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Geometry of Warped Product CR and Semi-Slant Submanifolds in Quasi-Para-Sasakian Manifolds

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Abstract. In the present paper we study the existence or non-existence of warped product semi-slant submanifolds in quasi-para-Sasakian manifolds and prove that there are no proper warped product semi-slant submanifolds in a quasi-para-Sasakian manifold such that totally geodesic and totally umbilical submanifolds of warped product are proper semi-slant and invariant (or anti-invariant), respectively.

1. **Introduction**

The concept of warped product manifolds was introduced by Bishop and O'Neill for constructing manifolds of non-positive curvature, as one of the most effective generalization of Riemannian product manifold [15]. About two decades ago, Chen extended the work of Bishop and O'Neill and studied the warped product CR-submanifold of Kaehler manifolds [3,4], this study was also extended by many geometers in different settings [2,13,14]. The existence or non-existence of warped product manifolds plays an important role in differential geometry as well as in physics. In [6], Blair introduced the notion of quasi-Sasakian manifolds that unifies Sasakian and cosymplectic manifolds. Tanno [19] also contributed some remarkable results on quasi-Sasakian structure. Recently, quasi-Sasakian structure have been studied in [1, 17, 18]). The geometry of almost paracontact manifold was studied by

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Kaneyuki and Williams in [16] as a natural generalization of natural odd-dimensional analogue to almost para-Hermitian structures. The study of almost paracontact metric manifolds was carried out in one of Zamkovoy's papers [20]. In [21], Olszak studied normal almost contact metric manifolds of dimension 3. In 2009, Welyczko [10] investigated curvature and torsion of Frenet-Legendre curves in 3-dimensional normal almost paracontact metric manifolds. Recently, 3-dimensional normal almost paracontact metric manifolds were studied in [5, 7, 8].

2. **Preliminaries**

Let $\bar{\mathcal{M}}$ be a $(2n+1)$-dimensional almost paracontact manifold with structure tensor $(f, \xi, \upsilon, <,>)$, where f, ξ and υ be a tensor field of type $(1,1)$, a vector field, and a 1-form, respectively on $\bar{\mathcal{M}}$ satisfying

$$f\xi = 0, \quad f^2 = I - \upsilon \otimes \xi, \quad \upsilon \circ f = 0, \tag{2.1}$$

$$\upsilon(\xi) = 1, \quad \upsilon(\mathcal{X}) =< \mathcal{X}, \xi >,$$

$$< f\cdot, f\cdot >= - <,> + \upsilon \otimes \upsilon, \tag{2.2}$$

where I is the identity on the tangent bundle $T\bar{\mathcal{M}}$ of $\bar{\mathcal{M}}$. We say that $\bar{\mathcal{M}}$ is a paracontact metric manifold if there exists a one-form υ such that

$$< \mathcal{X}, f\mathcal{Y} >= d\upsilon(\mathcal{X}, \mathcal{Y}) = \frac{1}{2}(\mathcal{X}\upsilon(\mathcal{Y}) - \mathcal{Y}\upsilon(\mathcal{X}) - \upsilon([\mathcal{X}, \mathcal{Y}]),$$

for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}(\bar{M})$, where $\mathfrak{X}(\bar{M})$ denotes the Lie algebra of vector fields on $\bar{M}$, and

$$< f\mathcal{X}, \mathcal{Y} > + < \mathcal{X}, f\mathcal{Y} >= 0 \tag{2.3}$$

for all vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\bar{\mathcal{M}}$.

Further, an almost paracontact metric manifold is called a quasi-para-Sasakian manifold if

$$(\bar{\nabla}_{\mathcal{X}} f)\mathcal{Y} = \upsilon(\mathcal{Y})\mathcal{F}\mathcal{X} - < \mathcal{F}\mathcal{X}, \mathcal{Y} > \xi, \tag{2.4}$$

and

$$\bar{\nabla}_{\mathcal{X}}\xi = -f\mathcal{F}\mathcal{X}, \quad f\mathcal{F}\mathcal{X} = \mathcal{F}f\mathcal{X}, \quad < \mathcal{F}\mathcal{X}, \mathcal{Y} >= - < \mathcal{X}, \mathcal{F}\mathcal{Y} >, \tag{2.5}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to the metric tensor $<,>$ and $\mathcal{F}$ is a tensor field of type $(1,1)$.

