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Applications of Two-Sided Bases on Ordered Ternary Semigroups

Jatuporn Sanborisoot[1,*], Wichayaporn Jantanan[2], Ronnason Chinram[3]

[1] *Algebra and Applications Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Mahasarakham 44150, Thailand*

[2] *Department of Mathematics, Faculty of Science, Buriram Rajabhat University, Buriram 31000, Thailand*

[3] *Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand*

[*] *Corresponding author:* jatuporn.san@msu.ac.th

Abstract. Ideal theory is an important area of research in ring theory and semigroup theory. Two-sided bases of ideals of semigroups are important in the study of ideal theory in semigroups. In this paper, we apply the notion of two-sided bases of ideals of semigroups to study the notion of two-sided bases of two-sided ideals of ordered ternary semigroups. The properties and examples of two-sided bases of an ordered ternary semigroup are also investigated and presented. Moreover, we also give the results for an ordered ternary semigroup containing two-sided bases.

1. Introduction

The theory of ternary algebraic systems was introduced by Lehmer [12], who investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroups was considered by Banach, who showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup [14]. In 2009, Iampan [6] investigated some properties of ordered ternary semigroups which are a generalization of ternary semigroups. Later, many authors have studied and discussed the structures of ordered ternary semigroups by several authors, for example [2,3,7,10,13,15].

On the other hand, Fabrici [5] introduced the concept of two-sided bases of a semigroup, which is based two-sided ideals generated by a non-empty subset of a semigroup. Later, Changphas and Sammaprab [1] extended the results obtained by Fabrici in [5] to ordered semigroups. In 2015, Thongkam and Changphas [16] introduced two-sided bases of a ternary semigroup and studied the structure of a ternary semigroup containing two-sided bases. Moreover, the concept of two-sided

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bases has been studied in other algebraic structures by several authors, for example, [8,9,11]. In this paper, the notions of two-sided bases of an ordered ternary semigroup are introduced and the properties and examples of two-sided bases of an ordered ternary semigroup are also presented. The structure of an ordered ternary semigroup containing two-sided bases will be studied. It is observed that the obtained results generalize the results of Thongkam and Changphas in [16].

2. Preliminaries

Before proving the main results, we need the following definitions.

Definition 2.1. [12] A *ternary semigroup* is an algebraic structure $(T, [\,])$ such that T is a non-empty set and $[\,] : T \times T \times T \to T$ is a ternary operation written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, satisfying the following conditions:

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in T$.

It is observed that every semigroup can be reduced to a ternary semigroup but a ternary semigroup need not necessarily reduce to a semigroup. We can illustrate it by the following example.

Example 2.1. [4] Let $T = \{-i, 0, i\}$. Then $(T, [\,])$ is a ternary semigroup under the usual multiplication of complex numbers, but $(T, \cdot)$ is not a semigroup under the usual multiplication of complex numbers because $(-i) \cdot i = 1 \notin T$.

Definition 2.2. [6] An *ordered ternary semigroup* is an algebraic structure $(T, [\,], \leq)$ such that $(T, [\,])$ is a ternary semigroup and $(T, \leq)$ is a partially ordered set satisfying the following conditions:

$$x \leq y \Rightarrow [xx_1x_2] \leq [yx_1x_2], [x_1xx_2] \leq [x_1yx_2] \text{ and } [x_1x_2x] \leq [x_1x_2y]$$

for all $x, x_1, x_2, y \in T$.

Throughout the paper T will denote an ordered ternary semigroup unless otherwise specified. For any non-empty subsets A, B and C of an ordered ternary semigroup T, we denote

$$[ABC] := \{[abc] \mid a \in A, b \in B, c \in C\}.$$

If $A = \{a\}$, then we write $[aBC]$ instead of $[\{a\}BC]$ and similarly if $B = \{b\}$ and $C = \{c\}$, we write $[AbC]$ and $[ABc]$, respectively. For the sake of simplicity, we write abc instead of $[abc]$ and ABC instead of $[ABC]$. For $A \subseteq T$, denote

$$(A] := \{t \in T \mid t \leq a \text{ for some } a \in A\}.$$

If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For any non-empty subsets A, B and C of an ordered ternary semigroup T, we have

(1) $A \subseteq (A]$ and $((A]] = (A]$;

(2) $A \subseteq B$ implies $(A] \subseteq (B]$;
(3) $(A](B](C] \subseteq (ABC]$;
(4) $(A] \cup (B] = (A \cup B]$.

