat{\varepsilon}(p) \sqsubseteq^+_\alpha \widehat{\varepsilon}(pq)$ and so $\min\{\mathrm{SUP}\widehat{\varepsilon}(p) + \alpha, 1\} \leq \mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha$. If $\min\{\mathrm{SUP}\widehat{\varepsilon}(p) + \alpha, 1\} = \mathrm{SUP}\widehat{\varepsilon}(p) + \alpha$, then $\mathrm{SUP}\widehat{\varepsilon}(p) \leq \mathrm{SUP}\widehat{\varepsilon}(pq)$. Thus

$$(f^{\widehat{\varepsilon}})^+_k(pq) = \mathrm{SUP}^+_k\widehat{\varepsilon}(pq) \geq \mathrm{SUP}^+_k\widehat{\varepsilon}(p) = (f^{\widehat{\varepsilon}})^+_k(p)$$

and so $(f^{\widehat{\varepsilon}})^+_k(pq) \geq (f^{\widehat{\varepsilon}})^+_k(p)$. On the other hand, suppose that $\min\{\mathrm{SUP}\widehat{\varepsilon}(p) + \alpha, 1\} = 1$. Then

$$\mathrm{SUP}\widehat{\varepsilon}(pq) + k \geq \mathrm{SUP}\widehat{\varepsilon}(pq) + \alpha \geq 1$$

and so

$$(f^{\widehat{\varepsilon}})^+_k(pq) = \mathrm{SUP}^+_k\widehat{\varepsilon}(pq) = 1 \geq (f^{\widehat{\varepsilon}})^+_k(p).$$

Hence $(f^{\widehat{\varepsilon}})^+_k(pq) \geq (f^{\widehat{\varepsilon}})^+_k(p)$. Therefore, $(f^{\widehat{\varepsilon}})^+_k$ is a FRI of $\mathcal{A}$ for all $k \in [\alpha, 1]$.

(3) ⇒ (2). It is directly obtained from taking $k = \alpha$.

(2) ⇒ (1). Assume that $(f^{\widehat{\varepsilon}})^+_\alpha$ is a FRI of $\mathcal{A}$. Then $(f^{\widehat{\varepsilon}})^+_\alpha(pq) \geq (f^{\widehat{\varepsilon}})^+_\alpha(p)$ for all $p, q \in \mathcal{A}$. Thus

$$\mathrm{SUP}^+_\alpha\widehat{\varepsilon}(pq) = (f^{\widehat{\varepsilon}})^+_\alpha(pq) \geq (f^{\widehat{\varepsilon}})^+_\alpha(p) = \mathrm{SUP}^+_\alpha\widehat{\varepsilon}(p)$$

for all $p, q \in \mathcal{A}$. Hence $\widehat{\varepsilon}(p) \sqsubseteq^+_\alpha \widehat{\varepsilon}(pq)$ for all $p, q \in \mathcal{A}$, which implies that $\widehat{\varepsilon}$ is a $\sup^+_\alpha$-HFRI of $\mathcal{A}$. □

Theorem 3.2. *For a HFS* $\widehat{\varepsilon}$ *on* $\mathcal{A}$*, the followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* $\sup^-_\beta$*-HFRI (resp.,* $\sup^-_\beta$*-HFLI,* $\sup^-_\beta$*-HFII,* $\sup^-_\beta$*-HFI) of* $\mathcal{A}$*,*

(2) $(f^{\widehat{\varepsilon}})^-_\beta$ *is a FRI (resp., FLI, FII, FI) of* $\mathcal{A}$*, and*

(3) $(f^{\widehat{\varepsilon}})^-_k$ *is a FRI (resp., FLI, FII, FI) of* $\mathcal{A}$ *for all* $k \in [\beta, 1]$.

Proof. (1) ⇒ (3). Assume that $\widehat{\varepsilon}$ is a $\sup^-_\beta$-HFRI of $\mathcal{A}$, $k \in [\beta, 1]$ and $p, q \in \mathcal{A}$. If $0 \geq \mathrm{SUP}\widehat{\varepsilon}(p) - \beta$, then $0 \geq \mathrm{SUP}\widehat{\varepsilon}(p) - k$ and so

$$(f^{\widehat{\varepsilon}})^-_k(pq) \geq 0 = \mathrm{SUP}^-_k\widehat{\varepsilon}(p) = (f^{\widehat{\varepsilon}})^-_k(p).$$

Thus $(f^{\widehat{\varepsilon}})^-_k(pq) \geq (f^{\widehat{\varepsilon}})^-_k(p)$. On the other hand, suppose that $\mathrm{SUP}\widehat{\varepsilon}(p) - \beta > 0$. Since $\widehat{\varepsilon}$ is a $\sup^-_\beta$-HFRI of $\mathcal{A}$, we have $\widehat{\varepsilon}(p) \sqsubseteq^-_\beta \widehat{\varepsilon}(pq)$. Then $\mathrm{SUP}\widehat{\varepsilon}(pq) - \beta \geq \mathrm{SUP}\widehat{\varepsilon}(p) - \beta$ and so $\mathrm{SUP}\widehat{\varepsilon}(pq) \geq \mathrm{SUP}\widehat{\varepsilon}(p)$. Thus

$$(f^{\widehat{\varepsilon}})^-_k(pq) = \mathrm{SUP}^-_k\widehat{\varepsilon}(pq) \geq \mathrm{SUP}^-_k\widehat{\varepsilon}(p) = (f^{\widehat{\varepsilon}})^-_k(p).$$

Hence $(f^{\widehat{\varepsilon}})^-_k(pq) \geq (f^{\widehat{\varepsilon}})^-_k(p)$. Therefore, $(f^{\widehat{\varepsilon}})^-_k$ is a FRI of $\mathcal{A}$ for all $k \in [\beta, 1]$.

(3) ⇒ (2). It is directly obtained from taking $k = \beta$.

(2) ⇒ (1). Assume that $(f^{\widehat{\varepsilon}})^-_\beta$ is a FRI of $\mathcal{A}$ and $p, q \in \mathcal{A}$. Then $(f^{\widehat{\varepsilon}})^-_\beta(pq) \geq (f^{\widehat{\varepsilon}})^-_\beta(p)$ and so

$$\mathrm{SUP}^-_\beta\widehat{\varepsilon}(pq) = (f^{\widehat{\varepsilon}})^-_\beta(pq) \geq (f^{\widehat{\varepsilon}})^-_\beta(p) = \mathrm{SUP}^-_\beta\widehat{\varepsilon}(p).$$

Hence $\widehat{\varepsilon}(p) \sqsubseteq^-_\beta \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup^-_\beta$-HFRI of $\mathcal{A}$. □

Lemma 3.1. *Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$. Then $\mathrm{SUP}_k^+\widehat{\omega}(p) = 1 - \mathrm{SUP}_k^-\widehat{\varepsilon}(p)$ and $\mathrm{SUP}_k^-\widehat{\omega}(p) = 1 - \mathrm{SUP}_k^+\widehat{\varepsilon}(p)$ for all $p \in \mathcal{A}$, $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $k \in [0, 1]$.*

