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Cooperative Investment Problem With an Authoritative Risk Determined by Central Bank

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Abstract. In this paper, we are interested in providing an analytic solution for cooperative investment risk. We reformulate cooperative investment risk by writing dual representations for each risk preference (Coherent risk measure). Finding an analytic solution for this problem for both cases individual and cooperative investment by using dual representation for each risk preference has a strong effect on the financial market. In addition, we formulate a problem that covers the risk minimization with an expected return maximization problem with risk constraint, for the general case of an arbitrary joint distribution for the asset return under certain conditions and assuming that all coherent risk measure is continuous from below. Thus, the optimal portfolio is written as the optimal Lagrange multiplier associated with an equality-constrained dual problem. Furthermore, a unique equilibrium allocation as a fair optimal allocation solution in terms of equilibrium price density function for each agent is also shown.

1. Introduction

Cooperative investment is considered a recent problem and it is not very old, all the work before (2013) was working in risk-sharing without portfolio optimization problem. Later, they focused on optimal risk-sharing which has become one of the central avenues of study for researchers, which is defined as similar to cooperative investment but is not concerned with portfolio optimization.
Cooperative investment synthesizes three key elements; (1) Modeling of agents' risk preferences. The fact that different agents have different utilities or different risk preferences for goods is the basis of all markets. In my paper we choose the coherent risk measure, not that there are many types of coherent risk measures, we focus on negative expected as definitions of coherent risk measure and we write dual representation for each risk preference for each agent (investor). Then, formulate an individual optimization problem

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(2) Formulating and solving a cooperative investment problem. In this paper we develop [3] by Akturk et al in 2020 who studied portfolio investment with two risks and we develop this paper and solve cooperative investment by considering an authority risk measure determined by the central bank. Hence, we solve a cooperative investment problem with three risk measure: the first risk measure represent the first agent, the second risk measure represent the second agent and an authority measure reflects the third risk measure. It can be formulated as follows: for a given uncertain outcome X where we have m agents, the question is how X can be partitioned into random $Y_i, i = 1, ..., m$, which is based on their risk-reward preferences, such that $\sum_{i=1}^{m} Y_i = X$ and whether each Y_i is acceptable for each agent $i$or not. At first glance, cooperative seems to offer no advantage over individual investment. However, the exact reason why cooperative investment has an advantage is that agents' shares may not be replicable in an incomplete financial market. In other words, sharing creates instruments that the one hand, satisfy individual risk preferences but, on the other hand, may not be replicable on the incomplete market, so each agent is strictly better at participating in cooperative investment than investing alone. Note that, the underlying asset returns X are in some L^p space with $L^p \in [1, +\infty]$ and they have an arbitrary joint distribution with possible correlation. Assuming that all risk measures are continuous from below so that the suprema in the dual representations are attained at the same dual probability measure, we derive a simple dual problem with a linear objective and a linear equality constraint in addition to domain constraints for the dual variables. Thus, at first, we write dual representation for each risk preference (Coherent risk measure) for each agent (investor). As shown in the examples; example 1, then create individual optimization problem and cooperative investment problem then find the optimal solution as shown in theorem(1), and theorem (2), respectively. In the last step, we find the equilibrium allocation in terms of equilibrium price by formulating the optimal problem in case of equilibrium with an initial endowment for each agent's 'investor'.

2. Literature Review

In 2013, Grechuk and Zabarankin [9] studied risk-sharing problems for agents with utility functions depending only on the expected value and a deviation measure of an uncertain payoff. Moreover, all of these works formulated and studied cooperative games with players using different deviation measures as numerical representations of their attitudes toward risk. Note that, cooperative investment is considered a recent problem and it is not very old, all the work before (2013) was working in risk measure without portfolio optimization problems. In 2015, Grechuk and Zabarankin [7], studied a cooperative game with a general deviation measure, they showed that a cooperative portfolio does not, in general, accommodate the risk preferences of all agents, whereas the risk preferences of each agent are satisfied at the stage of fair sharing of the cooperative portfolio's return. More over, In 2015, Grechuk and Zabarankin [7],

described the cooperative investment in a single period with the alternative utility function and alternative deviation measure, respectively. In 2016, Almualim [2] dynamic cooperative investment with the GARCH model and applied the GARCH model to the asset return. In 2017, Grechuk and Zabarankin [8], extended their work into cooperative investment in multi-period with synergy effect also suppose that U_i is monetary utility function and he solves the following problem $sup_{X\in\mathcal{F}}U^*(X)$, where $X^* = \sum_{i=1}^{m} Y_i$ is a maximizer to the investment problem. In, 2020, Akturk [3] studied Portfolio optimization with coherent risk measures with an authorized risk but not in cooperative investment. In 2021, Sarkar B. & Guchhait [10] studied the economic and environmental assessment of retailers within a supply chain management and they took into account the equilibrium condition of the forward and backward supply chain and their results found by the classical optimization technique. However, they did not study the problem with an authoritative risk determined by the Central Bank. Furthermore, in 2022, Sarkar B., et al. [12] focus on A multi-period multi-product inventory model that is tested through an artificial neural network for experiencing an uncertain environment. In addition, their result shows that the proposed approach is the best for cost optimization and time minimization through an artificial neural network. Furthermore, in 2022, Abdikerimova, et al. [4] designed their problem as a risk-sharing strategy that is based on mean-variance optimizations of participants' terminal reserves. They show convergence of the risk-sharing solution and the ratios of long-term reserves. As well as they also studied the impact of financial fairness on various risk-sharing strategies and their long-term limits, but they did not apply it to portfolio optimization problems.

In this paper, I develop [3] by Akturk in 2020, and joined it in case of cooperative investment. Hence, we create Cooperative investment with an authorized risk introduced by the central bank where the underlying asset returns X are in some L^∞. Then we start to solve three key elements of cooperative investment as follows:

1) formulate individual investment and it is different than [3] because in my case we need to add an expected return constraint for the investor and this is the first difference before. then

2) expend our problem to cooperative investment for two agents with an authority risk measure and its level determined by the central bank.

3) studying an equilibrium problem to find a fair equilibrium allocation to be satisfied and acceptable for each agent which I mean in this point the result from equilibrium allocation for each agent is better than the result from solving the investment problem alone.

3. Problem Formulation

3.1. **Problem reformulation. Firstly** : in case of individual problem. Let us start to model risk-aversion, let $\rho_1, \rho_2, \rho_3 : L^p \to R$ be a three arbitrary coherent risk measures. The aim of the portfolio manager for individual cases is to choose a portfolio $\omega \in W$ that minimizes the type 1 risk $\rho_1(\omega^T X)$ while controlling the type 2 risk $\rho_2(\omega^T X)$ with a fixed threshold level $r \in R$ that is while satisfying $\rho_2(\omega^T X) \le r$ which we refer to as the risk constraint (an external regulatory authority), and expected return level reflected by $E[\omega^T X]$. In the case of individual investment with each risk measure defined by negative expectation in this case we can formulate the individual investment as follows:

$$minimize \rho_1(\omega^T X) \; subjectto \; \rho_2(\omega^T X) \le r \, , E[\omega^T X] \ge \pi \, , \omega \in W \tag{3.1}$$

here $\rho_1(Y) = E[-Y]$ for each $Y \in L^p$, in our case the random vector X with arbitrary distribution and assuming that $\rho_1, \rho_2$are continuous from below, in this paper we characterize an optimal solution for (2.2) as a Lagrange multiplier of an associated dual problem. we need to address some literature reviews for portfolio optimization problems under an arbitrary joint distribution as follows: we assume $X \in L_n^p$ for a for a fixed $p \in [1, +\infty]$ and ρ_1, ρ_2 are continuous on L^p, see ([11],corollary 2.3), thus ρ_1, ρ_2 admit dual representations of the form:

$$\rho_1(Y) = max_{Q_1 \in \mathcal{Q}_1} E^{Q_1}[-Y]$$

and

$$\rho_2(Y) = max_{Q_2 \in \mathcal{Q}_2} E^{Q_2}[-Y]$$

for each $Y \in L^p$, where $\mathcal{Q}_1, \mathcal{Q}_2$ are convex subsets of $M_1^q(P)$ such that corresponding density set $D(\mathcal{Q}_1)$, $D(\mathcal{Q}_2)$, are convex $\sigma(L^q, L^p)$-compact subset of L^q. For each $j \in \{1, 2\}$, Let us define the continuous convex function $g_i : R^n \to R$ by

$$g_i(\omega) = \rho_j(\omega^T X) = max_{V \in D(\mathcal{Q}_j)} E[-V \omega^T X]$$

for each $\omega \in R^n$. We recall a few notations and facts from convex analysis. Let $\mathcal{X}$ be a Hausdorff locally convex topological linear space with topological dual $\mathcal{Y}$ and bilinear duality mapping $< ., . >: \mathcal{Y} \times \mathcal{X} \to R$

i $\mathcal{X} = R^n$ with the usual topology which yields $\mathcal{Y} = R^n$ together with $< x, y >= y^T x$ for every $x \in R^n$,$y \in R^n$.

ii $\mathcal{X} = L^q$ with $q \in [1, +\infty)$ with the weak topology $\sigma(L^q, L^p)$, which yields $\mathcal{Y} = L^p$ together with $< Y, U >= E[UY]$ for every $U \in L^q, Y \in L^p$.

iii $\mathcal{X} = L^\infty$ with weak topology $\sigma(L^q, L^p)$, which yields $\mathcal{Y} = L^p$ together with $< Y, U >= E[UY]$ for every $U \in L^q, Y \in L^p$.

Let $A \subset \mathcal{X}$ be a set. $cone(A)$:=$\{\lambda x | \lambda \ge 0, x \in A\}$, is called the conic hull of A. if A is convex then $cone(A)$ is a convex cone. For $x \in A$, the convex cone

$$\mathcal{N}_A := \{y \in \mathcal{Y} | \forall \hat{x} \in A :< y, x > \ge < y, \hat{x} >\}$$

is called the normal cone of A at x. The function $I_A : \mathcal{X} \to R \bigcup \{+\infty\}$ defined by $I_A(x) = 0$ for $x \in A$ and $I_A = +\infty$ for $x \in \mathcal{X} \backslash A$ is called indicator function of A. Note that A is convex if and only if I_A is convex. Let $g := \mathcal{X} \to R \bigcup \{+\infty\}$ be a function. For $x \in \mathcal{X}$, the set $\partial g := \{y \in \mathcal{Y} | \forall \hat{x} \in \mathcal{X} : g(\hat{x} \ge g(x)+ < y, \hat{x} - x >\}$ is called subdifferential of g at x . If A is a nonempty convex set then it is well-known that from Zalinescu,2002, [14]. $\partial I_A(x) = \mathcal{N}_A(x)$ for every $x \in A$ and $\partial I_A(x) = \phi$ for every $x \in \mathcal{X} \backslash A$. The function $g^* : \mathcal{Y} \to R \bigcup \{\pm\infty\}$ defined by $g^*(y) := sup_{x \in \mathcal{X}}(< y, x > -g(x))$ for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that g is lower semi continuous at x.

Now, we need to formulate a second constraint qualification, we also need the following. For $A \subset \mathcal{X}$, the set

$$qri(A) := \{x \in A | \mathcal{N}_A(x) \text{ is a subspace of } \mathcal{Y}\}$$

is called the quasi-relative interior of A see ([5]). When $\mathcal{X} = R^n$, hence, $qri(A)$ coincides with relative interior of A. In this case $qri(A) \neq \phi$ whenever A is nonempty, close, and convex. When $\mathcal{X} = L^q(q \in [1, +\infty])$ is considered with topology $\sigma(L^q, L^p)$ and A is nonempty,close and convex, one has $qri(A) \neq \phi$ see [5]. In particular, if $A - L^q_+ := \{U \in L^q | \mathcal{P}\{U \geq 0\} = 1\}$, then $qri(A) = \{U \in L^q | \mathcal{P}\{U > 0\} = 1\}$, see [5], while the usual interior of A can even be empty. (For $q < +\infty$, considering the strong and topologies on L^q yield the same quasi relative interior for a convex set, see [5].

Note that in our problem as mentioned in (3.1), we add constraint qualification which is called (Slater's condition) as an authority risk measure defined by ρ_2 to be able to study a dual problem with zero duality gap.

The main theorems in this paper are showing theorems and their proofs, by constructing a Lagrange dual problem for (3.1) and exploiting the dual representations of ρ_1, ρ_2. Moreover, the optimal solution for (3.1) can be calculated as the Lagrange multiplier of the equality constraint of the dual problem at optimality. where the dual problem is as follows;

$$\text{maximize } -r\nu - \lambda_1 \pi - \lambda_2 \tag{3.2}$$

$$\text{subject to } E[-UX] + \nu\, E[-VX] + \lambda_1 E[X] + \lambda_2\, 1 = 0$$

$$U \in \mathcal{D}(\mathcal{Q}_1), V \in cone(\mathcal{D}(\mathcal{Q}_2)), \nu \geq 0, \lambda_1, \lambda_2 \in R$$

optimal value for individual problem is shown in Theorem 1.

Secondly : reformulate the problem in the case of cooperative investment.

In this section we develop and present a novel technique for solving continuous portfolio optimization problems in cooperative investment cases. Now, we suppose the two agents (investors) agree to invest their joint capital into the risky instrument. Then, divide the random variable X by the amount of money investors (agents) get at the end of the investment period, where Y_1, and $Y2$ are the optimal allocation of the first and second agents, respectively. such that $X = Y_1 + Y_2$. Now, the portfolio optimization for individual investment for first and second investors is formulated as a problem (3.1), while the cooperative investment optimization problem with an external regulatory authority with a different risk reflected by ρ_3 imposes the risk constraint as an obligation for the portfolio manager. This also makes sense when the portfolio manager wishes to work with two risk measures in the case of individual investment and three risk measures in the case of cooperative investment. Furthermore, the principle one risk ρ_1 for the agent having higher seniority than the other risk ρ_2 which represents a risk constraint and controlled within a fixed threshold level $r \in R$, which is expressed as follows $\rho_2(\omega^T Y_i) \leq r$. Indeed, in the case of cooperative investment, an external regulatory authority with a different risk is reflected by $\rho_3(X)$, Where $X = Y_1 + Y2$ and

$X = \omega^T x$. Thus, we formulate cooperative investment for the continuous portfolio optimization problem with short selling as :

$$\begin{aligned} minmize\ \rho_1(\omega^T Y_1), \\ S.T. \\ \rho_2(\omega^T Y_2) \le r_1,\ \rho_3(\omega^T x) \le r_2 \\ E[\omega^T Y_1] \ge \pi_1, E[\omega_2^T Y_2] \ge \pi_2, \\ X = Y_1 + Y_2,\ \omega \in \mathcal{W} \end{aligned} \tag{3.3}$$