Definition 2.3. [7] A non-empty subset A of an ordered ternary semigroup T is called a *ternary subsemigroup* of T if $AAA \subseteq A$.

Definition 2.4. [15] A non-empty subset A of an ordered ternary semigroup T is called a *left (respectively, right, lateral) ideal* of T if

(1) $TTA \subseteq A$ (respectively, $ATT \subseteq A, TAT \subseteq A$) ;
(2) for $a \in A$ and $b \in T$, $b \leq a$ implies $b \in A$.

A non-empty subset A of T is called a *two-sided ideal* of T if A is both a left ideal and a right ideal of T. If A is a left, a right and a lateral ideal of T, then it is called an *ideal* of T.

A two-sided ideal A of T is said to be *proper* if $A \neq T$. A proper two-sided ideal A of T is said to be *maximal* if there is no a proper two-sided ideal B of T such that $A \subset B$. The symbol $\subset$ stands for proper inclusion for sets. The union of two-sided ideals of T is a two-sided ideal of T. The intersection of two-sided ideals of T is a two-sided ideal of T, if it is a non-empty.

Let T be an ordered ternary semigroup and A be a non-empty subset of T. In [7, 15], the intersection of all two-sided ideals of T containing A, denoted by $(A)_t$ is the smallest two-sided ideal of T containing A, that is

$$(A)_t = (A \cup TTA \cup ATT \cup TTATT].$$

For $a \in T$, we write $(a)_t$ simply for $(\{a\})_t$, and called the principal two-sided ideal of T generated by $a \in T$. Further we have,

$$(a)_t = (a \cup TTa \cup aTT \cup TTaTT].$$

We introduce the quasi-ordering on an ordered ternary semigroup T by for any $a, b \in T$,

$$a \leq_t b \Leftrightarrow (a)_t \subseteq (b)_t.$$

We write $a <_t b$ (i.e. $(a)_t \subset (b)_t$) if $a \leq_t b$ but $a \neq b$.

3. Main Results

In this section, the algebraic structure of an ordered ternary semigroup containing two-sided bases will be presented. We first give the definition of a two-sided base of an ordered ternary semigroup and cite some examples with the following definitions.

Definition 3.1. A non-empty subset A of an ordered ternary semigroup T is called a *two-sided base* of T if it satisfies the following conditions:

(1) $T = (A \cup TTA \cup ATT \cup TTATT]$, i.e., $T = (A)_t$;
(2) there exists no a proper subset B of A such that $T = (B)_t$.

Example 3.1. Let $T = \{a, b, c, d, e\}$. Define the ternary operation on T by $xyz = x * (y * z)$ for all $x, y, z \in T$, where the binary operation $*$ is defined by

$*$	a	b	c	d	e
a	a	e	e	a	e
b	d	b	b	d	b
c	d	b	b	d	b
d	d	b	b	d	b
e	a	e	e	a	e

and the partial order defined by

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (e,e), (b,a), (b,d), (b,e), (c,a), (c,d), (c,e), (d,a), (e,a)\}.$$

Then T is an ordered ternary semigroup [13]. We have that the singleton sets consisting of an element of T are two-sided bases of T.

Example 3.2. Let $T = \{a, b, c, d, e, f\}$. Define the ternary operation on T by $xyz = (x * y) * z$ for all $x, y, z \in T$ where the binary operation $*$ is defined by

$*$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	b	b	b
c	a	b	c	d	b	b
d	a	b	b	b	c	d
e	a	b	e	f	b	b
f	a	b	b	b	e	f

and the partial order defined by

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (a,b)\}.$$

Then T is an ordered ternary semigroup and $\{c, d\}, \{c, e\}, \{c, f\}, \{d, e\}, \{d, f\}, \{e, f\}$ are two-sided bases of T.