Proof. Let $p \in \mathcal{A}$, $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $k \in [0, 1]$. Then

$$\begin{aligned}
\mathrm{SUP}_k^+\widehat{\omega}(p) &= \min\{\mathrm{SUP}\widehat{\omega}(p) + k, 1\} \\
&= \min\{(1 - \mathrm{SUP}\widehat{\varepsilon}(p)) + k, 1\} \\
&= \min\{1 - (\mathrm{SUP}\widehat{\varepsilon}(p) - k), 1\} \\
&= 1 - \max\{\mathrm{SUP}\widehat{\varepsilon}(p) - k, 0\} \\
&= 1 - \mathrm{SUP}_k^-\widehat{\varepsilon}(p)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{SUP}_k^-\widehat{\omega}(p) &= \max\{\mathrm{SUP}\widehat{\omega}(p) - k, 0\} \\
&= \max\{(1 - \mathrm{SUP}\widehat{\varepsilon}(p)) - k, 1 - 1\} \\
&= \max\{1 - (\mathrm{SUP}\widehat{\varepsilon}(p) + k), 1 - 1\} \\
&= 1 - \min\{\mathrm{SUP}\widehat{\varepsilon}(p) + k, 1\} \\
&= 1 - \mathrm{SUP}_k^+\widehat{\varepsilon}(p).
\end{aligned}$$

Therefore, $\mathrm{SUP}_k^+\widehat{\omega}(p) = 1 - \mathrm{SUP}_k^-\widehat{\varepsilon}(p)$ and $\mathrm{SUP}_k^-\widehat{\omega}(p) = 1 - \mathrm{SUP}_k^+\widehat{\varepsilon}(p)$. □

Theorem 3.3. *For a HFS $\widehat{\varepsilon}$ on $\mathcal{A}$, the followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFRI (resp., $\sup_\alpha^+$-HFLI, $\sup_\alpha^+$-HFII, $\sup_\alpha^+$-HFI) of $\mathcal{A}$,*
(2) *$(\mathrm{f}^{\widehat{\varepsilon}^*})_\alpha^-$ is an AFRI (resp., AFLI, AFII, AFI) of $\mathcal{A}$,*
(3) *$(\mathrm{f}^{\widehat{\omega}})_\alpha^-$ is an AFRI (resp., AFLI, AFII, AFI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$, and*
(4) *$(\mathrm{f}^{\widehat{\omega}})_k^-$ is an AFRI (resp., AFLI, AFII, AFI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $k \in [\alpha, 1]$.*

Proof. (1) ⇒ (4). Assume that $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFRI of $\mathcal{A}$. By Proposition 3.3, we have that $\widehat{\varepsilon}$ is a $\sup_k^+$-HFRI of $\mathcal{A}$ for all $k \in [\alpha, 1]$. By Lemma 3.1, we get

$$(\mathrm{f}^{\widehat{\omega}})_k^-(p) = \mathrm{SUP}_k^-\widehat{\omega}(p) = 1 - \mathrm{SUP}_k^+\widehat{\varepsilon}(p) \geq 1 - \mathrm{SUP}_k^+\widehat{\varepsilon}(pq) = \mathrm{SUP}_k^-\widehat{\omega}(pq) = (\mathrm{f}^{\widehat{\omega}})_k^-(pq)$$

for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$, $k \in [\alpha, 1]$ and $p, q \in \mathcal{A}$. Hence $(\mathrm{f}^{\widehat{\omega}})_k^-$ is an AFRI of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $k \in [\alpha, 1]$.

(4) ⇒ (3) and (3) ⇒ (2). They are clear.

(2) ⇒ (1). Assume that $(\mathrm{f}^{\widehat{\varepsilon}^*})^-_\alpha$ is an AFRI of $\mathcal{A}$ and $p, q \in \mathcal{A}$. Then $(\mathrm{f}^{\widehat{\varepsilon}^*})^-_\alpha(p) \geq (\mathrm{f}^{\widehat{\varepsilon}^*})^-_\alpha(pq)$ and by Lemma 3.1, we have

$$\begin{aligned}\mathrm{SUP}^+_\alpha\widehat{\varepsilon}(p) &= 1 - \mathrm{SUP}^-_\alpha\widehat{\varepsilon}^*(p)\\ &= 1 - (\mathrm{f}^{\widehat{\varepsilon}^*})^-_\alpha(p)\\ &\leq 1 - (\mathrm{f}^{\widehat{\varepsilon}^*})^-_\alpha(pq)\\ &= 1 - \mathrm{SUP}^-_\alpha\widehat{\varepsilon}^*(pq)\\ &= \mathrm{SUP}^+_\alpha\widehat{\varepsilon}(pq).\end{aligned}$$

Hence $\widehat{\varepsilon}(p) \sqsubseteq^+_\alpha \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup^+_\alpha$-HFRI of $\mathcal{A}$. □

Theorem 3.4. *For a HFS $\widehat{\varepsilon}$ on $\mathcal{A}$, the followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a $\sup^-_\beta$-HFRI (resp., $\sup^-_\beta$-HFLI, $\sup^-_\beta$-HFII, $\sup^-_\beta$-HFI) of $\mathcal{A}$,*
(2) *$(\mathrm{f}^{\widehat{\varepsilon}^*})^+_\beta$ is an AFRI (resp., AFLI, AFII, AFI) of $\mathcal{A}$,*
(3) *$(\mathrm{f}^{\widehat{\omega}})^+_\beta$ is an AFRI (resp., AFLI, AFII, AFI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$, and*
(4) *$(\mathrm{f}^{\widehat{\omega}})^+_k$ is an AFRI (resp., AFLI, AFII, AFI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and for all $k \in [\beta, 1]$.*

Proof. (1) ⇒ (4). Assume that $\widehat{\varepsilon}$ is a $\sup^-_\beta$-HFRI of $\mathcal{A}$. By Proposition 3.7 and Lemma 3.1, we get

$$(\mathrm{f}^{\widehat{\omega}})^+_k(pq) = \mathrm{SUP}^+_k\widehat{\omega}(pq) = 1 - \mathrm{SUP}^-_k\widehat{\varepsilon}(pq) \leq 1 - \mathrm{SUP}^-_k\widehat{\varepsilon}(p) = \mathrm{SUP}^+_k\widehat{\omega}(p) = (\mathrm{f}^{\widehat{\omega}})^+_k(p)$$

for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$, $k \in [\beta, 1]$ and $p, q \in \mathcal{A}$. Hence $(\mathrm{f}^{\widehat{\omega}})^+_k$ is an AFRI of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $k \in [\beta, 1]$.

(4) ⇒ (3) and (3) ⇒ (2). They are clear.