The aim of the portfolio manager to choose a portfolio $\omega \in \mathcal{W}$ that minimizes the type-1 risk $\rho_1(\omega^T Y_1)$ while controlling the type-2 for second agents $\rho_2(\omega^T Y_2)$ within a fixed threshold level $r_1 \in R$, and controlling the type-3 risk $\rho_3(\omega^T X)$ within a fixed threshold level $r \in R$, note that when you need to choose $r_2 \in R$ is less than or equal to value as solving minimization individual problem for each agent. In Particular, For a random vector X with an arbitrary distribution and assuming that ρ_1, ρ_2, ρ_3 are continuous from below. Note that, this framework covers as special cases the problem of maximizing expected return subject to a risk constraint if we take $\rho_1(Y_1) = E[-Y_1]$ for each $Y \in L^p$ as well as the problem of minimizing (the type 1) risk while maintaining a high-enough expected return if we take $\rho_2(Y_2) = E[-Y_2]$ and $\rho_3(X) = E[-X]$ for each $Y_1, Y_2, X \in L^p$. So the Lagrange dual problem (3.3) for cooperative investment problem takes the more explicit form as follows

$$maximize\ -r_1\nu_1 - r_2\nu_2 - \lambda_1\pi_1 - \lambda_2\pi_2 - \lambda_3 \tag{3.4}$$

$$subject\ to\ E[-UY_1] + \nu_1\ E[-V_1Y_2] + \nu_2 E[V_2 x] + \lambda_1 E[Y_1] + \lambda_2 E[Y_2] + \lambda_3\ 1 = 0$$

$$U \in \mathcal{D}(\mathcal{Q}_1), V_1 \in \mathcal{D}(\mathcal{Q}_2), V_3 \in \mathcal{D}(\mathcal{Q}_3), \nu_{1,2} \in R, \lambda_{1,2,3} \in R$$

optimal value for cooperative Investment problem is shown in Theorem 2.

Remark :

According to the condition of Karush-Kuhn-Tucker condition for the problem and from [14], thus an optimal solution for (3.1),(3.3) is an optimal solution for their dual problem (3.2), (3.4) for individual investment problem and cooperative investment problem, respectively. Note that dual problem (3.2), and (3.4) is equal to (4.1), and (4.2) in the next section. According to (Akturk, Ararat, 2019) Slater's condition (as an external regulatory authority with a different risk perception reflected by ρ_2, and ρ_3 for individual and cooperative investment problems, respectively) already guarantees the existence of an optimal solution. We reformulated the dual problem and defined variable U, V, ν, λ and the relationship between them in the dual problem (3.2),(3.4), thus, the existence of an optimal for the dual problem is not guaranteed a prior. However, when we rewrite the dual problem and rewrite the objective (first line of the problem in both (3.2),(3.4)) these automatically imply the existence of an optimal solution for the Lagrange multiplier for the equality constraint in the dual problem (3.2),(3.4), which is shown to give an optimal for the original problem (3.1), (3.3) respectively, for more details see [14]. Consequently, we already find the optimal solution but when we change the value of r_2, r_3 as in the fixed level of risk for the second investor, this characterizes the set of all Pareto optimal allocations, which can be visualized as the efficient frontier.

3.2. **Fair Equilibrium allocation.** Now, the neutral question is how can we select a unique 'fair' point on the efficient frontier. In the next section, we will address the unique solution that satisfies each agent, thus, we need to find a special point which is called 'Equilibrium allocation'. Hence, the third step for solving the cooperative investment problem (3.3), is to find a fair point that is called " an equilibrium allocation" among all the points in the efficient frontier. Note that: to find the whole efficient frontier we need to change the value for r_1, r_2, π_1, pi_2 in the cooperative investment (3.3). the efficient frontier is the convex curve between two investors for the main problem in the case of the cooperative investment problem (3.3) "concave curve for its corresponding dual problem (3.4)". According to the theory of market Equilibrium, the price of assets will no longer be given in advance. Different agents demand by their preferences and their budgets. According to (Follmer, 2009), see [6].

3.2.1. *Steps for Finding Equilibrium allocation.* (**1**) We need the equilibrium allocation for each agent's investors by solving the utility maximization problem of agent $i \in I$ concerning price density φ.

$$maximize\ U_i(Y_i)\ s.t\ E(\varphi Yi) \le E(\varphi W_i),\ i \in I = 1, 2, ..., m \tag{3.5}$$

where $U_i = E[u_i(.)]$, and we can suppose there are no initial endowments, in this special case we can replace the condition $E(\varphi Y_i) \le 0$ and find an equilibrium allocation in terms of price density φ. Hence, to formulate each problem for each agent (investor) $i \in I = \{1, 2, 3, ..., m\}$ to find each equilibrium allocation in terms of price density φ

$$\begin{aligned} minmize_{Y_1}\ \rho_1(Y_i) \\ s.t\ \rho_2(X) \le r \\ E[Y_1] \ge \pi \\ E[\varphi Y_i] \le 0 \end{aligned} \tag{3.6}$$

and we reformulate it for each agent (investor $i \in I = \{1, 2, 3, ..., m\}$ as follows:

$$maximize_{Y_1} E[-UY_i] + \nu_1(E[-M_1X] + r) + \nu_2 E[-\varphi M_2 Y_i] + \lambda_1(E[Y_i] - \pi_1) \tag{3.7}$$

in our problem, we will say Y_i^{φ} solves the utility maximization problem for agent 'investor' $i \in I$ with respect to the price density φ. Thus, the key problem is whether φ can be chosen in such a way that the requested profiles Y_i^{φ},$i \in I$ form a feasible allocation. Moreover, according to [6] defined 'Arrow-Debreu -equilibrium' as follows

Definition : A price density φ^* together with a feasible allocation $(Y_i^*)_{i \in I}$ is called an Arrow-Debreu equilibrium if each Y_i^* solves the utility maximization problem of agent $i \in A$ concerning φ^*.

In particular, the initial endowments $W_i, i \in I$ are assumed to be non-negative. Moreover, we assume $P^{\varphi}[W_i > 0] \ne 0$ for all $i \in I$ and $E[X] < \infty$, where $\sum_{i \in I} W_i = X$. In our case, we have

$\sum_{i\in I} Y_i = X$ since we don't have an initial endowment. A function $\varphi \in L^1(\Omega, \mathcal{F}, P)$, such that $\varphi > 0$ $P-$ a.s, is a price density if

$$E[\varphi X] < \infty$$

more that this condition is satisfied as soon as φ is bounded, due to our assumption $E[X] < \infty$. Given a price density φ, each agent faces exactly the optimization problem in terms of price measure $P^\varphi \approx P$. Hence, if $(Y_i^*)_{i\in I}$ is an equilibrium allocation concerning the price density φ^*. Feasibility implies $0 \le Y_i^* \le X$ and so it follows as in the proof of (corollary 3.42), see [6] that

$$Y_i^* = I_i^+(c_i\varphi^*),\ i \in I$$

with positive constant $c_i > 0$. Indeed, according to [6], we have the inverse function of the strictly decreasing function in (4.2), then the optimal X^*, where $X^* = \sum_{i\in I} Y_i^*$. Thus, $X^* = I(c\varphi)$, where each equilibrium allocation $Y_i^* = I_i^+(c_i\varphi^*)$, $c = \sum_{i\in I} c_i$, and I^+ is simply the positive part of the function $I = (U')^{-1}$, so its the inverse of restriction of U' to $[0, \infty]$. where In our problem after rewriting dual representation for each risk preference for each agent (investor). Hence, our problem will be written as follows:

$$maximize \mathcal{U}_i(Y_i)\ s.t\ E(\varphi Yi) \le 0,\ i \in I = 1, 2, ..., m \tag{3.8}$$

where,

$$\mathcal{U} = E[-UY_i] + \nu_1(E[-M_1X] + r) + \nu_2 E[-\varphi M_2 Y_i] + \lambda_1(E[Y_i] - \pi_1)$$

(**2**) then, joint equilibrium allocation for each agent 'investor' and solve the feasibility problem to find the equilibrium price.

Let us start to formulate equilibrium problem; Consider a finite set I of economic agents and convex set $\mathcal{X}$ of admissible claim. Suppose at the initial time $t = 0$ each agent $i \in I$ in our case in this paper $i = 1, 2$ two investors, so each agent 'investor ' has no initial endowment w_i, $i = 1, 2$ whose discount payoff at time $t = 1$, furthermore, Agents may want to exchange since there is no initial endowment w_i, hence admissible claim $Y_i \in \mathcal{X}$. Consequently, This could lead to a new allocation Y_i, $i \in I = \{1, 2\}$ ad the total demand matches the overall supply.