We now prove some elementary results.

Lemma 3.1. *Let T be an ordered ternary semigroup. For any $a, b \in T$, if $a \leq b$, then $a \leq_t b$.*

Proof. Let $a, b \in T$ such that $a \leq b$. We will show that $a \leq_t b$, it suffices to show that $(a)_t \subseteq (b)_t$. Suppose that $x \in (a)_t = (a \cup TTa \cup aTT \cup TTaTT]$. Then $x \leq a$ or $x \in (TTa]$ or $x \in (aTT]$ or $x \in (TTaTT]$. We consider four cases:

Case 1: $x \leq a$. Then $x \leq a \leq b$ for some $b \in (b)_t$. Thus, $x \in ((b)_t] = (b)_t$.

Case 2: $x \in (TTa]$. Then $x \leq y$ for some $y \in TTa$. Since $y \in TTa$, we set $y = sta$ for some $s, t \in T$. Since $a \leq b$, we have $sta \leq stb$. Thus, $x \leq y = sta \leq stb$ and so $x \leq stb$ for some $stb \in TTb \subseteq (b)_t$. Hence, $x \in (b)_t$.

Case 3: $x \in (aTT]$. Then $x \leq z$ for some $z \in aTT$. Since $z \in aTT$, we set $z = as_1t_1$ for some $s_1, t_1 \in T$. Since $a \leq b$, we have $as_1t_1 \leq bs_1t_1$. Thus, $x \leq z = as_1t_1 \leq bs_1t_1$ and so $x \leq bs_1t_1$ for some $bs_1t_1 \in bTT \subseteq (b)_t$. Hence, $x \in (b)_t$.

Case 4: $x \in (TTaTT]$. Then $x \leq w$ for some $w \in TTaTT$. Since $w \in TTaTT$, we set $w = s_2s_3at_2t_3$ for some $s_2, s_3, t_2, t_3 \in T$. Since $a \leq b$, we have $s_2s_3a \leq s_2s_3b$. Since $s_2s_3a \leq s_2s_3b$, then $s_2s_3at_2t_3 \leq s_2s_3bt_2t_3$. Thus, $x \leq w = s_2s_3at_2t_3 \leq s_2s_3bt_2t_3$ and so $x \leq s_2s_3bt_2t_3$ for some $s_2s_3bt_2t_3 \in TTbTT \subseteq (b)_t$. Hence, $x \in (b)_t$.

From all four cases, we conclude that $(a)_t \subseteq (b)_t$, i.e., $a \leq_t b$. □

Lemma 3.2. *Let A be a two-sided base of an ordered ternary semigroup T. For any a, b in A, if $a \in (TTb \cup bTT \cup TTbTT]$, then $a = b$.*

Proof. Let $a, b \in A$ such that $a \in (TTb \cup bTT \cup TTbTT]$ and $a \neq b$. Setting $B = A \setminus \{a\}$. Then $b \in B$. We will show that $(A)_t \subseteq (B)_t$, it suffices to show that $A \subseteq (B)_t$. Let $x \in A$. We have $x \neq a$ or $x = a$. If $x \neq a$, then $x \in B \subseteq (B)_t$. If $x = a$, then

$$x = a \in (TTb \cup bTT \cup TTbTT] \subseteq (b)_t \subseteq (B)_t.$$

So, we have $A \subseteq (B)_t$. It follows that $(A)_t \subseteq (B)_t$. Thus, $T = (A)_t \subseteq (B)_t \subseteq T$, and so $T = (B)_t$. This is a contradiction. Hence, $a = b$. □

The following theorem characterizes when a non-empty subset of an ordered ternary semigroup is a two-sided base.