(2) ⇒ (1). Assume that $(\mathrm{f}^{\widehat{\varepsilon}^*})^-_\beta$ is an AFRI of $\mathcal{A}$. By Lemma 3.1, we get

$$\begin{aligned}\mathrm{SUP}^-_\beta\widehat{\varepsilon}(pq) &= 1 - \mathrm{SUP}^+_\beta\widehat{\varepsilon}^*(pq)\\ &= 1 - (\mathrm{f}^{\widehat{\varepsilon}^*})^+_\beta(pq)\\ &\geq 1 - (\mathrm{f}^{\widehat{\varepsilon}^*})^+_\beta(p)\\ &= 1 - \mathrm{SUP}^+_\beta\widehat{\varepsilon}^*(p)\\ &= \mathrm{SUP}^-_\beta\widehat{\varepsilon}(p)\end{aligned}$$

for all $p, q \in \mathcal{A}$. Thus $\widehat{\varepsilon}(p) \sqsubseteq^-_\beta \widehat{\varepsilon}(pq)$ for all $p, q \in \mathcal{A}$, which implies that $\widehat{\varepsilon}$ is a $\sup^-_\beta$-HFRI of $\mathcal{A}$. □

Theorem 3.5. *For a HFS $\widehat{\varepsilon}$ on $\mathcal{A}$, the followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a $\sup^+_\alpha$-HFRI (resp., $\sup^+_\alpha$-HFLI, $\sup^+_\alpha$-HFII, $\sup^+_\alpha$-HFI) of $\mathcal{A}$,*
(2) *$((\mathrm{f}^{\widehat{\varepsilon}})^+_\alpha, (\mathrm{f}^{\widehat{\varepsilon}^*})^-_\alpha)$ is a PFRI (resp., PFLI, PFII, PFI) of $\mathcal{A}$,*
(3) *$((\mathrm{f}^{\widehat{\varepsilon}})^+_\alpha, (\mathrm{f}^{\widehat{\omega}})^-_\alpha)$ is a PFRI (resp., PFLI, PFII, PFI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$, and*
(4) *$((\mathrm{f}^{\widehat{\varepsilon}})^+_k, (\mathrm{f}^{\widehat{\omega}})^-_k)$ is a PFRI (resp., PFLI, PFII, PFI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $k \in [\alpha, 1]$.*

Proof. It follows from Theorems 3.1 and 3.3. □

Theorem 3.6. *For a HFS $\widehat{\varepsilon}$ on $\mathcal{A}$, the followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a $\sup_{\beta}^{-}$-HFRI (resp., $\sup_{\beta}^{-}$-HFLI, $\sup_{\beta}^{-}$-HFII, $\sup_{\beta}^{-}$-HFI) of $\mathcal{A}$,*

(2) *$((\mathsf{f}^{\widehat{\varepsilon}})_{\beta}^{-}, (\mathsf{f}^{\widehat{\varepsilon}^{*}})_{\beta}^{+})$ is a PFRI (resp., PFLI, PFII, PFI) of $\mathcal{A}$,*

(3) *$((\mathsf{f}^{\widehat{\varepsilon}})_{\beta}^{-}, (\mathsf{f}^{\widehat{\omega}})_{\beta}^{+})$ is a PFRI (resp., PFLI, PFII, PFI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$, and*

(4) *$((\mathsf{f}^{\widehat{\varepsilon}})_{k}^{-}, (\mathsf{f}^{\widehat{\omega}})_{k}^{+})$ is a PFRI (resp., PFLI, PFII, PFI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $k \in [\beta, 1]$.*

Proof. It follows from Theorems 3.2 and 3.4. □

3.4. **Hesitant fuzzy sets and hybrid sets.** In this part, we characterize $\sup_{\alpha}^{+}$-HFRIs, $\sup_{\alpha}^{+}$-HFLIs, $\sup_{\alpha}^{+}$-HFIIs, $\sup_{\alpha}^{+}$-HFIs, $\sup_{\beta}^{-}$-HFRIs, $\sup_{\beta}^{-}$-HFLIs, $\sup_{\beta}^{-}$-HFIIs and $\sup_{\beta}^{-}$-HFIs of semigroups in terms of HFSs and hybrid sets.

Theorem 3.7. *For a HFS $\widehat{\varepsilon}$ on $\mathcal{A}$, the followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a $\sup_{\alpha}^{+}$-HFRI (resp., $\sup_{\alpha}^{+}$-HFLI, $\sup_{\alpha}^{+}$-HFII, $\sup_{\alpha}^{+}$-HFI) of $\mathcal{A}$, and*

(2) *$\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ is a HFRI (resp., HFLI, HFII, HFI) of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0, 1-\alpha])$.*

Proof. (1) ⇒ (2). Assume that $\widehat{\varepsilon}$ is a $\sup_{\alpha}^{+}$-HFRI of $\mathcal{A}$. Let $\blacktriangledown \in \wp([0, 1-\alpha])$, $p, q \in A$ and $k \in \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(p)$. Then $\widehat{\varepsilon}(p) \sqsubseteq_{\alpha}^{+} \widehat{\varepsilon}(pq)$ and $\mathsf{SUP}\widehat{\varepsilon}(p) \geq k \in \blacktriangledown$. Thus

$$\begin{aligned}\mathsf{SUP}\widehat{\varepsilon}(pq) &= (\mathsf{SUP}\widehat{\varepsilon}(pq) + \alpha) - \alpha \\ &\geq \mathsf{SUP}_{\alpha}^{+}\widehat{\varepsilon}(pq) - \alpha \\ &\geq \mathsf{SUP}_{\alpha}^{+}\widehat{\varepsilon}(p) - \alpha \\ &= \min\{\mathsf{SUP}\widehat{\varepsilon}(p), 1-\alpha\} \\ &\geq k\end{aligned}$$

which implies that $k \in \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(pq)$. Hence $\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(p) \subseteq \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(pq)$. Therefore, $\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ is a HFRI of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0, 1-\alpha])$.

(2) ⇒ (1). Assume that $\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ is a HFRI of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0, 1-\alpha])$. Let $p, q \in \mathcal{A}$ and $\blacktriangledown = [0, 1-\alpha]$. Then

$$\mathsf{SUP}_{\alpha}^{+}\widehat{\varepsilon}(p) - \alpha = \min\{\mathsf{SUP}\widehat{\varepsilon}(p), 1-\alpha\} \in \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(p) \subseteq \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(pq).$$

Thus

$$\mathsf{SUP}_{\alpha}^{+}\widehat{\varepsilon}(pq) - \alpha = \min\{\mathsf{SUP}\widehat{\varepsilon}(pq), 1-\alpha\} \geq \mathsf{SUP}_{\alpha}^{+}\widehat{\varepsilon}(p) - \alpha.$$

Hence $\mathsf{SUP}_{\alpha}^{+}\widehat{\varepsilon}(pq) \geq \mathsf{SUP}_{\alpha}^{+}\widehat{\varepsilon}(p)$ which implies that $\widehat{\varepsilon}(p) \sqsubseteq_{\alpha}^{+} \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup_{\alpha}^{+}$-HFRI of $\mathcal{A}$. □

Theorem 3.8. *For a HFS $\widehat{\varepsilon}$ on $\mathcal{A}$, the followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a $\sup_{\beta}^{-}$-HFRI (resp., $\sup_{\beta}^{-}$-HFLI, $\sup_{\beta}^{-}$-HFII, $\sup_{\beta}^{-}$-HFI) of $\mathcal{A}$,*

(2) $\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ *is a HFRI (resp., HFLI, HFII, HFI) of* $\mathcal{A}$ *for all* $\blacktriangledown \in \wp((\beta, 1])$.