Definition [6]:

A collection $Y_i, i \in I = \{1, 2\} \subset \mathcal{X}$ is called a feasible allocation if it satisfies the market clearing condition

$$\sum_{i\in I} Y_i = X,\ P - a.s$$

The budget constraints will be determined by a linear pricing rule of the form

$$\Phi(X) := E[\varphi X],\ X \in \mathcal{X}$$

where φ is a price density, and $\mathcal{F}$ feasible set, i.e an integrable function $(\Omega, \mathcal{F})$, such that $\varphi > 0$ P-a.s and $E[Y_i|\varphi] < \infty$ for all $i \in I$. To any such φ we can associate a normalized price measure $P^\varphi \approx P$

with density $\varphi E[\varphi]^{-1}$.
Note that the market clearing condition

$$X = \sum_{i \in I} Y_i^* = \sum_{i \in I} I_i^+(c_i \varphi^*)$$

Consequently, we can write the feasibility problem as follows:

$$\begin{gathered} \textit{Find } P \\ \textit{subject to} \\ Y_1 + Y_2 = X_r(\omega) \\ \sum (PX_r(\omega) = 0 \\ \rho_2(Y_2) = r \end{gathered} \tag{3.9}$$

this problem can be solved as follows:

$$\begin{gathered} \textit{minimize } 0 \\ \textit{subject to} \\ Y_1 + Y_2 = X_r(\omega) \\ \sum (PX_r(\omega) = 0 \\ \rho_2(Y_2) = r \end{gathered} \tag{3.10}$$

To any such φ we can associate a normalized price measure $P^{\varphi} \approx P$ with density $\varphi E[\varphi]^{-1}$, see [6]. Note that, the aim for solving the feasibility problem is to get to the end fair point which is on an efficient frontier for more details for applying this in the real market: we solve the feasibility problem we have the value for price then plug the value for price in each equilibrium allocation since it is written in terms of price density.

4. Main Results

4.1. **Theorems and Proofs: Theorem 1** : The optimal value for the individual problem (3.1) is equal to the optimal value for the corresponding dual problem

$$\begin{gathered} \textit{maximize } -r\nu - \lambda_1 \pi - \lambda_2 \\ \textit{subject to } E[-Ux] + \nu\, E[-Vx] + \lambda_1 E[x] + \lambda_2\, 1 = 0 \\ U \in \mathcal{D}(Q_1), V \in cone(\mathcal{D}(Q_2)), \nu \geq 0, \lambda_1, \lambda_2 \in R \end{gathered} \tag{4.1}$$

and optimal value denote it by

$$p = \sup_{\nu \geq, \lambda_i \in R} d(\nu, \lambda_1, \lambda_2) \ fori = 1, 2$$

for each $\nu \geq 0$, and $\lambda_i,\ i = 1, 2 \in R$.

Proof : Let us denote p the optimal value of the problem (3.1) Since the optimal value for problem (3.1) is the optimal value of the Lagrange dual problem (4.1), that is

$$p = sup_{\nu \geq 0, \lambda_1, 2 \in R} d(\nu, \lambda_1, \lambda_2)$$

where, for each $\nu \geq 0, \lambda_1, \lambda_2 \in R$, thus

$$d(\nu, \lambda_1, \lambda_2) = \inf_{\omega \in R^n} (\rho_1(\omega^T x) + \nu(\rho_2(\omega^T x) - r) + \lambda_1(E[\omega^T x] - \pi) + \lambda_2(1^T \omega - 1))$$

By using dual representation of ρ_1, ρ_2, we fix $\nu \geq 0$, $\lambda_1, \lambda_2 \in R$

$$\begin{aligned} &d(\nu, \lambda_1, \lambda_2) \\ &= inf_{\omega \in R^n} (\max_{U \in \mathcal{D}(Q_1)} E[-U\omega^T x]) + \nu\, max_{V \in \mathcal{D}(Q_2)} E[-V\omega^T x]) + \lambda_1(E[\omega^T x]) + \lambda_2(1^T \omega) - r\nu - \lambda_1 \pi - \lambda_2 \end{aligned}$$

let $f(\omega, U, V) := E[-U\omega^T x] + \nu\, E[-V\omega^T x] + \lambda_1 E[\omega^T x] + \lambda_2 1^T \omega$ for each $\omega \in R^n$, $U \in \mathcal{D}(Q_1), V \in \mathcal{D}(Q_2)$. Note that $\omega \to f(\omega, U, V)$ is convex(affine) and continuous, $(U, V) \to f(\omega, U, V)$ is concave (affine) and $\sigma(L^q, L^p)$-continuous (continuous), and $\mathcal{D}(Q_1) \times \mathcal{D}(Q_2)$ is $\sigma(L^q, L^p)$-compact. Hence, From classical minimax theorem see [13] ensures that

$$\begin{aligned} &d(\nu, \lambda_1, \lambda_2) \\ &= \sup_{(U,V) \in \mathcal{D}(Q_1) \times \mathcal{D}(Q_2)} inf_{\omega \in R^n} (E[-U\omega^T x] + \nu\, E[-V\omega^T x] + \lambda_1 E[\omega^T x] + \lambda_2 1^T \omega) - r\nu - \lambda_1 \pi - \lambda_2 \end{aligned}$$

Clearly, for every $(U, V) \in \mathcal{D}(Q_1) \times \mathcal{D}(Q_2)$

$$\begin{aligned} &\inf_{\omega} \in R^n (E[-Ux] + \nu\, E[-Vx] + \lambda_1 E[x] + \lambda_2 1)^T \omega \\ &= \begin{cases} 0, & \text{if } E[-Ux] + \nu\, E[-Vx] + \lambda_1 E[x] + \lambda_2\, 1 = 0 \\ -\infty, & \text{else} \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned} &d(\nu, \lambda_1, \lambda_2) \\ &= \begin{cases} -r\nu - \lambda_1 \pi - \lambda_2, & \text{if } \exists (U, V) \in Q_1) \times \mathcal{D}(Q_2 : E[-Ux] + \nu\, E[-Vx] + \lambda_1 E[x] + \lambda_2\, 1 = 0 \\ -\infty, & \text{else} \end{cases} \end{aligned}$$

So the Lagrange dual problem (3.1) for individual cases takes the more explicit form as follows

$$maximize\ - r\nu - \lambda_1 \pi - \lambda_2 \tag{4.2}$$

$$subject\ to\ E[-Ux] + \nu\, E[-Vx] + \lambda_1 E[x] + \lambda_2\, 1 = 0$$

$$U \in \mathcal{D}(Q_1), V \in \mathcal{D}(Q_2), \nu \geq 0, \lambda_1, \lambda_2 \in R$$

Now, we make some changes in variables to avoid the multiplication of variables ν, V as follows; if $M \in cone(\mathcal{D}(Q_2))$, then their exist $\nu \geq 0$ and $V \in \mathcal{D}(Q_2)$

such that $M = \nu V$: we simply take $\nu = E[M]$ and $V = \frac{M}{\nu}$ if $\nu > 0$ and aribatary $V \in \mathcal{D}(Q_2)$ if $\nu = 0$. Conversely, if $\nu \geq 0$ and $V \in \mathcal{D}(Q_2)$, then $M = \nu V \in cone(\mathcal{D}(Q_2))$. These observations allow us to reformulate a dual problem (4.2) as (4.1). Note that both problems have p as their optimal value. Let $(U^*, M^*, \lambda_1^*, \lambda_2^*) \in L^q \times L^p \times R \times R$ be an optimal solution for (4.1), see [5], corollary