By applying f to (2.5) and using (2.1), we obtain

$$\mathcal{F}\mathcal{X} = \upsilon(\mathcal{F}\mathcal{X})\xi - f(\bar{\nabla}_{\mathcal{X}}\xi). \tag{2.6}$$

Also by replacing $\mathcal{X}$ by ξ in (2.5) it follows that

$$\bar{\nabla}_{\xi}\xi = 0. \tag{2.7}$$

Using (2.4), (2.6) and (2.7) we infer

$$\mathcal{F}\xi = \upsilon(\mathcal{F}\xi)\xi, \tag{2.8}$$

and

$$(\bar{\nabla}_{\xi} f)\mathcal{X} = 0 \tag{2.9}$$

for any $\mathcal{X} \in \Gamma(T\bar{\mathcal{M}})$.

If $\mathcal{M}$ is a contact CR-submanifold of $\bar{\mathcal{M}}$ and the projections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are denoted by $\mathcal{P}$ and $\mathcal{Q}$, respectively; then for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$\mathcal{X} = \mathcal{P}\mathcal{X} + \mathcal{Q}\mathcal{X} + \upsilon(\mathcal{X})\xi. \tag{2.10}$$

Now we put

$$B\lambda + C\lambda = f\lambda, \tag{2.11}$$

where $B\lambda$ and $C\lambda$ are tangential and normal part of $f\lambda$ on $\mathcal{M}$.

Next we define the tensor field of type $(1, 1)$ on $\mathcal{M}$ by

$$f\mathcal{X} = fP\mathcal{X}, \tag{2.12}$$

and the $\Gamma(T\mathcal{M}^{\perp})$-valued 2-form ω by

$$\omega\mathcal{X} = fQ\mathcal{X}. \tag{2.13}$$

Since D is invariant by f, then it follows from (2.11) and (2.12) that B is $\Gamma(D^{\perp})$-valued and t is $\Gamma(D)$-valued, respectively.

By using (2.1), (2.10), (2.12) and (2.13), we obtain

$$\omega\mathcal{X} + t\mathcal{X} = f\mathcal{X}, \tag{2.14}$$

and

$$t^3 + t = 0; C^3 + C = 0. \tag{2.15}$$

Then by (2.15) we conclude that t and C are f-structure in sense of Yano [11] on $T\mathcal{M}$ and $T\mathcal{M}^{\perp}$, respectively.

Now suppose $<, >$ be the induced metric and ξ be tangent to $\mathcal{M}$. Further, we suppose ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle $T\mathcal{M}$ and the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{X}}\mathcal{Y}, \tag{2.16}$$

$$\bar{\nabla}_{\mathcal{X}}\lambda = -\Lambda_{\lambda}\mathcal{X} + \nabla^{\perp}_{\mathcal{X}}\lambda \tag{2.17}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and any vector field λ normal to $\mathcal{M}$, where σ and Λ_{λ} are the second fundamental form and the shape operator for the immersion of $\mathcal{M}$ into $\bar{\mathcal{M}}$. The second fundamental form σ and shape operator Λ_{λ} are related by

$$< \sigma(\mathcal{X}, \mathcal{Y}), \lambda >=< \Lambda_{\lambda}\mathcal{X}, \mathcal{Y} > \tag{2.18}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$.

Furthermore, for any $\mathcal{Z} \in \Gamma(T\bar{\mathcal{M}})$, we put

$$\mathcal{F}\mathcal{Z} = \alpha\mathcal{Z} + \beta\mathcal{Z}, \tag{2.19}$$

where $\alpha\mathcal{Z}$ and $\beta\mathcal{Z}$ are the tangent part and the normal part of $\mathcal{F}\mathcal{Z}$, respectively.
From (2.3) we have

$$< t\mathcal{X}, \mathcal{Y} > + < \mathcal{X}, t\mathcal{Y} >= 0. \tag{2.20}$$

In account of (2.6), (2.11), (2.12) and (2.16) we obtain

$$\alpha\mathcal{X} = \upsilon(\mathcal{X})\upsilon(\mathcal{F}\mathcal{X})\xi - t(\nabla_{\mathcal{X}}\xi) - B\sigma(\mathcal{X}, \xi), \tag{2.21}$$

and

$$\beta\mathcal{X} = -\omega(\nabla_{\mathcal{X}}\xi) - C\sigma(\mathcal{X}, \xi). \tag{2.22}$$

Proposition 2.1. *If $\mathcal{M}$ is a contact CR-submanifold of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$, then $\Gamma(T\mathcal{M})$ is invariant with respect to the action of f if and only if we have*

$$\omega(\nabla_{\mathcal{X}}\xi) = 0, \tag{2.23}$$

and

$$C\sigma(\mathcal{X}, \xi) = 0. \tag{2.24}$$

Proof. From (2.22) it follows that $\mathcal{F}$ is a tensor field of type $(1, 1)$ on $\mathcal{M}$ if and only if

$$\omega(\nabla_{\mathcal{X}}\xi) + C\sigma(\mathcal{X}, \xi) = 0. \tag{2.25}$$

Then (2.23) and (2.24) follows from (2.25) (since $< \omega\mathcal{Y}, C\lambda >= 0$ for any $\mathcal{Y} \in \Gamma(T\mathcal{M})$).