Proof. (1) $\Rightarrow$ (2). Assume that $\widehat{\varepsilon}$ is a $\sup_{\beta}^{-}$-HFRI of $\mathcal{A}$. Let $\blacktriangledown \in \wp((\beta, 1])$, $p, q \in \mathcal{A}$ and $k \in \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(p)$. Then $\mathsf{SUP}\widehat{\varepsilon}(p) \geq k > \beta$, $k \in \blacktriangledown$ and $\widehat{\varepsilon}(p) \sqsubseteq_{\beta}^{-} \widehat{\varepsilon}(pq)$. Thus

$$\begin{aligned}\max\{\mathsf{SUP}\widehat{\varepsilon}(pq), \beta\} &= \mathsf{SUP}_{\beta}^{-}\widehat{\varepsilon}(pq) + \beta \\ &\geq \mathsf{SUP}_{\beta}^{-}\widehat{\varepsilon}(p) + \beta \\ &= \max\{\mathsf{SUP}\widehat{\varepsilon}(p), \beta\} \\ &\geq k \\ &> \beta,\end{aligned}$$

that is, $\mathsf{SUP}\widehat{\varepsilon}(pq) \geq k$. Hence $k \in \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(pq)$ and so $\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(p) \subseteq \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(pq)$. Therefore, $\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ is a HFRI of $\mathcal{A}$ for all $\blacktriangledown \in \wp((\beta, 1])$.

(2) $\Rightarrow$ (1). Assume that $\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}$ is a HFRI of $\mathcal{A}$ for all $\blacktriangledown \in \wp((\beta, 1])$ and $p, q \in \mathcal{A}$. If $\mathsf{SUP}\widehat{\varepsilon}(p) \leq \beta$, then

$$\mathsf{SUP}_{\beta}^{-}\widehat{\varepsilon}(p) = 0 \leq \mathsf{SUP}_{\beta}^{-}\widehat{\varepsilon}(pq)$$

and so $\widehat{\varepsilon}(p) \sqsubseteq_{\beta}^{-} \widehat{\varepsilon}(pq)$. On the other hand, suppose that $\mathsf{SUP}\widehat{\varepsilon}(p) > \beta$. Let $\blacktriangledown = (\beta, 1]$. Then

$$\mathsf{SUP}_{\beta}^{-}\widehat{\varepsilon}(p) + \beta = \max\{\mathsf{SUP}\widehat{\varepsilon}(p), \beta\} = \mathsf{SUP}\widehat{\varepsilon}(p) \in \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(p) \subseteq \mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}(pq).$$

Thus

$$\mathsf{SUP}_{\beta}^{-}\widehat{\varepsilon}(pq) + \beta \geq (\mathsf{SUP}\widehat{\varepsilon}(pq) - \beta) + \beta = \mathsf{SUP}\widehat{\varepsilon}(pq) \geq \mathsf{SUP}_{\beta}^{-}\widehat{\varepsilon}(p) + \beta.$$

Hence $\mathsf{SUP}_{\beta}^{-}\widehat{\varepsilon}(p) \leq \mathsf{SUP}_{\alpha}^{-}\widehat{\varepsilon}(pq)$ and so $\widehat{\varepsilon}(p) \sqsubseteq_{\beta}^{-} \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup_{\beta}^{-}$-HFRI of $\mathcal{A}$. □

Theorem 3.9. *For a HFS* $\widehat{\varepsilon}$ *on* $\mathcal{A}$*, the followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* $\sup_{\alpha}^{+}$*-HFRI (resp.,* $\sup_{\alpha}^{+}$*-HFLI,* $\sup_{\alpha}^{+}$*-HFII,* $\sup_{\alpha}^{+}$*-HFI) of* $\mathcal{A}$,

(2) $(\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}, (\mathsf{f}^{\widehat{\varepsilon}^{*}})_{\alpha}^{-})$ *is a HyRI (resp., HyLI, HyII, HyI) of* $\mathcal{A}$ *over* $[0, 1]$ *for all* $\blacktriangledown \in \wp([0, 1-\alpha])$,

(3) $(\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}, (\mathsf{f}^{\widehat{\omega}})_{\alpha}^{-})$ *is a HyRI (resp., HyLI, HyII, HyI) of* $\mathcal{A}$ *over* $[0, 1]$ *for all* $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ *and* $\blacktriangledown \in \wp([0, 1-\alpha])$*, and*

(4) $(\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}, (\mathsf{f}^{\widehat{\omega}})_{k}^{-})$ *is a HyRI (resp., HyLI, HyII, HyI) of* $\mathcal{A}$ *over* $[0, 1]$ *for all* $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$, $k \in [\alpha, 1]$ *and* $\blacktriangledown \in \wp([0, 1-\alpha])$.

Proof. It follows from Theorems 3.3 and 3.7. □

Theorem 3.10. *For a HFS* $\widehat{\varepsilon}$ *on* $\mathcal{A}$*, the followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* $\sup_{\beta}^{-}$*-HFRI (resp.,* $\sup_{\beta}^{-}$*-HFLI,* $\sup_{\beta}^{-}$*-HFII,* $\sup_{\beta}^{-}$*-HFI) of* $\mathcal{A}$,

(2) $(\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}, (\mathsf{f}^{\widehat{\varepsilon}^{*}})_{\beta}^{+})$ *is a HyRI (resp., HyLI, HyII, HyI) of* $\mathcal{A}$ *over* $[0, 1]$ *for all* $\blacktriangledown \in \wp((\beta, 1])$,

(3) $(\mathcal{H}_{\mathsf{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}, (\mathsf{f}^{\widehat{\omega}})_{\beta}^{+})$ *is a HyRI (resp., HyLI, HyII, HyI) of* $\mathcal{A}$ *over* $[0, 1]$ *for all* $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ *and* $\blacktriangledown \in \wp((\beta, 1])$*, and*

(4) $(\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},\blacktriangledown)}, (f^{\widehat{\omega}})_k^+)$ *is a HyRI (resp., HyLI, HyII, HyI) of* $\mathcal{A}$ *over* $[0,1]$ *for all* $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$, $k \in [\beta, 1]$ *and* $\blacktriangledown \in \wp((\beta, 1])$.

Proof. It follows from Theorems 3.4 and 3.8. □

3.5. **Interval-valued fuzzy sets and cubic sets.** In this part, we characterize $\sup_\alpha^+$-HFRIs, $\sup_\alpha^+$-HFLIs, $\sup_\alpha^+$-HFIIs, $\sup_\alpha^+$-HFIs, $\sup_\beta^-$-HFRIs, $\sup_\beta^-$-HFLIs, $\sup_\beta^-$-HFIIs, and $\sup_\beta^-$-HFIs of semigroups in terms of IvFSs and cubic sets.