Theorem 3.1. *A non-empty subset A of an ordered ternary semigroup T is a two-sided base of T if and only if A satisfies the following conditions:*

(1) *for any x in T there exists a in A such that $x \leq_t a$;*
(2) *for any a, b in A, if $a \neq b$, then neither $a \leq_t b$ nor $b \leq_t a$.*

Proof. Assume that A is a two-sided base of T. Then $T = (A)_t$. Let $x \in T$. Thus, $x \in (A)_t$. Then there exists $a \in A$ such that $x \in (a)_t$. It follows that $(x)_t \subseteq (a)_t$. Hence, $x \leq_t a$ where $a \in A$. This shows that (1) holds. Next, let $a, b \in A$ such that $a \neq b$. Suppose that $a \leq_t b$. Then $(a)_t \subseteq (b)_t$. We set $B = A \setminus \{a\}$. Then $b \in B$. Let $x \in T$, by (1), there exists $c \in A$ such that $x \leq_t c$ i.e., $(x)_t \subseteq (c)_t$. Since $c \in A$, we have $c \neq a$ or $c = a$. If $c \neq a$, then $c \in B$. Thus, $x \in (x)_t \subseteq (c)_t \subseteq (B)_t$. Hence, $T \subseteq (B)_t$ and so $T = (B)_t$. This is a contradiction. If $c = a$, then

$$x \in (x)_t \subseteq (c)_t = (a)_t \subseteq (b)_t \subseteq (B)_t$$

and so $x \in (B)_t$. Thus, $T \subseteq (B)_t$. Hence, $T = (B)_t$. This is a contradiction. The case $b \leq_t a$ is proved similarly. Thus, (2) follows.

Conversely, assume that the conditions (1) and (2) hold. To show that A is a two-sided base of T, we let $x \in T$, by (1), there exists $a \in A$ such that $x \leq_t a$ and so $x \in (x)_t \subseteq (a)_t \subseteq (A)_t$. Thus, $T = (A)_t$. Next, suppose that there exists a proper subset B of A such that $T = (B)_t$. We let $a \in A \setminus B$. Then

$a \in A \subseteq T = (B)_t$. By (1), there exists $b \in B \subseteq A$ such that $a \leq_t b$. This contradicts to (2). Therefore, A is a two-sided base of T. □

Theorem 3.2. *Let A be a two-sided base of an ordered ternary semigroup T such that $(a)_t = (b)_t$ for some $a \in A$ and $b \in T$. If $a \neq b$, then T contains at least two two-sided bases.*

Proof. Assume that $a \neq b$. Suppose that $b \in A$. Since $a \in (a)_t = (b)_t$, we have $a \in (b]$ and $a \in (TTb \cup bTT \cup TTbTT]$. If $a \in (b]$, then $a \leq b$. By Lemma 3.1, we obtain $a \leq_t b$. This is a contradiction. If $a \in (TTb \cup bTT \cup TTbTT]$, by Lemma 3.2, we obtain $a = b$. This contradicts the assumption. Thus, $b \in T \setminus A$. We set $B = (A \setminus \{a\}) \cup \{b\}$. Then $B \neq A$. We will show that B is a two-sided base of T. To show that B satisfies (1) in Theorem 3.1, we let $x \in T$. Since A is a two-sided base of T, there exists $c \in A$ such that $x \leq_t c$. If $c \neq a$, then $c \in B$. If $c = a$, we have $(c)_t = (a)_t$. Thus, $(x)_t \subseteq (c)_t = (a)_t = (b)_t$ and so $x \leq_t b$ where $b \in B$. Next, to show that B satisfies (2) in Theorem 3.1. Let $y, z \in B$ such that $y \neq z$. If $y = b$ and $z = b$, then $y = z$. This contradicts to $y \neq z$. If y, z are distinct from b, then $y, z \in A$. Since A is a two-sided base of T, then neither $y \leq_t z$ nor $z \leq_t y$. We consider two cases: $y = b$ or $z = b$. If $y = b$ and $y \neq z$, we have $z \in A$. Suppose that $y \leq_t z$. Then $a \leq_t b = y \leq_t z$ where $a, z \in A$. This is a contradiction. Suppose that $z \leq_t y$. Then $z \leq_t y = b \leq_t a$ where $a, z \in A$. This is a contradiction. Hence neither $y \leq_t z$ nor $z \leq_t y$. The case $z = b$ can be proved in the same manner. Thus, B satisfies (2) in Theorem 3.1. Therefore, B is a two-sided base of T. □

The following corollary is a consequence of Theorem 3.2.