Corollary 2.1. *If $\mathcal{M}$ is a contact CR-submanifold of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$ such that $\Gamma(T\mathcal{M})$ is invariant with respect to the action of $\mathcal{F}$, then both the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are invariant with respect to the action of $\mathcal{F}$.*

Proof. Let $\mathcal{X} \in \Gamma(\mathcal{D})$, then by using the third relation of (2.5) and (2.8) we obtain

$$< \mathcal{F}\mathcal{X}, \xi >= - < \mathcal{X}, \mathcal{F}\xi >= \upsilon(\mathcal{F}\xi) < \mathcal{X}, \xi >= 0.$$

On the other hand, by using (2.2), the second relation of (2.5) and the invariace of $\mathcal{D}$ with respect to the action of f we infer

$$< \mathcal{F}\mathcal{X}, \mathcal{Z} >=< \mathcal{F}f\mathcal{X}', \mathcal{Z} >= - < \mathcal{F}\mathcal{X}', f\mathcal{Z} >= 0,$$

where $\mathcal{X}' \in \Gamma(\mathcal{D})$ and $\mathcal{Z} \in \Gamma(\mathcal{D}^{\perp})$. Hence $\mathcal{D}$ is invariant by $\mathcal{F}$. In a similar way it follows that $\mathcal{D}^{\perp}$ is invariant by the action of $\mathcal{F}$.

The Riemannian connections ∇ and $\nabla^{\perp}$ allow us to define the usual covariant derivatives as

$$(\nabla_{\mathcal{X}}t)\mathcal{Y} = \nabla_{\mathcal{X}}t\mathcal{Y} - t\nabla_{\mathcal{X}}\mathcal{Y}, \tag{2.26}$$

and

$$(\nabla_{\mathcal{X}}\omega)\mathcal{Y} = \nabla^{\perp}_{\mathcal{X}}\omega\mathcal{Y} - \omega\nabla_{\mathcal{X}}\mathcal{Y}. \tag{2.27}$$

Now, the canonical structures t and ω on a submanifold $\mathcal{M}$ are said to be parallel if $\nabla t = 0$ and $\nabla\omega = 0$, respectively. On a CR-submanifold of a quasi-para-Sasakian manifold, it follows from (2.5) and (2.16) that

$$\nabla_{\mathcal{X}}\xi = -f\mathcal{F}\mathcal{X}, \tag{2.28}$$

and

$$\sigma(\mathcal{X}, \xi) = 0 \tag{2.29}$$

for each $\mathcal{X} \in T\mathcal{M}$. Furthermore, from (2.29) we obtain

$$\Lambda_{\omega}\xi = 0; \quad \upsilon(\Lambda_{\omega})\mathcal{X} = 0. \tag{2.30}$$

Lemma 2.1. *For a contact CR-submanifold $\mathcal{M}$ of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$, we infer*

$$(\nabla_{\mathcal{X}}t)\mathcal{Y} = \Lambda_{\omega\mathcal{Y}}\mathcal{X} + B\sigma(\mathcal{X}, \mathcal{Y}) + \upsilon(\mathcal{Y})\alpha\mathcal{X} - <\mathcal{F}\mathcal{X}, \mathcal{Y}>\xi, \tag{2.31}$$

$$(\nabla_{\mathcal{X}}\omega)\mathcal{Y} = C\sigma(\mathcal{X}, \mathcal{Y}) - \sigma(\mathcal{X}, t\mathcal{Y}) + \upsilon(\mathcal{Y})\beta\mathcal{X}. \tag{2.32}$$

Proof. By using (2.4), (2.16)-(2.19), (2.26) and (2.27), we obtain

$$\begin{aligned}(\alpha\mathcal{X} + \beta\mathcal{X})\upsilon(\mathcal{Y}) - <\mathcal{F}\mathcal{X}, \mathcal{Y}>\xi &= (\nabla_{\mathcal{X}}t)\mathcal{Y} + (\nabla_{\mathcal{X}}\omega)\mathcal{Y} - \Lambda_{\omega\mathcal{Y}}\mathcal{X} \\ &\quad -B\sigma(\mathcal{X}, \mathcal{Y}) - C\sigma(\mathcal{X}, \mathcal{Y}) + \sigma(\mathcal{X}, t\mathcal{Y})\end{aligned}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M})$. By equating the tangential and the normal parts in above relation, (2.31) and (2.32), respectively follows.