Let ξ and η be FSs of $\mathcal{A}$ such that $\xi \leq \eta$, the followings are true.

(1) $[\xi, \xi]$, $[\xi_\beta^-, \eta]$, $[\xi, \eta_\alpha^+]$ and $[\xi_\beta^-, \eta_\alpha^+]$ are IvFSs on $\mathcal{A}$.

(2) If $\alpha \geq \beta$, then $[\xi_\alpha^-, \eta_\beta^-]$ is an IvFS on $\mathcal{A}$.

(3) If $\alpha \leq \beta$, then $[\xi_\alpha^+, \eta_\beta^+]$ is an IvFS on $\mathcal{A}$.

(4) If $\breve{\lambda}$ is an IvFS on $\mathcal{A}$ and $\breve{\lambda} = [\xi, \eta]$, then $\breve{\lambda}^L = \xi$ and $\breve{\lambda}^U = \eta$.

Theorem 3.11. *For a HFS* $\widehat{\varepsilon}$ *on* $\mathcal{A}$, *the followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* $\sup_\alpha^+$*-HFRI (resp.,* $\sup_\alpha^+$*-HFLI,* $\sup_\alpha^+$*-HFII,* $\sup_\alpha^+$*-HFI) of* $\mathcal{A}$,

(2) $[(f^{\widehat{\varepsilon}})_\alpha^+, (f^{\widehat{\varepsilon}})_k^+]$ *is an IvFRI (resp., IvFLI, IvFII, IvFI) of* $\mathcal{A}$ *for all* $k \in [\alpha, 1]$,

(3) $[(f^{\widehat{\varepsilon}})_k^+, (f^{\widehat{\varepsilon}})_\beta^+]$ *is an IvFRI (resp., IvFLI, IvFII, IvFI) of* $\mathcal{A}$ *for all* $k, \beta \in [\alpha, 1]$ *with* $k \leq \beta$,

(4) $\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1-\alpha])}$ *is an IvFRI (resp., IvFLI, IvFII, IvFI) of* $\mathcal{A}$.

Proof. (1) ⇔ (2) and (1) ⇔ (3). It follows from Theorem 3.1.

(1) ⇒ (4). Assume that $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFRI of $\mathcal{A}$ and $p, q \in \mathcal{A}$. Then $\widehat{\varepsilon}(p) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(pq)$ and so $\text{SUP}_\alpha^+\widehat{\varepsilon}(p) \leq \text{SUP}_\alpha^+\widehat{\varepsilon}(pq)$. Thus

$$\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1-\alpha])}(p) = [0, \text{SUP}_\alpha^+\widehat{\varepsilon}(p) - \alpha] \precsim [0, \text{SUP}_\alpha^+\widehat{\varepsilon}(pq) - \alpha] = \mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1-\alpha])}(pq).$$

Therefore, $\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1-\alpha])}$ is an IvFRI of $\mathcal{A}$.

(4) ⇒ (1). Assume that $\mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1-\alpha])}$ is an IvFRI of $\mathcal{A}$ and $p, q \in \mathcal{A}$. Then

$$[0, \text{SUP}_\alpha^+\widehat{\varepsilon}(p) - \alpha] = \mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1-\alpha])}(p) \precsim \mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1-\alpha])}(pq) = [0, \text{SUP}_\alpha^+\widehat{\varepsilon}(pq) - \alpha].$$

Thus

$$\text{SUP}_\alpha^+\widehat{\varepsilon}(p) = (\text{SUP}_\alpha^+\widehat{\varepsilon}(p) - \alpha) + \alpha \leq (\text{SUP}_\alpha^+\widehat{\varepsilon}(pq) - \alpha) + \alpha = \text{SUP}_\alpha^+\widehat{\varepsilon}(pq).$$

Hence $\widehat{\varepsilon}(p) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup_\alpha^+$-HFRI of $\mathcal{A}$. □

Theorem 3.12. *For a HFS* $\widehat{\varepsilon}$ *on* $\mathcal{A}$, *the followings are equivalent:*

(1) $\widehat{\varepsilon}$ *is a* $\sup_\beta^-$*-HFRI (resp.,* $\sup_\beta^-$*-HFLI,* $\sup_\beta^-$*-HFII,* $\sup_\beta^-$*-HFI) of* $\mathcal{A}$,

(2) $[(f^{\widehat{\varepsilon}})_k^-, (f^{\widehat{\varepsilon}})_\beta^-]$ *is an IvFRI (resp., IvFLI, IvFII, IvFI) of* $\mathcal{A}$ *for all* $k \in [\beta, 1]$, *and*

(3) $[(f^{\widehat{\varepsilon}})_k^-, (f^{\widehat{\varepsilon}})_\alpha^-]$ *is an IvFRI (resp., IvFLI, IvFII, IvFI) of* $\mathcal{A}$ *for all* $k, \alpha \in [\beta, 1]$ *with* $\alpha \leq k$.

Proof. It follows from Theorem 3.2. □

Theorem 3.13. *Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$ and $\breve{\lambda} = \mathcal{H}_{\text{SUP}}^{(\widehat{\varepsilon},[0,1-\alpha])}$. Then the followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a $\sup_{\alpha}^{+}$-HFRI (resp., $\sup_{\alpha}^{+}$-HFLI, $\sup_{\alpha}^{+}$-HFII, $\sup_{\alpha}^{+}$-HFI) of $\mathcal{A}$,*

(2) *$\langle \breve{\lambda}, (f^{\widehat{\varepsilon}^*})_{\alpha}^{-} \rangle$ is a CuRI (resp., CuLI, CuII, CuI) of $\mathcal{A}$,*

(3) *$\langle \breve{\lambda}, (f^{\widehat{\omega}})_{\alpha}^{-} \rangle$ is a CuRI (resp., CuLI, CuII, CuI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$,*

(4) *$\langle \breve{\lambda}, (f^{\widehat{\omega}})_{k}^{-} \rangle$ is a CuRI (resp., CuLI, CuII, CuI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$ and $k \in [\alpha, 1]$.*

Proof. It follows from Theorems 3.3 and 3.11. □

Theorem 3.14. *Let $\widehat{\varepsilon}$ be a HFS on $\mathcal{A}$, $k \in [\beta, 1]$ and $\breve{\lambda} = [(f^{\widehat{\varepsilon}})_{k}^{-}, (f^{\widehat{\varepsilon}})_{\beta}^{-}]$. Then the followings are equivalent:*

(1) *$\widehat{\varepsilon}$ is a $\sup_{\beta}^{-}$-HFRI (resp., $\sup_{\beta}^{-}$-HFLI, $\sup_{\beta}^{-}$-HFII, $\sup_{\beta}^{-}$-HFI) of $\mathcal{A}$,*

(2) *$\langle \breve{\lambda}, (f^{\widehat{\varepsilon}^*})_{\beta}^{+} \rangle$ is a CuRI (resp., CuLI, CuII, CuI) of $\mathcal{A}$,*

(3) *$\langle \breve{\lambda}, (f^{\widehat{\omega}})_{\beta}^{+} \rangle$ is a CuRI (resp., CuLI, CuII, CuI) of $\mathcal{A}$ for all $\widehat{\omega} \in \mathcal{SC}(\widehat{\varepsilon})$.*

Proof. It follows from Theorems 3.4 and 3.12. □

4. Characterizing semigroups by $\sup_{\alpha}^{+}$-type and $\sup_{\beta}^{-}$-type of HFSs

In this section, we characterize intra-regular, completely regular, left (right) regular, left (right) simple and simple semigroups and groups in terms of $\sup_{\alpha}^{+}$-type and $\sup_{\beta}^{-}$-type of HFSs.