4.8), there is a strong duality with corresponding Lagrange dual problem that relaxes the equality constraint, that is, we have

$$p = \inf_{\omega\in R^n} \sup_{U\in\mathcal{D}(Q_1),M\in cone(\mathcal{D}(Q_2)),\lambda_{1,2}\in R} (-rE[M] - \lambda_1\pi - \lambda_2 - \omega^T(E[Ux] + E[Mx] - \lambda_1 E[X] - \lambda_2 1))$$

$$p = \inf_{\omega\in R^n} \sup_{U\in\mathcal{D}(Q_1),M\in cone(\mathcal{D}(Q_2)),\lambda_{1,2}\in R} (-rE[M] - \lambda_1\pi - \lambda_2 + E[-U\omega^T x] + E[-M\omega^T x] + \lambda_1 E[\omega^T x] + \lambda_2\omega^T 1)$$

also from [5], corollary 4.8, ensures that there exists an optimal Lagrange multiplier $\omega^* R^n$. By the first-order condition concerning $U = U^*$, thus we have that

$$0 \in -(\omega^*)^T x - \mathcal{N}_{\mathcal{D}(Q_1)}(U^*)$$

this means

$$E[-U^*(\omega^*)^T x] \geq E[-U'(\omega^*)^T x]$$

for every $U' \in \mathcal{D}(Q_1)$, that is

$$\rho_1((\omega^*)^T x) = E[-U^*(\omega^*)^T x]$$

We conclude that $U^* \in \psi(\omega^*)$, where $\psi(\omega^*)$ defines as $\psi_j(\omega^*) := argmax_{V\in\mathcal{D}(Q)_j} E[-Vx^T\omega]$, see [3], Lemma 3.4, Hence,

$$E[-U^* x] \in \partial g_1(\omega^*) \tag{4.3}$$

In the same way, the first-order condition concerning $M = M^*$ yields

$$E[-M^*((\omega^*)^T x + r)] \geq E[-M'((\omega^*)^T x + r)]$$

for every $M' \in cone(\mathcal{D}_2)$, that is

$$E[-M^*((\omega^*)^T x + r)] = \max_{M'\in cone(\mathcal{D}(Q)_2)} E[-M'((\omega^*)^T x + r)] \tag{4.4}$$

Since$cone(\mathcal{D}_2)$ is a cone, the quantity $\sup_{M'\in cone(\mathcal{D}(Q)_2)} E[-M'((\omega^*)^T x + r)]$ can either take the value 0 or $+\infty$, Since $E[-M'((\omega^*)^T x + r)]$ is a finite number, both sides of (5.3) must equal to zero, thus we obtain

$$0 = \max_{M'\in cone(\mathcal{D}(Q)_2)} E[-M'((\omega^*)^T x + r)] = (\sup_{\lambda'\geq 0}\lambda')(\max_{V'\in\mathcal{D}(Q_2)} E[-V'((\omega^*)^T x + r)]) \tag{4.5}$$

$$= +\infty.\rho_2((\omega^*)^T x + r) = +\infty(\rho_2(\omega^*)^T x) - r)$$

Moreover , we have from optimality $\rho_2(\omega^*)^T x) = r$.
Let $\nu^* = E[M^*]$, and suppose first that $\nu^* > 0$ and let $V^* := \frac{M^*}{\nu^*} \in \mathcal{D}(Q_2)$ Then,

$$E[-M^*((\omega^*)^T x + r)] = \nu^* E[(\omega^*)^T x + r] = 0$$

so that , $E[-V^*(\omega^*)^T x] = r$, Hence

$$E[-V^*(\omega^*)^T x] = r = \rho_2(\omega^*)^T x) = \max_{V'\in\mathcal{D}(Q_2)} E[-V'(\omega^*)^T x]$$

that is $V^* \in \psi_2(\omega^*)$, Actually

$$E[-V^* x] \in \partial g_2(\omega^*)$$

Furthermore, suppose that $\nu^* = 0$ that is $M^* = 0$ $p-$ almost sure. Let us pack some $V^* \in \psi_2(\omega^*)$ arbitrarily. since $\psi_2(\omega^*) \neq \phi$ because ρ_2 is assumed to be continuous from below, thus, in both cases, we may write $M^* = \nu^* V^*$ and we can write

$$E[-M^*x] = \nu^* E[-V^*x] \in \nu^* \partial g_2(\omega^*) \tag{4.6}$$

Now, from feasibility of$(U^*, M^*, \lambda_1^*, \lambda_2^*)$ for dual problem (4.1), we have

$$E[-U^*x] + E[-M^*x] + \lambda_1^*\pi + \lambda_2^* 1 = E[-U^*x] + \nu^* E[-V^*] + \lambda_1^*\pi + \lambda_2^* 1 = 0 \tag{4.7}$$

Consequently, from (4.3),(4.6)and (4.7) we obtain

$$0 \in \partial g_1(\omega^*) + \nu^* \partial g_2(\omega^*) + \lambda_1^*\pi + \lambda_2^* 1$$

Finally, According to first-order condition concerning $\lambda_{1,2} = \lambda_{1,2}^*$, respectively. Also, we got

$$1^T \omega^* = 1$$

where $\omega^* \in \mathcal{W}$.

Therorm 2 : The optimal value for the cooperative investment problem (3.3) is equal to the optimal value for the corresponding dual problem

$$maximize\ -r_1\nu_1 - r_2\nu_2 - \lambda_1\pi_1 - \lambda_2\pi_2 - \lambda_3 \tag{4.8}$$

$$subject\ to\ E[-UY_1] + \nu_1\ E[-V_1Y_2] + \nu_2 E[V_2x] + \lambda_1 E[Y_1] + \lambda_2 E[Y_2] + \lambda_3\ 1 = 0$$

$$U \in \mathcal{D}(\mathcal{Q}_1), V_1 \in \mathcal{D}(\mathcal{Q}_2), V_3 \in \mathcal{D}(\mathcal{Q}_3), \nu_{1,2} \in R, \lambda_{1,2,3} \in R$$

and optimal value denote it by

$$p = \sup_{\nu_i \geq 0, \lambda_1, \lambda_2, \lambda_3 \in R} d(\nu_1, \nu_2, \lambda_1, \lambda_2, \lambda_3), For\ i = 1, 2$$

for each $\nu_1, \nu_2 \geq 0$,$\lambda_1, \lambda_2, \lambda_3 \in R$.

Note that the proof for case cooperative investment with an authorized risk measure for theorem (2) is similar to the proof of the theorem (1) just we have more constraints since the problem is two agents (investors) managing their risk and taking in account the authorized risk determined by the central bank.

Proof : Let us denote p the optimal value of the problem (3.3) Since the optimal value for the problem (3.3) is the optimal value of corresponding the Lagrange dual problem for (4.8), that is

$$p = \sup_{\nu_i \geq 0, \lambda_1, \lambda_2, \lambda_3 \in R} d(\nu_1, \nu_2, \lambda_1, \lambda_2, \lambda_3), For\ i = 1, 2$$

where, for each $\nu_1, \nu_2 \geq 0$,$\lambda_1, \lambda_2, \lambda_3 \in R$, thus

$$\begin{aligned} d(\nu_1, \nu_2, \lambda_1, \lambda_2, \lambda_3) = \inf_{\omega \in R^n} (&\rho_1(\omega^T Y_1) + \nu_1(\rho_2(\omega^T Y_2) - r_1) + \nu_2(\rho_3(\omega^T x) - r_2) + \lambda_1(E[\omega^T Y_1] - \pi_1) \\ &+ \lambda_2(E[\omega^T Y_2] - \pi_2) + \lambda_3(1^T\omega - 1)) \end{aligned}$$