The covariant derivatives of B and C are given respectively by

$$(\nabla_{\mathcal{X}}B)\lambda = \nabla_{\mathcal{X}}B\lambda - B(\nabla^{\perp}_{\mathcal{X}}\lambda), \tag{2.33}$$

and

$$(\nabla^{\perp}_{\mathcal{X}}C)\lambda = \nabla^{\perp}_{\mathcal{X}}C\lambda - C(\nabla^{\perp}_{\mathcal{X}}\lambda) \tag{2.34}$$

for any $\mathcal{X} \in \Gamma(T\mathcal{M})$ and $\geq \in \Gamma(T\mathcal{M}^{\perp})$.

Lemma 2.2. *For a contact CR-submanifold $\mathcal{M}$ of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$, we infer*

$$(\nabla_{\mathcal{X}}B)\lambda = \Lambda_{C\lambda}\mathcal{X} - t(\Lambda_{\lambda}\mathcal{X}) - <\mathcal{F}\mathcal{X}, \lambda>\xi, \tag{2.35}$$

and

$$(\nabla^{\perp}_{\mathcal{X}}C)\lambda = -\sigma(\mathcal{X}, B\lambda) - \omega(\Lambda_{\lambda}\mathcal{X}) \tag{2.36}$$

for any $\mathcal{X} \in \Gamma(T\mathcal{M})$ and $\lambda \in \Gamma(T\mathcal{M}^{\perp})$.

Lemma 2.3. *For a contact CR-submanifold $\mathcal{M}$ of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$, we infer*

$$\Lambda_{f\mathcal{X}}\mathcal{Y} = \Lambda_{f\mathcal{Y}}\mathcal{X}, \tag{2.37}$$

and

$$< \sigma(\mathcal{U},\mathcal{V}), f\mathcal{Z} >=< \nabla_{\mathcal{U}}\mathcal{Z}, f\mathcal{V} > \tag{2.38}$$

for all $\mathcal{U} \in \Gamma(T\mathcal{M}), \mathcal{V} \in \Gamma(\mathcal{D})$ and $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}^{\perp})$.

Proof. By using (2.2), (2.4) and (2.16)-(2.18), we have

$$< \Lambda_{f\mathcal{X}}\mathcal{Y},\mathcal{U} >=< \sigma(\mathcal{Y},\mathcal{U}), f\mathcal{X} >=< \bar{\nabla}_{\mathcal{U}}\mathcal{Y}, f\mathcal{X} > - < \nabla_{\mathcal{U}}\mathcal{Y}, f\mathcal{X} >$$
$$=< \nabla_{\mathcal{U}}\mathcal{Y}, f\mathcal{X} >= - < f(\nabla_{\mathcal{U}}\mathcal{Y}), \mathcal{X} >= - < -(\bar{\nabla}_{\mathcal{U}}f)\mathcal{Y} + \bar{\nabla}_{\mathcal{U}}f\mathcal{Y}, \mathcal{X} >$$
$$+ < \upsilon(\mathcal{Y})\mathcal{F}\mathcal{U} - < \mathcal{F}\mathcal{U}, \mathcal{Y} > \xi, \mathcal{X} > - < \bar{\nabla}_{\mathcal{U}}f\mathcal{Y}, \mathcal{X} >$$
$$- < -\Lambda_{f\mathcal{Y}}\mathcal{U} + \nabla^{\perp}_{\mathcal{U}}f\mathcal{Y}, \mathcal{X} >=< \Lambda_{f\mathcal{Y}}\mathcal{U}, \mathcal{X} >=< \Lambda_{f\mathcal{Y}}\mathcal{X},\mathcal{U} > .$$