Corollary 3.1. *Let A be a two-sided base of an ordered ternary semigroup T, and let $a \in A$. If $(a)_t = (b)_t$ for some $b \in T$ and $a \neq b$, then b is an element of a two-sided base of T which is different from A.*

Theorem 3.3. *Any two two-sided bases of an ordered ternary semigroup T have the same cardinality.*

Proof. Let A and B be two two-sided bases of an ordered ternary semigroup T. Let $a \in A$. Since B is a two-sided base of T, by Theorem 3.1(1), there exists $b \in B$ such that $a \leq_t b$. Similarly, since A is a two-sided base of T, there exists $a' \in A$ such that $b \leq_t a'$. Thus, $a \leq_t b \leq_t a'$, $a \leq_t a'$. By Theorem 3.1(2), $a = a'$. This implies $(a)_t = (b)_t$. Define a mapping $f : A \to B$ by $f(a) = b$ for all a in A.

To show that f is well-defined. Let $a_1, a_2 \in A$ such that $a_1 = a_2$, $f(a_1) = b_1$, $f(a_2) = b_2$ for some $b_1, b_2 \in B$. Thus, $(a_1)_t = (a_2)_t$, $(a_1)_t = (b_1)_t$, $(a_2)_t = (b_2)_t$ and so $(a_1)_t = (a_2)_t = (b_1)_t = (b_2)_t$. This implies $b_1 = b_2$. Hence, f is well-defined. Next, to show that f is one-to-one. Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$ for some $b \in B$. We have $a_1 \leq_t b$ and $a_2 \leq_t b$. Since A is a two-sided base of T, there exists $a_3 \in A$ such that $b \leq_t a_3$. Thus, $a_1 \leq_t a_3$ and $a_2 \leq_t a_3$ where $a_1, a_2, a_3 \in A$. By Theorem 3.1(2), we obtain $a_1 = a_3 = a_2$. Hence, f is one-to-one. Finally, we will show that f is onto. Let $b \in B$. Since A is a two-sided base of T, there exists $a \in A$ such that $b \leq_t a$. Similarly, since B is a two-sided base of T, we have $a \leq_t b'$ for some $b' \in B$. Thus, $b \leq_t a \leq_t b'$, $b \leq_t b'$. By Theorem 3.1(2), $b = b'$. Hence, $a \leq b' = b$ and $b \leq_t a$. This implies $(a)_t = (b)_t$. Therefore, f is onto. □

In Example 3.1, it is observed that $\{c\}$ is a two-sided base of T, but it is not a ternary subsemigroup of T. This shows that a two-sided base of an ordered ternary semigroup need not be a ternary subsemigroup in general. The following theorem gives necessary and sufficient conditions for a two-sided base of T to be a ternary subsemigroup of T.

Theorem 3.4. *Let A be a two-sided base of an ordered ternary semigroup T. Then A is a ternary subsemigroup of T if and only if $A = \{a\}$ with $aaa = a$.*

Proof. Assume that A is a ternary subsemigroup of T. Let $a, b \in A$. Then $aab \in A$. We set $aab = c$ for some $c \in A$. We have $c = aab \in TTb \subseteq (TTb \cup bTT \cup TTbTT]$. By Lemma 3.2, $c = b$. Similarly, $c = aab \in aTT \subseteq (TTa \cup aTT \cup TTaTT]$. By Lemma 3.2, $c = a$. Thus, $a = c = b$. Therefore, $A = \{a\}$ with $aaa = a$. The converse statement is obvious. □