By using Dual representation of ρ_1, ρ_2, ρ_3, we fix $\nu_1, \nu_2 \geq 0$, $\lambda_1, \lambda_2, \lambda_3 \in R$

$$d(\nu_1, \nu_2, \lambda_1, \lambda_2, \lambda_3) = \inf_{\omega \in R^n} (\max_{U \in \mathcal{D}(\mathcal{Q}_1)} E[-U\omega^T Y_1]) + \nu_1 \max_{V_1 \in \mathcal{D}(\mathcal{Q}_2)} E[-V_1\omega^T Y_2]) + \nu_2 \max_{V_2 \in \mathcal{D}(\mathcal{Q}_3)} E[-V_2\omega^T x])$$

$$+\lambda_1(E[\omega^T Y_1]) + \lambda_2(E[\omega^T Y_2]) + \lambda_3(1^T\omega)) - r_1\nu_1 - r_2\nu_2 - \lambda_1\pi_1\lambda_2\pi_2 - \lambda_3$$

Let $f(\omega, U, V_1, V_2) := E[-U\omega^T Y_1] + \nu_1\ E[-V_1\omega^T Y_2] + \nu_2\ E[-V_2\omega^T x] + \lambda_1 E[\omega^T X] + \lambda_1 E[\omega^T Y_1] + \lambda_2 E[\omega^T Y_2] + \lambda_3 1^T\omega$ for each $\omega \in R^n$, $U \in \mathcal{D}(Q_1), V_1 \in \mathcal{D}(Q_2), V_3 \in \mathcal{D}(Q_3)$ Note that $\omega \to f(\omega, U, V_1, V_2)$ is convex(affine) and continuous, $(U, V_1, V_2) \to f(\omega, U, V_1, V_2)$ is concave (affine) and $\sigma(L^q, L^p, L^s)$ -continuous (continuous), and $\mathcal{D}(Q_1) \times \mathcal{D}(Q_2) \times \mathcal{D}(Q_3)$ is $\sigma(L^q, L^p, L^s)$-compact. Hence, From the classical min-max theorem see [13] ensures that

$$d(\nu_1, \nu_2, \lambda_1, \lambda_2, \lambda_3) = \sup_{(U,V_1,V_2)\in\mathcal{D}(Q_1)\times\mathcal{D}(Q_2)\times\mathcal{D}(Q_3)} \inf_{\omega\in R^n} (E[-U\omega^T Y_1] + \nu_1\ E[-V_1\omega^T Y_2] + \nu_2\ E[-V_2\omega^T x]$$

$$+\lambda_1 E[\omega^T Y_1] + \lambda_2 E[\omega^T Y_2] + \lambda_3\ 1^T\omega) - r_1\nu_1 - r_2\nu_2 - \lambda_1\pi_1\lambda_2\pi_2 - \lambda_3$$

Clearly, for every $(U, V_1, V_2) \in \mathcal{D}(Q_1) \times \mathcal{D}(Q_2) \times \mathcal{D}(Q_3)$

$$\inf_{\omega} \in R^n(E[-UY_2] + \nu_1\ E[-V_1 Y_2] + \nu_2\ E[-V_1 x] + \lambda_1 E[Y_1] + +\lambda_2 E[Y_2] + \lambda_3 1)^T\omega$$

$$= \begin{cases} 0, & \text{if } E[-UY_1] + \nu_1\ E[-V_1 Y_2] + \nu_2 E[-V_2 x] + \lambda_1 E[Y_1] + \lambda_2 E[Y_2] + \lambda_3\ 1 = 0 \\ -\infty, & \text{else} \end{cases}$$

It follows that

$$d(\nu_1, \nu_2, \lambda_1, \lambda_2, \lambda_3)$$

$$= \begin{cases} -r_1\nu_1 - r_2\nu_2 - \lambda_1\pi_1 - \lambda_2\pi_2 - \lambda_3, & \text{if } \exists (U, V_1, V_2) \in Q_1) \times \mathcal{D}(Q_2 \times \mathcal{D}(Q_3 : \\ & E[-UY_1] + \nu_1\ E[-V_1 Y_2] + \nu_2\ E[-V_2 x] + \lambda_1 E[Y_1] + \lambda_2 E[Y_2] + \lambda_3\ 1 = 0 \\ -\infty, & \text{else} \end{cases}$$

So the Lagrange dual problem (4.8) for cooperative investment problem takes the more explicit form as follows

$$maximize\ -r_1\nu_1 - r_2\nu_2 - \lambda_1\pi_1 - \lambda_2\pi_2 - \lambda_3 \tag{4.9}$$

$$subject\ to\ E[-UY_1] + \nu_1\ E[-V_1 Y_2] + \nu_2 E[V_2 x] + \lambda_1 E[Y_1] + \lambda_2 E[Y_2] + \lambda_3\ 1 = 0$$

$$U \in \mathcal{D}(Q_1), V_1 \in \mathcal{D}(Q_2), V_3 \in \mathcal{D}(Q_3), \nu_{1,2} \in R, \lambda_{1,2,3} \in R$$

Now, we make some changes in variables to avoid the multiplication of variables ν_1, V_1, ν_2, V_2, as follows; if $M_1 \in cone(\mathcal{D}(Q_2))$, then their exist $\nu_1 \geq 0$ and $V_1 \in \mathcal{D}(Q_2)$ such that $M_1 = \nu_1 V_1$: we simply take $\nu_1 = E[M_1]$ and $V_1 = \frac{M_1}{\nu_1}$ if $\nu_1 > 0$ and arbitrary $V_1 \in \mathcal{D}(Q_2)$ if $\nu_1 = 0$. Conversely, if $\nu_1 \geq 0$ and $V_1 \in \mathcal{D}(Q_2)$, then $M_1 = \nu_1 V_1 \in cone(\mathcal{D}(Q_2))$. Similarly, variables if $M_2 \in cone(\mathcal{D}(Q_3))$, then their exist $\nu_2 \geq 0$ and $V_2 \in \mathcal{D}(Q_2)$ such that $M_2 = \nu_2 V_2$: we simply take $\nu_2 = E[M_2]$ and $V_2 = \frac{M_2}{\nu_2}$ if $\nu_2 > 0$ and arbitrary $V_2 \in \mathcal{D}(Q_3)$ if $\nu_2 = 0$. Conversely, if $\nu_2 \geq 0$ and $V_2 \in \mathcal{D}(Q_3)$, then $M_2 = \nu_2 V_2 \in cone(\mathcal{D}(Q_3))$ let These observations allow us to reformulate a dual problem (4.9) as (4.8). Note that both problems have p as their optimal value.

Let $(U^*, M_1^*, M_2^*, \lambda_1^*, \lambda_2^*, \lambda_3^*) \in L^q \times L^p \times L^s \times R \times R \times R$ be an optimal solution for (4.2), see [5], corollary 4.8, there is strong duality with corresponding Lagrange dual problem that relaxes the equality constraint, that is, we have

$$p = \inf_{\omega\in R^n} \sup_{U\in\mathcal{D}(Q_1),M_1\in cone(\mathcal{D}(Q_2)),M_2\in cone(\mathcal{D}(Q_3)),\lambda_{1,2,3}\in R} (-r_1E[M_1] - r_2E[M_2]$$

$$-\lambda_1\pi_1 - -\lambda_2\pi_2 - \lambda_3 - \omega^T(E[UY_1] + E[M_1Y_2] + E[M_2x] - \lambda_1E[Y_1] - \lambda_2E[Y_2] - \lambda_3 1))$$

$$p = \inf_{\omega\in R^n} \sup_{U\in\mathcal{D}(Q_1),M_1\in cone(\mathcal{D}(Q_2)),M_1\in cone(\mathcal{D}(Q_3)),\lambda_{1,2,3}\in R} (-r_1E[M_1] - r_2E[M_2]$$

$$lk - \lambda_1\pi_1 - \lambda_2\pi_2 - \lambda_3 + E[-U\omega^TY_1] + E[-M_1\omega^TY_2] + E[-M_2\omega^Tx] + \lambda_1E[\omega^TY_1] + \lambda_2E[\omega^TY_2] + \lambda_3\omega^T1)$$