Since $\upsilon(\mathcal{Y}) = \upsilon(\mathcal{X}) = 0$, therefore we find (2.37).
Next, by using (2.2), (2.4) and (2.16), we obtain

$$< \sigma(\mathcal{U},\mathcal{V}), f\mathcal{Z} >=< \bar{\nabla}_{\mathcal{U}}\mathcal{V}, f\mathcal{Z} > - < \mathcal{V}, \bar{\nabla}_{\mathcal{U}}f\mathcal{Z} >$$
$$- < \mathcal{V}, (\bar{\nabla}_{\mathcal{U}}f)\mathcal{Z} + f(\bar{\nabla}_{\mathcal{U}}\mathcal{Z}) > - < \mathcal{V}, \upsilon(\mathcal{Z})\mathcal{F}\mathcal{U} - < \mathcal{F}\mathcal{U}, \mathcal{Z} > \xi >$$
$$- < \mathcal{V}, f(\bar{\nabla}_{\mathcal{U}}\mathcal{Z}) >=< f\mathcal{V}, \bar{\nabla}_{\mathcal{U}}\mathcal{Z} >=< f\mathcal{V}, \nabla_{\mathcal{U}}\mathcal{Z} >$$

which leads to (2.38).

A submanifold $\mathcal{M}$ of an almost para contact metric manifold $\bar{\mathcal{M}}$ is said to be invariant if $\mathcal{F}$ is identically zero, that is, $f\mathcal{X} \in T\mathcal{M}$ and anti-invariant if t is identically zero, that is, $f\mathcal{X} \in T^{\perp}\mathcal{M}$, for any $\mathcal{X} \in TM$.
For each non-zero vector $\mathcal{X}$ tangent to $\mathcal{M}$ at any point x such that $\mathcal{X}$ is not proportional to ξ, we denote by $\theta(\mathcal{X})$, the angle between $f\mathcal{X}$ and $T_x\mathcal{M}$ for all $x \in \mathcal{M}$.

Definition 2.1. *A submanifold N is said to be slant if the angle $\theta(\mathcal{X})$ is constant for all $\mathcal{X} \in T_{\mathcal{X}}N - \{\xi\}$ and $x \in N$. The angle θ is called a slant angle or Wirtinger angle. Obviously, if $\theta = 0$, then N is invariant; and if $\theta = \pi/2$, then $\mathcal{M}$ is an anti-invariant submanifold. If the slant angle of N is different from 0 and $\pi/2$ then it is called proper slant.*

A characterization of slant submanifolds is given by the following theorem:

Theorem 2.1. *[9] Let N be slant submanifold of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$ such that ξ is tangent to N. Then N is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$t^2\mathcal{X} = \mu(\mathcal{X} - \upsilon(\mathcal{X}))\xi. \tag{2.39}$$

Furthermore, if θ is the slant angle of N, then $\mu = \cos^2\theta$.

Corollary 2.2. *Let N be a slant submanifold with slant angle θ of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$ such that ξ is tangent to N. Then we have*

$$< t\mathcal{Z}, t\mathcal{W} >= \cos^2\theta\{- < \mathcal{Z}, \mathcal{W} > +\upsilon(\mathcal{Z})\upsilon(\mathcal{W})\}, \tag{2.40}$$

$$< \omega\mathcal{Z}, \omega\mathcal{W} >= \sin^2\theta\{- < \mathcal{Z}, \mathcal{W} > +\upsilon(\mathcal{Z})\upsilon(\mathcal{W})\} \tag{2.41}$$

for any $\mathcal{Z}, \mathcal{W}$ tangent to N.

3. Warped product semi-slant submanifolds a quasi-para-Sasakian manifold

For two Riemannian manifolds $(N_1, <,>_1)$ and $(N_2, <,>_2)$ and a positive differentiable function δ on N_1, the warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_\delta N_2 = (N_1 \times N_2, <,>)$, where

$$<,>=<,>_1 + \delta^2 <,>_2 . \tag{3.1}$$

More explicitly, if the vector fields $\mathcal{X}$ and $\mathcal{Y}$ are tangent to $N_1 \times_\delta N_2$ at (x, y), then

$$< \mathcal{X}, \mathcal{Y} >=<,>_1 (\pi_1 * \mathcal{X}, \pi_1 * \mathcal{Y}) + \delta^2(x) <,>_2 (\pi_2 * \mathcal{X}, \pi_2 * \mathcal{Y}), \tag{3.2}$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $N_1 \times_\delta N_2$ onto N_1 and N_2, respectively, and $*$ stands for derivative map.

If $\widetilde{\mathcal{M}} = N_1 \times_\delta N_2$ is a warped product manifold, this means that N_1 and N_2 are totally geodesic and totally umbilical submanifolds of $\widetilde{\mathcal{M}}$, respectively.