Theorem 3.5. *Let T be an ordered ternary semigroup and let B^* be the union of all two-sided bases of T. If $T \setminus B^*$ is a non-empty, then it is a two-sided ideal of T.*

Proof. Assume that $T \setminus B^*$ is a non-empty. Let $a \in T \setminus B^*$ and $x, y \in T$. We will show that $xya \in T \setminus B^*$, we assume that $xya \notin T \setminus B^*$. Then $xya \in B^*$. Thus, $xya \in A$ for some a two-sided base A of T. Hence, there is $b \in A$ such that $b = xya$. This implies $b \in (a)_t$ and so $(b)_t \subseteq (a)_t$. That is, $b \leq_t a$. If $(b)_t = (a)_t$, by Corollary 3.1, $a \in B^*$. This is a contradiction. Thus, $(b)_t \neq (a)_t$. Since A is a two-sided base of T, there exists $c \in A$ such that $a \leq_t c$. If $b = c$, then $(a)_t \subseteq (c)_t = (b)_t \subseteq (a)_t$. Thus, $(a)_t = (b)_t$. This is a contradiction. Hence, $b \neq c$. By $b \leq_t a \leq_t c$, where $b \neq c$ and $b, c \in A$. This contradicts condition (2) of Theorem 3.1. Thus, $xya \in T \setminus B^*$. Similarly, we can show that $axy \in T \setminus B^*$. Next, let $x \in T \setminus B^*$ and $y \in T$ such that $y \leq x$. To show that $y \in T \setminus B^*$. Suppose that $y \in B^*$. Then $y \in A$ for some a two-sided base A of T. Since A is a two-sided base of T, there exists $z \in A$ such that $x \leq_t z$. Since $y \leq x$, by Lemma 3.1, $y \leq_t x$. Thus, $y \leq_t x \leq_t z$ where $y, z \in A$. If $y = z$, then $x \leq_t z = y \leq_t x$. By Theorem 3.1(2), $x = y$. This is a contradiction. Thus, $y \neq z$ and $y \leq_t z$. This contradicts condition (2) of Theorem 3.1. Hence, $y \in T \setminus B^*$. Therefore, $T \setminus B^*$ is a two-sided ideal of T. □

Theorem 3.6. *Let B^* be the union of all two-sided bases of an ordered ternary semigroup T such that $\emptyset \neq B^* \subset T$. Let I^* be a proper two-sided ideal of T containing every proper two-sided ideal of T. Then the following statements are equivalent:*

(1) *$T \setminus B^*$ is a maximal two-sided ideal of T ;*
(2) *$B^* \subseteq (a)_t$ for every $a \in B^*$;*
(3) *$T \setminus B^* = I^*$;*
(4) *every two-sided base of T has only one element.*

Proof. (1) $\Leftrightarrow$ (2). Assume that $T \setminus B^*$ is a maximal two-sided ideal of T. Suppose that $B^* \nsubseteq (a)_t$ for some $a \in B^*$. Since $B^* \nsubseteq (a)_t$, then there exists $x \in B^*$ such that $x \notin (a)_t$. Thus, $x \notin T \setminus B^*$. Hence $(T \setminus B^*) \cup (a)_t \neq T$ and so $(T \setminus B^*) \cup (a)_t$ is a proper two-sided ideal of T such that $(T \setminus B^*) \subset (T \setminus B^*) \cup (a)_t$. This contradicts the maximality of $T \setminus B^*$. Therefore, $B^* \subseteq (a)_t$ for every $a \in B^*$.