also from [5], corollary 4.8, ensures that there exist an optimal Lagrange multiplier $\omega^* \in R^n$. By the first-order condition with respect to $U = U^*$, thus we have that

$$0 \in -(\omega^*)^TY_1 - \mathcal{N}_{\mathcal{D}(Q_1)}(U^*)$$

this means

$$E[-U^*(\omega^*)^TY_1] \geq E[-U'(\omega^*)^TY_1]$$

for every $U' \in \mathcal{D}(Q_1)$, that is

$$\rho_1((\omega^*)^TY_1) = E[-U^*(\omega^*)^TY_1]$$

We conclude that $U^* \in \psi(\omega^*)$, where $\psi(\omega^*)$ defines as $\psi_j(\omega^*) := argmax_{V\in\mathcal{D}(Q)_j}E[-VY_1^T\omega]$, see [3], Lemma 3.4),

$$E[-U^*x] \in \partial g_1(\omega^*) \tag{4.10}$$

In the same way, the first order condition concerning $M_i = M_i^*, for\ i = 1, 1$ yields

$$E[-M_1^*((\omega^*)^TY_2 + r_1)] \geq E[-M_1'((\omega^*)^TY_2 + r_1)]$$

and

$$E[-M_2^*((\omega^*)^Tx + r_2)] \geq E[-M_2'((\omega^*)^Tx + r_2)]$$

for every $M_1' \in cone(\mathcal{D}_2)$ and $M_2' \in cone(\mathcal{D}_3)$, that is

$$E[-M_1^*((\omega^*)^TY_2 + r_1)] = \max_{M_1'\in cone(\mathcal{D}(Q)_2)} E[-M_1'((\omega^*)^TY_2 + r_1)] \tag{4.11}$$

and

$$E[-M_2^*((\omega^*)^Tx + r_2)] = \max_{M_2'\in cone(\mathcal{D}(Q)_3)} E[-M_2'((\omega^*)^Tx + r_2)] \tag{4.12}$$

Since $cone(\mathcal{D}_2)$ is a cone, the quantity $\sup_{M_1'\in cone(\mathcal{D}(Q)_2)} E[-M_1'((\omega^*)^TY_2 + r_1)]$ can either take the value 0 or $+\infty$, and $cone(\mathcal{D}_3)$ is a cone, the quantity $\sup_{M_2'\in cone(\mathcal{D}(Q)_3)} E[-M_2'((\omega^*)^Tx + r_2)]$ can either take the value 0 or $+\infty$, Since $E[-M_1'((\omega^*)^TY_2 + r_1)]$ and $E[-M_2'((\omega^*)^Tx + r_2)]$ are a finite number, both sides of (5.10) must equal to zero, thus we obtain

$$0 = \max_{M_1'\in cone(\mathcal{D}(Q)_2)} E[-M_1'((\omega^*)^TY_2 + r_1)] = (\sup_{\lambda_1'\geq 0} \lambda_1')(\max_{V_1'\in\mathcal{D}(Q_2)} E[-V_1'((\omega^*)^TY_2 + r_1)]) \tag{4.13}$$

$$= +\infty.\rho_2((\omega^*)^TY_2 + r_1) = +\infty(\rho_2(\omega^*)^TY_2) - r_1)$$

as well

$$0 = \max_{M_2' \in cone(\mathcal{D}(Q)_3)} E[-M_2'((\omega^*)^T x + r_2)] = (\sup_{\lambda_2' \geq 0} \lambda_2')(\max_{V_2' \in \mathcal{D}(Q_3)} E[-V_2'((\omega^*)^T x + r_2)]) \tag{4.14}$$

$$= +\infty.\rho_3((\omega^*)^T x + r_2) = +\infty(\rho_3(\omega^*)^T x) - r_2)$$

Moreover , we have from optimality $\rho_2(\omega^*)^T Y_2) = r_1$ and $\rho_3(\omega^*)^T x) = r_2$

Let $\nu_i^* = E[M_i^*],\ for\ i = 1,2$, and suppose first that $\nu_i^* > 0,\ for\ i = 1,2$ and let $V_1^* := \frac{M_1^*}{\nu_1^*} \in \mathcal{D}(Q_2)$ and

$V_2^* := \frac{M_1^*}{\nu_1^*} \in \mathcal{D}(Q_3)$ Then,

$$E[-M_1^*((\omega^*)^T Y_2 + r_1)] = \nu_1^* E[(\omega^*)^T Y_2 + r_1] = 0$$

$$E[-M_2^*((\omega^*)^T x + r_2)] = \nu_2^* E[(\omega^*)^T x + r_2] = 0$$

so that , $E[-V_1^*(\omega^*)^T Y_2] = r_1$, $E[-V_2^*(\omega^*)^T x] = r_2$ Hence

$$E[-V_1^*(\omega^*)^T Y_2] = r = \rho_2(\omega^*)^T Y_2) = \max_{V_1' \in \mathcal{D}(Q_2)} E[-V_1'(\omega^*)^T Y_2]$$

that is $V_1^* \in \psi_2(\omega^*)$,

$$E[-V_2^*(\omega^*)^T x] = r = \rho_2(\omega^*)^T x) = \max_{V_2' \in \mathcal{D}(Q_3)} E[-V_2'(\omega^*)^T x]$$

that is $V_2^* \in \psi_3(\omega^*)$, Actually

$$E[-V_1^* Y_2] \in \partial g_2(\omega^*)$$

$$E[-V_2^* x] \in \partial g_3(\omega^*)$$

Furthermore, suppose that $\nu_i^* = 0,\ for\ i = 1,2$ that is $M_i^* = 0\ for\ i = 1,2\ p-$ almost sure. Let us pack some $V_2^* \in \psi_2(\omega^*)$,$V_3^* \in \psi_3(\omega^*)$, arbitrarily. Since $\psi_2(\omega^*) \neq \phi$, $\psi_3(\omega^*) \neq \phi$, because ρ_2, ρ_3 are assumed to be continuous from below. Thus, in both cases we may write $M_i^* = \nu_i^* V_i^*,\ for\ i = 1,2$ and we can write

$$E[-M_1^* Y_2] = \nu_1^* E[-V_1^* Y_2] \in \nu_1^* \partial g_2(\omega^*) \tag{4.15}$$

and

$$E[-M_2^* x] = \nu_2^* E[-V_2^* x] \in \nu_2^* \partial g_3(\omega^*) \tag{4.16}$$

Now, from feasibility of$(U^*, M_1^*, M_2^*, \lambda_1^*, \lambda_2^*, \lambda_3^*)$ for dual problem (4.2), we have

$$\begin{aligned} & E[-U^* x] + E[-M_1^* Y_2] + E[-M_3^* x] + \lambda_1^* \pi_1 + \lambda_2^* \pi_2 + \lambda_3^* 1 \\ & = E[-U^* x] + \nu_1^* E[-V_1^* Y_2] + \nu_1^* E[-V_2^* x] + \lambda_1^* \pi_1 + \lambda_1^* \pi_1 + \lambda_3^* 1 \\ & = 0 \end{aligned} \tag{4.17}$$

Consequently, from (4.10),(4.15), (4.16) and (4.17) we obtain

$$0 \in \partial g_1(\omega^*) + \nu_1^* \partial g_2(\omega^*) + \nu_2^* \partial g_3(\omega^*) + \lambda_1^* \pi_1 + \lambda_2^* \pi_2 + \lambda_3^* 1$$

Finally, According to first order condition with respect to $\lambda_{1,2,3} = \lambda^*_{1,2,3}$, respectively. Also. we got

$$1^T \omega^* = 1$$

where $\omega^* \in \mathcal{W}$.