For warped product manifolds, we have the following proposition [12, 15]:

Proposition 3.1. *On a warped product manifold $\widetilde{\mathcal{M}} = N_1 \times_\delta N_2$, we have*

(1) $\nabla_{\mathcal{X}}\mathcal{Y} \in \Gamma(TN_1)$ *is the lift of* $\nabla_{\mathcal{X}}\mathcal{Y}$ *on* N_1,

(2) $\nabla_{\mathcal{U}}\mathcal{X} = \nabla_{\mathcal{X}}\mathcal{U} = \mathcal{X}(ln\delta)\mathcal{U}$,

(3) $\nabla_{\mathcal{U}}\mathcal{V} = \nabla'_{\mathcal{U}}\mathcal{V} - < \mathcal{U}, \mathcal{V} > \nabla ln_\delta$

for any $X, Y \in \Gamma(TN_1)$ and $U, V \in \Gamma(TN_2)$, where ∇ and ∇' denote the Levi-Civita connections on $\mathcal{M}$ and N_2, respectively.

Let us suppose that $\bar{\mathcal{M}}$ be a quasi-para-Sasakian manifold and $N_1 \times_\delta N_2$ be a warped product semi-slant submanifold of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$. Such submanifolds are always tangent to the structure vector field ξ. If the manifolds N_θ and N_T (resp., $N^\perp$) are slant and invariant (resp., anti-invariant) submanifolds of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$, then their warped product semi-slant submanifolds may be given by one of the following forms:

(i) $N_\theta \times_\delta N_T$, (ii) $N_\theta \times_\delta N_\perp$, (iii) $N_T \times_\delta N_\theta$, (iv) $N_\perp \times_\delta N_\theta$.

Here, we are concerned with cases (i) and (ii).

Theorem 3.1. *If $\bar{\mathcal{M}}$ is a quasi-para-Sasakian manifold, then there do not exist proper warped product semi-slant submanifolds $N_\theta \times_\delta N_T$ such that N_θ is a proper slant submanifold, N_T is an invariant submanifold of $\bar{\mathcal{M}}$ and ξ is tangent to N .*

Proof. Let $N_\theta \times_\delta N_T$ be a proper warped product semi-slant submanifold of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$. For any $\mathcal{X}, \mathcal{Y} \in \Gamma(TN_\theta)$ and $\mathcal{U}, \mathcal{V} \in \Gamma(TN_T)$, we have

$$(\bar{\nabla}_{\mathcal{X}} f)\mathcal{U} = \bar{\nabla}_{\mathcal{X}} f\mathcal{U} - f(\bar{\nabla}_{\mathcal{X}}\mathcal{U}). \tag{3.3}$$

Thus, from (2.4), (2.11), (2.14) and (2.16) we obtain

$$\upsilon(\mathcal{U})\mathcal{F}\mathcal{X} - <\mathcal{F}\mathcal{X}, \mathcal{U}> \xi = \sigma(\mathcal{X}, t\mathcal{U}) - B\sigma(\mathcal{X}, \mathcal{U}) - C\sigma(\mathcal{X}, \mathcal{U}).$$

This means that

$$B\sigma(\mathcal{X}, \mathcal{U}) = 0, \tag{3.4}$$

and

$$C\sigma(\mathcal{X}, \mathcal{U}) - \sigma(\mathcal{X}, t\mathcal{U}) = 0. \tag{3.5}$$

On the other hand, by interchanging roles of $\mathcal{U}$ and $\mathcal{X}$ in (3.3), we conclude

$$t\mathcal{X}log(\delta)\mathcal{U} = \Lambda_{\omega\mathcal{X}}\mathcal{U} + \mathcal{X}log(\delta)t\mathcal{U} + B\sigma(\mathcal{U}, \mathcal{X}), \tag{3.6}$$

and

$$\nabla^{\perp}_{\mathcal{U}}\omega\mathcal{X} + \sigma(\mathcal{U}, t\mathcal{X}) - C\sigma(\mathcal{U}, \mathcal{X}) = 0. \tag{3.7}$$