A semigroup $\mathcal{A}$ is called

(1) intra-regular if for each $w \in \mathcal{A}$, there exist $p, q \in \mathcal{A}$ such that $w = pw^2q$,

(2) completely regular if for each $p \in \mathcal{A}$ there exists $q \in \mathcal{A}$ such that $p = pqp$ and $pq = qp$,

(3) left regular if for each $p \in \mathcal{A}$ there exists $q \in \mathcal{A}$ such that $p = qp^2$,

(4) right regular if for each $p \in \mathcal{A}$ there exists $q \in \mathcal{A}$ such that $p = p^2q$,

(5) left simple if $\mathcal{A} = \mathcal{B}$ for each left ideal $\mathcal{B}$ of $\mathcal{A}$,

(6) right simple if $\mathcal{A} = \mathcal{B}$ for each right ideal $\mathcal{B}$ of $\mathcal{A}$,

(7) simple if $\mathcal{A} = \mathcal{B}$ for each ideal $\mathcal{B}$ of $\mathcal{A}$,

(8) group if it is both left simple and right simple.

It is well-known that $\mathcal{A}$ is completely regular if and only if it is both left and right regular.

Proposition 4.1. *Let $\widehat{\varepsilon}$ be a HFS on an intra-regular semigroup $\mathcal{A}$. Then $\widehat{\varepsilon}$ is a $\sup_{\alpha}^{+}$-HFII (resp., $\sup_{\beta}^{-}$-HFII) of $\mathcal{A}$ if and only if $\widehat{\varepsilon}$ is a $\sup_{\alpha}^{+}$-HFI (resp., $\sup_{\beta}^{-}$-HFI) of $\mathcal{A}$.*

Proof. ($\Rightarrow$). Assume that $\widehat{\varepsilon}$ is a $\sup_{\alpha}^{+}$-HFII of $\mathcal{A}$ and $p, q \in \mathcal{A}$. Then there exist $w_1, w_2, w_3, w_4 \in \mathcal{A}$ such that $p = w_1p^2w_2$ and $q = w_3q^2w_4$. Thus $\widehat{\varepsilon}(p) \sqsubseteq_{\alpha}^{+} \widehat{\varepsilon}(w_1p^2w_2q) = \widehat{\varepsilon}(pq)$ and $\widehat{\varepsilon}(q) \sqsubseteq_{\alpha}^{+} \widehat{\varepsilon}(pw_3q^2w_4) = \widehat{\varepsilon}(pq)$. Therefore, $\widehat{\varepsilon}$ is a $\sup_{\alpha}^{+}$-HFI of $\mathcal{A}$.
($\Leftarrow$). It follows from Proposition 3.4. □

Theorem 4.1. *Let $\mathcal{A}$ be a semigroup. The followings are equivalent:*

(1) $\mathcal{A}$ *is intra-regular,*
(2) $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^+$*-HFI* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,
(3) $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^+$*-HFII* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,
(4) $\widehat{\varepsilon}(p) \cong_k^- \widehat{\varepsilon}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^-$*-HFI* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,
(5) $\widehat{\varepsilon}(p) \cong_k^- \widehat{\varepsilon}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^-$*-HFII* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,
(6) $\widehat{\varepsilon}(p) \cong \widehat{\varepsilon}(p^2)$ *for each* sup*-HFI* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,
(7) $\widehat{\varepsilon}(p) \cong \widehat{\varepsilon}(p^2)$ *for each* sup*-HFII* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$.

Proof. (7) $\Rightarrow$ (6). It follows from Proposition 3.4.

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6) and (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7). They follow from Proposition 3.10.

(1) $\Rightarrow$ (2). Assume that (1) holds, $k \in [0,1]$, $\widehat{\varepsilon}$ is a $\sup_k^+$-HFI of $\mathcal{A}$ and $p \in \mathcal{A}$. There exist $q, w \in \mathcal{A}$ such that $p = qp^2w$. Thus

$$\widehat{\varepsilon}(p) \sqsubseteq_k^+ \widehat{\varepsilon}(p^2) \sqsubseteq_k^+ \widehat{\varepsilon}(p^2w) \sqsubseteq_k^+ \widehat{\varepsilon}(qp^2w) = \widehat{\varepsilon}(p).$$

Hence $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$. Therefore, $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$ for each $k \in [0,1]$, $\sup_k^+$-HFI $\widehat{\varepsilon}$ of $\mathcal{A}$ and $p \in \mathcal{A}$.

(1) $\Rightarrow$ (3). It is similar to prove (1) $\Rightarrow$ (2).

(7) $\Rightarrow$ (1). Assume that (7) holds and $p \in \mathcal{A}$. Then $J[p^2] = \{p^2\} \cup \mathcal{A}p^2 \cup p^2\mathcal{A} \cup \mathcal{A}p^2\mathcal{A}$ is an interior ideal of $\mathcal{A}$ and by Theorem 2.9, $\widehat{\chi}_{J[p^2]}$ is a sup-HFII of $\mathcal{A}$. Since $p^2 \in J[p^2]$, we get $\widehat{\chi}_{J[p^2]}(p) \cong \widehat{\chi}_{J[p^2]}(p^2) = [0,1]$ which implies that $\widehat{\chi}_{J[p^2]}(p) = [0,1]$. Thus $p \in J[p^2]$ and so $p \in \mathcal{A}p^2\mathcal{A}$. Hence $\mathcal{A}$ is intra-regular.

(6) $\Rightarrow$ (1). It is similar to prove (7) $\Rightarrow$ (1). □

Proposition 4.2. *Let* $\widehat{\varepsilon}$ *be a HFS of an intra-regular semigroup* $\mathcal{A}$*. Then the followings are true:*

(1) $\widehat{\varepsilon}(pq) \cong_\alpha^+ \widehat{\varepsilon}(qp)$ *for each* $\sup_\alpha^+$*-HFII of* $\mathcal{A}$ *and* $p, q \in \mathcal{A}$,
(2) $\widehat{\varepsilon}(pq) \cong_\alpha^+ \widehat{\varepsilon}(qp)$ *for each* $\sup_\alpha^+$*-HFI of* $\mathcal{A}$ *and* $p, q \in \mathcal{A}$.