We conclude that ω^* is the optimal solution for problem (3.2)or (3.4) so from the condition of Karush-Kuhn-

Tucker condition for the problem and from [14] at $\omega = \omega^*$ is the same optimal solution for problems (3.1) and (3.3), respectively. Moreover, once we have the optimal ω^* we get $X = \omega^* x$ we can find the division Y_1 and Y_2 where $X = Y_1 + Y_2$. see [8].

4.2. **Examples and numerical results.** We choose the risk measure as a coherent risk measure. In the following examples, we show how to write dual representations for each risk preference for each agent (investor). In the first example, two investors will choose risk measure as negative expected value second example; second example investors choose average-value at-risk and in the third example one of the investors choose negative risk and others will choose average-value-at-risk.

Example 1 : see(Two-CVAR)

Let $p = 1$ and take $\rho(Y_1) = E[-Y_1]$ for every $Y_1 \in L^1$, it is easy to check that ρ satisfies properties for coherent risk measure above. while the dual representation for each investor (agents) risk preferences, we simply have $\mathcal{Q}_1 = \{\mathcal{P}\}$ so that $\mathcal{D}_1(\mathcal{Q}_1) = \{1\} \subset L^\infty$. While second investors will be of the form $\rho(Y_2) = E[-Y_2]$ for every $Y_2 \in L^2$, It is easy to check that ρ satisfies properties for coherent risk measure above. While the dual representation (2.1) for each investor (agents), we simply have $\mathcal{Q}_2 = \{\mathcal{P}\}$ so that $\mathcal{D}_2(\mathcal{Q}_2) = \{1\} \subset L^\infty$

Example 2 :

(A verage-at-risk) Let $\phi \in (0, 1)$ be a probability level. The average value-at-risk at Level ϕ for first investor

$Y_1 \in L^1$ is defined as

$$AV@R_\phi(Y_1) := \frac{1}{\phi}\int_0^\phi V@R_u(Y_1)du$$

It is well-known that $AV@R_\phi$ is a law-invariant coherent risk measure on L^1. In the dual representation in (3.1), we may take $\mathcal{Q}_1 = \{Q \in \mathcal{M}_1(\mathcal{P}) | \mathcal{P}\{\frac{Q}{\mathcal{P}} \leq \frac{1}{\phi}\} = 1\}$so that

$$\mathcal{D}_1(\mathcal{Q}_1) = \{V \in L^\infty | \mathcal{P}\{0 \leq V \leq \frac{1}{\phi}\} = 1\}$$

. While for second investor will be in the same form

$$AV@R_\phi(Y_2) := \frac{1}{\phi}\int_0^\phi V@R_u(Y_2)du$$

It is well-known that $AV@R_\phi$ is a law-invariant coherent risk measure on L^1. In the dual representation in (3.1), we may take $\mathcal{Q}_1 = \{Q \in \mathcal{M}_1(\mathcal{P}) | \mathcal{P}\{\frac{Q}{\mathcal{P}} \leq \frac{1}{\phi}\} = 1\}$so that

$$\mathcal{D}_2(\mathcal{Q}_2) = \{V \in L^\infty | \mathcal{P}\{0 \leq V \leq \frac{1}{\phi}\} = 1\}.$$

Example 3 :

The first investor choose negative Expected value and the second investor will choose Average value-at-risk. Let $p = 1$ and take $\rho(Y_1) = E[-Y_1]$ for every $Y_1 \in L^1$, it is easy to check that ρ satisfies properties for coherent risk measure above. While the dual representation (3.1) for each investor (agents), we simply have $\mathcal{Q}_\infty = \{\mathcal{P}\}$ so that $\mathcal{D}_1(\mathcal{Q}_1) = \{1\} \subset L^\infty$. The measure for the second investor will be as follows:

$$AV@R_\phi(Y_2) := \frac{1}{\phi} \int_0^\phi V@R_u(Y_2) du$$

It is well-known that $AV@R_\phi$ is a law-invariant coherent risk measure on L^1. In the dual representation in (3.1), we may take $\mathcal{Q}_2 = \{Q \in \mathcal{M}_1(\mathcal{P}) | \mathcal{P}\{\frac{Q}{\mathcal{P}} \leq \frac{1}{\phi}\} = 1\}$so that

$$\mathcal{D}_2(\mathcal{Q}_2) = \{V \in L^\infty | \mathcal{P}\{0 \leq V \leq \frac{1}{\phi}\} = 1\}.$$

For more details see [3].

Numerical Experiment For the financial market model, Let us assume that one risk-free asset and n risky asset. Also, the initial endowment of agent $i \in I = \{1, 2, ..., m\}$ is given by a portfolio $\bar{\omega} \in R^{n+1}$ so that the discount payoff at time $t = 1$ is

$$Y_i = \frac{\bar{\omega}.\bar{S}}{1+r},\ i \in I = \{1, 2, ..., m\}$$

, the market portfolio is given by $X = \frac{\bar{\omega}.\bar{S}}{1+r}$, with $\bar{\omega} := \sum_{i \in I} \bar{\omega}_i = (\omega^0, \omega)$, and $\bar{S} = (S^0, S)$ is asset Price. Hence, in our problem for cooperative investment. Just we need to replace each Y_i and X in the equilibrium allocation for each investor $Y_i^* = I^+(c_i \varphi^*)$ as follows:

Algorithm in real Market :

Step1: finding derivative of $\mathcal{U}$ in (3.7) for each investor (agent $i = 1, 2$) in terms of y_i, respectively.

Step2: by solving cooperative investment (3.3) we get the value of ω.

Step3: the value of the derivative in step 1 and the value of ω in step 3 plug them in the system (3.9) to get equilibrium price P.

Step4: plug the value of equilibrium price P in equilibrium allocation Y_i^* where we can find it as the positive inverse of derivative of $\mathcal{U}_i$ for each investor(agent $i = 1, 2$ at equilibrium price that we find it by solving the feasible problem (3.9). Note that, solving the problem (3.3) in CVX-MATLAB we write inv-pose for derivative of $\mathcal{U}_i$ to write Y_i in the program.

Real Experiment solving Individual Investment (IV) (3.1) and cooperative investment (CI) (3.3) with one risk-free =0.01 and 3 risky assets (APA, BA, BK) weekly historical data downloading from Yahoo finance S&P 500 (January 2022 to May 2022) where $r_1 = 0.0025, r_2 = 0.001$ and $\pi_1 = 0.025, \pi_2 = 0.05$. We got the result as follows: note that we wrote the coherent risk measure as a negative risk which is expected shortfall at 100

Risk measure	optimal value for CI	optimal value for IV	(CI-IV) $\times$ 100
$\rho_1(y)$	+0.0014842	+0.0445431	-0.043 %
$\rho_2(y)$	+0.00110	+0.0253	-0.024 %

we can changing the value or r_1 and fixed the value of $r_2, \pi_1, \pi 2$ in order to get the whole efficient frontier. then solve the feasibility problem (3.9) to get the equilibrium price and then equilibrium allocation as follows. $y_1^* = 0.0014822, Y_2^* = 0.001002$ which is still better than the optimal value for individual investment as shown in [3].

5. Conclusion

In this paper, we reformulate cooperative investment risk by writing a dual representation for each risk preference (coherent risk measure) for each agent (investor) and first, finding an analytic solution for the problem for both cases individual and cooperative investment problems which represented in theorems 1 and 2. Second, numerical experiments support our result by getting better investment in the case of cooperative investment. Hence, we conclude that the cooperative investment still has better results since sharing creates instruments that on the one hand, satisfy individual risk preferences but, on the other hand, are not replicable in an incomplete market, so each agent is strictly better in participating in cooperative investment than investing alone.

This research can be extended in at least two directions. First, solving cooperative investment with inflation effect in case of initial endowment exist and without. Second, a case study of applying cooperative investment in Saudi Arabia's Financial market.

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