From (3.6), we arrive at

$$\begin{aligned} t\mathcal{X}log(\delta) <\mathcal{U}, \mathcal{U}> &= <\Lambda_{\omega\mathcal{X}}\mathcal{U}, \mathcal{U}> + <B\sigma(\mathcal{U}, \mathcal{X}), \mathcal{U}> \\ &= <\sigma(\mathcal{U}, \mathcal{U}), \omega\mathcal{X}> + <B\sigma(\mathcal{U}, \mathcal{X}), \mathcal{U}> \\ &= <\sigma(\mathcal{U}, \mathcal{U}), \omega\mathcal{X}> - <\sigma(\mathcal{X}, \mathcal{U}), f\mathcal{U}> \\ &= <\sigma(\mathcal{U}, \mathcal{U}), \omega\mathcal{X}> . \end{aligned} \tag{3.8}$$

On the other hand, since the ambient space $\bar{\mathcal{M}}$ is a quasi-para-Sasakian manifold, then by using (3.5) and (3.7) we get

$$Ch(\mathcal{Z}, \xi) = 0 \tag{3.9}$$

for any $\mathcal{Z} \in \Gamma(TN)$.

By using (3.5) and (3.7), we get $\omega\mathcal{X} = C\sigma(\mathcal{X}, \xi) = 0$. Thus we have $t\mathcal{X}log(\delta) <\mathcal{U}, \mathcal{U}> = 0$, this implies that $t\mathcal{X}log(\delta) = 0$, that is, the warping function δ is constant on N_θ. □

Theorem 3.2. *If $\bar{\mathcal{M}}$ is a quasi-para-Sasakian manifold, then there do not exist proper warped product semi-slant submanifolds $N_\theta \times_\delta N_\perp$ such that N_θ is a proper slant submanifold, $N_\perp$ is an invariant submanifold of $\bar{\mathcal{M}}$ and ξ is tangent to N .*

Proof. Let $N_\theta \times_\delta N_\perp$ be a proper warped product semi-slant submanifold of a quasi-para-Sasakian manifold $\bar{\mathcal{M}}$ such that ξ is tangent to N. For any $\mathcal{X}, \mathcal{Y} \in \Gamma(TN_\theta)$ and $\mathcal{U}, \mathcal{V} \in \Gamma(TN_\perp)$, we have

$$(\bar{\nabla}_{\mathcal{X}} f)\mathcal{U} = \bar{\nabla}_{\mathcal{X}} f\mathcal{U} - f(\bar{\nabla}_{\mathcal{X}}\mathcal{U}).$$

Using (2.4), (2.14), (2.16), (2.17) and Proposition 3.1, the above equation takes the form

$$\upsilon(\mathcal{U})\mathcal{F}\mathcal{X} - g(\mathcal{F}\mathcal{X},\mathcal{U})\xi = -\Lambda_{\omega\mathcal{U}}\mathcal{X} + \nabla^{\perp}_{\mathcal{X}}\omega\mathcal{U} - \mathcal{X}(log\delta)\omega\mathcal{U} \tag{3.10}$$
$$-f\sigma(\mathcal{X},\mathcal{U}).$$

This means that

$$\Lambda_{\omega\mathcal{U}}\mathcal{X} + B\sigma(\mathcal{X},\mathcal{U}) = 0, \tag{3.11}$$

and

$$\nabla^{\perp}_{\mathcal{X}}\omega\mathcal{U} - \mathcal{X}(log\delta)\omega\mathcal{U} - C\sigma(\mathcal{X},\mathcal{U}) = 0. \tag{3.12}$$

By interchanging roles of $\mathcal{X}$ and $\mathcal{U}$ in (3.10), we arrive at

$$\begin{aligned}\upsilon(\mathcal{U})\mathcal{F}\mathcal{X} - < \mathcal{F}\mathcal{X},\mathcal{U} > \xi \;&=\; t\mathcal{X}log(\delta)\mathcal{U} + \sigma(\mathcal{U}, t\mathcal{X}) - \Lambda_{\omega\mathcal{X}}\mathcal{U} \\ &\quad +\nabla^{\perp}_{\mathcal{U}}\omega\mathcal{X} - \mathcal{X}log(\delta)\omega\mathcal{U} - B\sigma(\mathcal{U},\mathcal{X}) \\ &\quad -C\sigma(\mathcal{U},\mathcal{X}).\end{aligned} \tag{3.13}$$

Equating the tangential and normal components in (3.13), we find

$$t\mathcal{X}log(\delta)\mathcal{U} = \Lambda_{\omega\mathcal{X}}\mathcal{U} + B\sigma(\mathcal{U},\mathcal{X}), \tag{3.14}$$

and

$$\sigma(\mathcal{U}, t\mathcal{X}) + \nabla^{\perp}_{\mathcal{U}}\omega\mathcal{X} - \mathcal{X}log(\delta)\omega\mathcal{U} - C\sigma(\mathcal{U},\mathcal{X}) = 0, \tag{3.15}$$

respectively.
From (3.14), we find

$$< \Lambda_{\omega\mathcal{X}}\mathcal{U}, t\mathcal{Y} > + < B\sigma(\mathcal{U},\mathcal{X}), t\mathcal{Y} >= 0. \tag{3.16}$$