Conversely, assume that $B^* \subseteq (a)_t$ for every $a \in B^*$. Since $\varnothing \neq B^* \subset T$, we have $T \setminus B^* \neq \varnothing$. By Theorem 3.5, $T \setminus B^*$ is a proper two-sided ideal of T. We will show that $T \setminus B^*$ is a maximal two-sided ideal of T. Let A be a proper two-sided ideal of T such that $T \setminus B^* \subset A \subset T$. Then there exists $x \in A$ such that $x \notin T \setminus B^*$. Thus, $A \cap B^*$ is non-empty. Let $b \in A \cap B^*$. Since $b \in B^*$ and $b \in A$, we have $B^* \subseteq (b)_t$ and $(b)_t \subseteq A$. Thus, $B^* \subseteq A$ and so $T = (T \setminus B^*) \cup B^* \subseteq A \subseteq T$. Hence, $T = A$. This is a contradiction. Therefore, $T \setminus B^*$ is a maximal two-sided ideal of T.

$(3) \Leftrightarrow (4)$. Assume that $T \setminus B^* = I^*$. Then $T \setminus B^*$ is a maximal two-sided ideal of T. Let $a \in B^*$. Using $(1) \Leftrightarrow (2)$, we obtain $B^* \subseteq (a)_t$. Suppose that $T \setminus B^* \nsubseteq (a)_t$ for some $a \in B^*$. Then $(a)_t \neq T$ such that $(a)_t$ is a proper two-sided ideal of T. Thus, $a \in (a)_t \subseteq I^* = T \setminus B^*$. Hence, $a \in T \setminus B^*$. This is a contradiction. Thus, $T \setminus B^* \subseteq (a)_t$. Since $B^* \subseteq (a)_t$ and $T \setminus B^* \subseteq (a)_t$ for all $a \in B^*$, we obtain

$$T = (T \setminus B^*) \cup B^* \subseteq (a)_t \subseteq T.$$

This implies $T = (a)_t$. Hence, for any $a \in B^*$, $\{a\}$ is a two-sided base of T. Next, let A be a two-sided base of T and let $a, b \in A$. Then $A \subseteq B^*$. Thus, $a, b \in B^*$ and so $T = (a)_t$. Since $b \in T = (a)_t = (a \cup TTa \cup aTT \cup TTaTT]$, we have $b \in (a]$ or $b \in (TTa \cup aTT \cup TTaTT]$. If $b \in (a]$, then $b \leq a$, by Lemma 3.1, we obtain $b \leq_t a$. By Theorem 3.1(2), $b = a$. If $b \in (TTa \cup aTT \cup TTaTT]$, by Lemma 3.2, $b = a$. Hence, A has only one element.

Conversely, assume that every two-sided base of T has only one element. We will show that $T \setminus B^* = I^*$. It suffices to show that $A \subseteq T \setminus B^*$ for every a proper two-sided ideal A of T. Suppose that A is a proper two-sided ideal of T such that $A \nsubseteq T \setminus B^*$. Then there exists $a \in A$ such that $a \notin T \setminus B^*$. Since $a \in A$, this implies $(a)_t \subseteq A$. Since $a \notin T \setminus B^*$, then $a \in B^*$, by assumption, $T = (a)_t$. Thus, $T = (a)_t \subseteq A \subset T$. This implies $T = A$. This is a contradiction. Hence, $A \subseteq T \setminus B^*$. Therefore, $T \setminus B^* = I^*$.

$(1) \Leftrightarrow (3)$. Assume that $T \setminus B^*$ is a maximal two-sided ideal of T. We will show that $T \setminus B^* = I^*$. Let A be a two-sided ideal of T such that $A \nsubseteq T \setminus B^*$. Then there exists $a \in A$ such that $a \notin T \setminus B^*$. Since $a \in A$, it follows that $(a)_t \subseteq A$. Since $a \notin T \setminus B^*$, then $a \in B^*$. Using $(1) \Leftrightarrow (2)$, we obtain $B^* \subseteq (a)_t$. Thus, $B^* \subseteq (a)_t \subseteq A$. Let $x \in T$. Then there exists $y \in B^*$ such that $x \leq_t y$. Since $y \in B^*$, it follows that $(y)_t \subseteq B^*$. By $x \in (x)_t \subseteq (y)_t \subseteq B^* \subseteq A$. Hence, $T \subseteq A$ and so $T = A$. Therefore, $T \setminus B^* = I^*$. The converse statement is obvious. □

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