Proof. (1). Let $\widehat{\varepsilon}$ be a $\sup_\alpha^+$-HFII of $\mathcal{A}$ and $p, q \in \mathcal{A}$. By Theorem 4.1, we have

$$\widehat{\varepsilon}(qp) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(p(qp)q) = \widehat{\varepsilon}((pq)^2) \cong_\alpha^+ \widehat{\varepsilon}(pq) \sqsubseteq_\alpha^+ \widehat{\varepsilon}(q(pq)p) = \widehat{\varepsilon}((qp)^2) \cong_\alpha^+ \widehat{\varepsilon}(qp).$$

Thus $\widehat{\varepsilon}(pq) \cong_\alpha^+ \widehat{\varepsilon}(qp)$.

(2). It follows from (1) and Proposition 4.1. □

Similarly we can prove the following theorem.

Proposition 4.3. *Let* $\widehat{\varepsilon}$ *be a HFS of an intra-regular semigroup* $\mathcal{A}$*. Then the followings are true:*

(1) $\widehat{\varepsilon}(pq) \cong_\beta^- \widehat{\varepsilon}(qp)$ *for each* $\sup_\beta^-$*-HFII of* $\mathcal{A}$ *and* $p, q \in \mathcal{A}$,
(2) $\widehat{\varepsilon}(pq) \cong_\beta^- \widehat{\varepsilon}(qp)$ *for each* $\sup_\beta^-$*-HFI of* $\mathcal{A}$ *and* $p, q \in \mathcal{A}$.

Theorem 4.2. *Let* $\mathcal{A}$ *be a semigroup. The followings are equivalent:*

(1) $\mathcal{A}$ *is left regular,*

(2) $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^+$*-HFLI of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,

(3) $\widehat{\varepsilon}(p) \cong_k^- \widehat{\varepsilon}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^-$*-HFLI of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,

(4) $\widehat{\varepsilon}(p) \cong \widehat{\varepsilon}(p^2)$ *for each* sup*-HFLI of* $\mathcal{A}$ *and* $p \in \mathcal{A}$.

Proof. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). It follows from Proposition 3.10.

(1) $\Rightarrow$ (2). Assume that (1) holds, $k \in [0,1]$, $\widehat{\varepsilon}$ is a $\sup_k^+$-HFLI of $\mathcal{A}$ and $p \in \mathcal{A}$. Since $\mathcal{A}$ is left regular, there exists $q \in \mathcal{A}$ such that $p = qp^2$. Thus, by $\widehat{\varepsilon}$ is a $\sup_k^+$-HFLI of $\mathcal{A}$, we obtain

$$\widehat{\varepsilon}(p) \sqsubseteq_k^+ \widehat{\varepsilon}(p^2) \sqsubseteq_k^+ \widehat{\varepsilon}(qp^2) = \widehat{\varepsilon}(p).$$

Hence $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$. Therefore $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$ for each $k \in [0,1]$, $\sup_k^+$-HFLI of $\mathcal{A}$ and $p \in \mathcal{A}$.

(4) $\Rightarrow$ (1). Assume that (4) holds and $p \in \mathcal{A}$. Then $L[p^2] = \{p^2\} \cup \mathcal{A}p^2$ is a left ideal of $\mathcal{A}$ and by Theorem 2.11, we get that $\widehat{\chi}_{L[p^2]}$ is a sup-HFLI of $\mathcal{A}$. Since $p^2 \in L[p^2]$, we have $\widehat{\chi}_{L[p^2]}(p) \cong \widehat{\chi}_{L[p^2]}(p^2) = [0,1]$ and so $\widehat{\chi}_{L[p^2]}(p) = [0,1]$. Thus $p \in L[p^2]$ which implies that $p \in \mathcal{A}p^2$. Therefore, $\mathcal{A}$ is left regular. □

The left-right dual of Theorem 4.2 reads as follows:

Theorem 4.3. *Let* $\mathcal{A}$ *be a semigroup. The followings are equivalent:*

(1) $\mathcal{A}$ *is right regular,*

(2) $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^+$*-HFRI* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,

(3) $\widehat{\varepsilon}(p) \cong_k^- \widehat{\varepsilon}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^-$*-HFRI* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,

(4) $\widehat{\varepsilon}(p) \cong \widehat{\varepsilon}(p^2)$ *for each* sup*-HFRI* $\widehat{\varepsilon}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$.

Theorem 4.4. *Let* $\mathcal{A}$ *be a semigroup. The followings are equivalent:*

(1) $\mathcal{A}$ *is completely regular,*

(2) $\widehat{\varepsilon}(p) \cong_k^+ \widehat{\varepsilon}(p^2)$ *and* $\widehat{\omega}(p) \cong_k^+ \widehat{\omega}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^+$*-HFRI* $\widehat{\varepsilon}$, $\sup_k^+$*-HFLI* $\widehat{\omega}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,

(3) $\widehat{\varepsilon}(p) \cong_k^- \widehat{\varepsilon}(p^2)$ *and* $\widehat{\omega}(p) \cong_k^- \widehat{\omega}(p^2)$ *for each* $k \in [0,1]$, $\sup_k^-$*-HFRI* $\widehat{\varepsilon}$, $\sup_k^-$*-HFLI* $\widehat{\omega}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$,

(4) $\widehat{\varepsilon}(p) \cong \widehat{\varepsilon}(p^2)$ *and* $\widehat{\omega}(p) \cong \widehat{\omega}(p^2)$ *for each* sup*-HFRI* $\widehat{\varepsilon}$, sup*-HFLI* $\widehat{\omega}$ *of* $\mathcal{A}$ *and* $p \in \mathcal{A}$.

Proof. It follows from Theorems 4.2 and 4.3. □

A HFS $\widehat{\varepsilon}$ on $\mathcal{A}$ is called

(1) constant if $\widehat{\varepsilon}(p) = \widehat{\varepsilon}(q)$ for all $p, q \in \mathcal{A}$,

(2) $\sup_\alpha^+$-constant if $\widehat{\varepsilon}(p) \cong_\alpha^+ \widehat{\varepsilon}(q)$ for all $p, q \in \mathcal{A}$,

(3) $\sup_\beta^-$-constant if $\widehat{\varepsilon}(p) \cong_\beta^- \widehat{\varepsilon}(q)$ for all $p, q \in \mathcal{A}$, and

(4) sup-constant if $\widehat{\varepsilon}(p) \cong \widehat{\varepsilon}(q)$ for all $p, q \in \mathcal{A}$.

Then it can be easily seen the following conditions:

(1) if $\widehat{\varepsilon}$ is constant, then $\widehat{\varepsilon}$ is sup-constant,

(2) if $\widehat{\varepsilon}$ is sup-constant, then $\widehat{\varepsilon}$ is both $\sup_{\alpha}^{+}$-constant and $\sup_{\beta}^{-}$-constant.

Theorem 4.5. *Let $\mathcal{A}$ be a semigroup. The followings are equivalent:*

(1) *$\mathcal{A}$ is left simple,*

(2) *$\widehat{\varepsilon}$ is $\sup_k^{+}$-constant for every $k \in [0,1]$ and $\sup_k^{+}$-HFLI $\widehat{\varepsilon}$ of $\mathcal{A}$,*

(3) *$\widehat{\varepsilon}$ is $\sup_k^{-}$-constant for every $k \in [0,1]$ and $\sup_k^{-}$-HFLI $\widehat{\varepsilon}$ of $\mathcal{A}$,*

(4) *$\widehat{\varepsilon}$ is sup-constant for every sup-HFLI $\widehat{\varepsilon}$ of $\mathcal{A}$.*

Proof. (2) ⇔ (3) ⇔ (4). It follows from Proposition 3.10.