Since the ambient space $\bar{\mathcal{M}}$ is a quasi-para-Sasakian manifold, ξ is tangent to N and using (2.2), we obtain

$$\begin{aligned}< B\sigma(\mathcal{X},\mathcal{U}), t\mathcal{Y} > \;&=\; < f\sigma(\mathcal{X},\mathcal{U}), f\mathcal{Y} > \\ &=\; - < \sigma(\mathcal{X},\mathcal{U}), \mathcal{Y} > +\upsilon(\mathcal{Y})\upsilon(\sigma(\mathcal{X},\mathcal{U})) \\ &=\; 0.\end{aligned}$$

This implies that

$$< B\sigma(\mathcal{X},\mathcal{U}), t\mathcal{Y} >=< \sigma(\mathcal{U}, t\mathcal{Y}), \omega\mathcal{X} >= 0. \tag{3.17}$$

Thus we have

$$< \sigma(\mathcal{U}, t\mathcal{Y}), f\mathcal{X} >= 0 \tag{3.18}$$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(TN_\theta)$.
Moreover, making use of (3.11) and (3.18), we get

$$< \sigma(\mathcal{X}, t\mathcal{Y}), f\mathcal{U} >= 0. \tag{3.19}$$

By using the Gauss-Weingarten formulas and considering that N_θ is totally geodesic in N, we arrive at

$$\begin{aligned} < \sigma(\mathcal{X}, t\mathcal{Y}), f\mathcal{U} > &= < \bar{\nabla}_{t\mathcal{Y}}\mathcal{X}, f\mathcal{U}) = - < f(\bar{\nabla}_{t\mathcal{Y}}\mathcal{X}), \mathcal{U} > \\ &= - < \bar{\nabla}_{t\mathcal{Y}} f\mathcal{X} - (\bar{\nabla}_{t\mathcal{Y}} f)\mathcal{X}, \mathcal{U} > \\ &= - < \bar{\nabla}_{t\mathcal{Y}} t\mathcal{X}, \mathcal{U} > - < \bar{\nabla}_{t\mathcal{Y}} \omega\mathcal{X}, \mathcal{U} > \\ &\quad + < \upsilon(\mathcal{X})\mathcal{F}t\mathcal{Y}, \mathcal{U} > - < \mathcal{F}t\mathcal{Y}, \mathcal{X} >< \xi, \mathcal{U} > \\ &= < \Lambda_{\omega\mathcal{X}} t\mathcal{Y}, \mathcal{U} > -\upsilon(\mathcal{U}) < \mathcal{F}t\mathcal{Y}, \mathcal{X} > \\ &= < \sigma(t\mathcal{Y}, \mathcal{U}), \omega\mathcal{X} > -\upsilon(\mathcal{U}) < \mathcal{F}t\mathcal{Y}, \mathcal{X} > \\ &= \upsilon(\mathcal{U}) < t\mathcal{Y}, \mathcal{F}\mathcal{X} > . \end{aligned} \tag{3.20}$$

Thus from (3.19) and (3.20), we conclude

$$\upsilon(\mathcal{U}) < t\mathcal{Y}, \mathcal{F}\mathcal{X} > = < \sigma(\mathcal{X}, t\mathcal{Y}, f\mathcal{U} > = 0. \tag{3.21}$$

Here, if $\upsilon(\mathcal{U}) = 0$, then by using (2.32) and (3.12), we leads to

$$\mathcal{X} log(\delta)\omega\mathcal{U} = \upsilon(\nabla_{\mathcal{X}}\mathcal{U}) = - < -f\mathcal{F}\mathcal{X}, \mathcal{U} > = 0.$$

This is impossible. Because U is a non-zero vector field and $N_\perp \neq 0$. Thus $< t\mathcal{X}, t\mathcal{Y} > = cos^2\theta\{- < \mathcal{X}, \mathcal{Y} > +\upsilon(\mathcal{X})\upsilon(\mathcal{Y})\} = 0$, this implies that the slant angle θ is either identically $\pi/2$ or the warping function δ is constant on N_θ . This completes the proof. □

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