(1) ⇒ (2). Assume that (1) holds, $k \in [0,1]$ and $\widehat{\varepsilon}$ is a $\sup_k^{+}$-HFLI of $\mathcal{A}$. Let $p, q \in \mathcal{A}$. Since $\mathcal{A}$ is left simple, we have $p \in \mathcal{A} = \mathcal{A}q$ and $q \in \mathcal{A} = \mathcal{A}p$. Thus $p = w_1 q$ and $q = w_2 p$ for some $p, q \in \mathcal{A}$. Since $\widehat{\varepsilon}$ is a $\sup_k^{+}$-HFLI of $\mathcal{A}$, we get

$$\widehat{\varepsilon}(p) \sqsubseteq_k^{+} \widehat{\varepsilon}(w_2 p) = \widehat{\varepsilon}(q) \sqsubseteq_k^{+} \widehat{\varepsilon}(w_1 q) = \widehat{\varepsilon}(p).$$

Then $\widehat{\varepsilon}(p) \cong_k^{+} \widehat{\varepsilon}(q)$. Hence $\widehat{\varepsilon}$ is $\sup_k^{+}$-constant. Therefore, we obtain that $\widehat{\varepsilon}$ is $\sup_k^{+}$-constant for every $k \in [0,1]$ and $\sup_k^{+}$-HFLI $\widehat{\varepsilon}$ of $\mathcal{A}$.

(4) ⇒ (1). Assume that (4) holds. Let L be a left ideal of $\mathcal{A}$ and $w \in L$. Then, by Theorem 2.11, we have $\widehat{\chi}_L$ is sup-HFLI of $\mathcal{A}$. By assumption (4), we get that $\widehat{\chi}_L$ is sup-constant. Thus $\widehat{\chi}_L(p) \cong \widehat{\chi}_L(w) = [0,1]$ foll $p \in \mathcal{A}$ which implies that $\widehat{\chi}_L(p) = [0,1]$ for all $p \in \mathcal{A}$. Hence $\mathcal{A} = L$. Therefore, $\mathcal{A}$ is left simple. □

The left-right dual of Theorem 4.5 reads as follows:

Theorem 4.6. *Let $\mathcal{A}$ be a semigroup. The followings are equivalent:*

(1) *$\mathcal{A}$ is right simple,*

(2) *$\widehat{\varepsilon}$ is $\sup_k^{+}$-constant for every $k \in [0,1]$ and $\sup_k^{+}$-HFRI $\widehat{\varepsilon}$ of $\mathcal{A}$,*

(3) *$\widehat{\varepsilon}$ is $\sup_k^{-}$-constant for every $k \in [0,1]$ and $\sup_k^{-}$-HFRI $\widehat{\varepsilon}$ of $\mathcal{A}$,*

(4) *$\widehat{\varepsilon}$ is sup-constant for every sup-HFRI $\widehat{\varepsilon}$ of $\mathcal{A}$.*

The following theorem can be seen in a similar way as in the proof of Theorem 4.5.

Theorem 4.7. *Let $\mathcal{A}$ be a semigroup. The followings are equivalent:*

(1) *$\mathcal{A}$ is simple,*

(2) *$\widehat{\varepsilon}$ is $\sup_k^{+}$-constant for every $k \in [0,1]$ and $\sup_k^{+}$-HFI $\widehat{\varepsilon}$ of $\mathcal{A}$,*

(3) *$\widehat{\varepsilon}$ is $\sup_k^{-}$-constant for every $k \in [0,1]$ and $\sup_k^{-}$-HFI $\widehat{\varepsilon}$ of $\mathcal{A}$,*

(4) *$\widehat{\varepsilon}$ is sup-constant for every sup-HFI $\widehat{\varepsilon}$ of $\mathcal{A}$.*

From Theorems 4.5 and 4.6, we have the following theorem.

Theorem 4.8. *Let $\mathcal{A}$ be a semigroup. The followings are equivalent:*

(1) *$\mathcal{A}$ is group,*

(2) $\widehat{\varepsilon}$ *and* $\widehat{\omega}$ *are* $\sup_k^+$*-constant for every* $k \in [0,1]$, $\sup_k^+$*-HFLI* $\widehat{\varepsilon}$ *and* $\sup_k^+$*-HFRI* $\widehat{\omega}$ *of* $\mathcal{A}$,

(3) $\widehat{\varepsilon}$ *and* $\widehat{\omega}$ *are* $\sup_k^-$*-constant for every* $k \in [0,1]$, $\sup_k^-$*-HFLI* $\widehat{\varepsilon}$ *and* $\sup_k^-$*-HFRI* $\widehat{\omega}$ *of* $\mathcal{A}$,

(4) $\widehat{\varepsilon}$ *and* $\widehat{\omega}$ *are* sup*-constant for every* sup*-HFLI* $\widehat{\varepsilon}$ *and* sup*-HFRI* $\widehat{\omega}$ *of* $\mathcal{A}$.

5. Conclusions and Future Works

In present paper, we have introduced the concepts of $\sup_\alpha^+$-HFRIs (resp., $\sup_\alpha^+$-HFLIs, $\sup_\alpha^+$-HFIIs, $\sup_\alpha^+$-HFIs) and $\sup_\beta^-$-HFRIs (resp., $\sup_\beta^-$-HFLIs, $\sup_\beta^-$-HFIIs, $\sup_\beta^-$-HFIs) which are generalizations of the concepts of sup-HFRIs (resp., sup-HFLIs, sup-HFIIs, sup-HFIs) of semigroups, and discussed their some properties. Furthermore, the concepts have been established by FSs, Łukasiewicz fuzzy sets, Łukasiewicz anti-fuzzy sets, PFSs, HFSs, hybrid sets, IvFSs and cubic sets. Finally, we have characterized intra-regular, left (right) regular, completely regular, left (right) simple and simple semigroups in terms of $\sup_\alpha^+$-type and $\sup_\beta^-$-type of hesitant fuzzy sets.

The following are objectives for study and research in semigroups and other algebras:

- to introduce and study $\sup_\alpha^+$-type and $\sup_\beta^-$-type of HFSs based on bi-ideals of semigroups,
- to introduce and study $\sup_\alpha^+$-type and $\sup_\beta^-$-type of HFSs based on ideal theory in BCK/BCI-algebras, ternary semigroups, Γ-semigroups and LA-semigroups,
- to introduce and study $\sup_\alpha^+$-type and $\sup_\beta^-$-type of HFSs based on substructures of GE-algebras, BRK-algebras, BE-algebras and IUP-algebras [5, 6, 11, 26],
- to apply this study to the concept of rough sets according to Ansari's study [2, 3].

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